

Algebraic Topology

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Unit 1 – Introduction to Algebraic Topology

1.1 Overview of algebraic topology

Algebraic topology bridges the gap between geometry and algebra, using mathematical structures to analyze shapes and spaces. It assigns groups and other algebraic objects to topological spaces, helping us understand their properties and relationships.

This field revolutionizes our approach to geometry, allowing us to classify spaces and detect subtle differences. By connecting abstract algebra with concrete spatial concepts, algebraic topology opens up new ways to solve complex problems in math and beyond.

Fundamentals of Algebraic Topology

Goals and Methods

- Algebraic topology aims to study topological spaces using algebraic structures and techniques
- Assigns algebraic invariants (groups, rings, homology) to topological spaces in a way that preserves certain properties
- Allows for the classification and comparison of spaces
- Methods involve constructing algebraic objects from topological spaces and continuous maps between them
- Algebraic objects include groups, rings, or modules
- Algebraic invariants, such as homology and homotopy groups, are used to distinguish between non-homeomorphic spaces and detect certain topological properties
- For example, the fundamental group can detect the presence of "holes" in a space
- Computations often involve techniques from abstract algebra
- Exact sequences, chain complexes, and commutative diagrams are commonly used tools

Importance and Applications

- Algebraic topology provides a powerful framework for understanding the structure and properties of topological spaces
- It allows for the classification of spaces up to homeomorphism or homotopy equivalence
- Two spaces are homeomorphic if there exists a continuous bijection with a continuous inverse between them
- Two spaces are homotopy equivalent if there exist continuous maps between them that compose to the identity up to homotopy
- Algebraic invariants can detect obstructions to the existence of certain continuous maps or homeomorphisms

- For example, the fundamental group can show that the torus and the sphere are not homeomorphic
- Techniques from algebraic topology have found applications in various fields
- In physics, topological quantum field theories use ideas from algebraic topology
- In data analysis, persistent homology is used to study the "shape" of data sets

Connections of Algebraic Topology

Connections to Other Fields of Mathematics

- Algebraic topology has strong connections to various branches of mathematics
- Geometry: Techniques from algebraic topology are used to study manifolds, which are central objects in differential geometry and geometric topology
- Algebra: Algebraic topology utilizes concepts and tools from abstract algebra, such as group theory, ring theory, and module theory
- Analysis: Homological algebra, which deals with the study of chain complexes and their homology, is a key tool in algebraic topology and has applications in functional analysis
- Algebraic topology has influenced the development of category theory
- Many constructions in algebraic topology, such as homology and cohomology, are functorial
- Category theory provides a language for studying the relationships between different algebraic invariants

Interdisciplinary Applications

- Ideas from algebraic topology have been applied to various fields outside of mathematics
- Physics: Topological quantum field theories use algebraic topology to study quantum systems and their properties
- Computer Science: Persistent homology, a technique inspired by algebraic topology, is used in data analysis to study the "shape" of data sets
- Biology: Topological data analysis has been applied to study the structure of biological networks and to identify patterns in gene expression data
- Algebraic topology provides a framework for understanding the global structure and connectivity of complex systems
- It can reveal hidden patterns and relationships that may not be apparent from local information alone

Objects of Study in Algebraic Topology

Topological Spaces

- Topological spaces are the primary objects of study in algebraic topology
- A topological space is a set equipped with a topology, which specifies a notion of nearness or continuity

- Examples include metric spaces (where distance is defined), manifolds (locally Euclidean spaces), simplicial complexes (built from simplices), and CW complexes (built from cells)
- Continuous maps between topological spaces are central to algebraic topology
- A continuous map preserves the topological structure
- Continuous maps induce homomorphisms between algebraic invariants, allowing for the comparison of spaces

Algebraic Structures

- Algebraic structures, such as groups, rings, and modules, are associated with topological spaces to capture their properties and enable computations
- Homotopy groups, particularly the fundamental group, measure the "holes" in a topological space and the ways in which loops can be deformed
- The fundamental group assigns to each space a group that encodes information about the loops in the space
- Higher homotopy groups capture higher-dimensional hole-like features
- Homology groups are another important class of algebraic invariants
- Homology assigns abelian groups to a topological space, capturing its "hole" structure in various dimensions
- Homology groups are computed using chain complexes, which consist of abelian groups and boundary maps between them
- Cohomology groups are dual to homology groups and provide additional algebraic information about a space
- Cohomology groups are defined using cochains, which are dual to chains, and coboundary maps
- Cup products in cohomology give cohomology rings, which have a rich algebraic structure

1.2 Motivation and historical context

Algebraic topology emerged in the late 19th century as mathematicians applied algebraic methods to topological problems. Key figures like Henri Poincaré and Emmy Noether laid the groundwork, introducing fundamental concepts like homology groups and abstract algebra.

This field aims to solve topological problems using algebraic methods, providing more computable solutions. It's driven by the desire to understand global properties of spaces and has applications in physics, chemistry, and dynamical systems.

Historical Development of Algebraic Topology

Emergence and Key Figures

- Algebraic topology emerged in the late 19th and early 20th centuries as mathematicians sought to apply algebraic methods to topological problems
- Henri Poincaré, often considered the founder of algebraic topology, introduced the fundamental group and homology groups in his seminal work "Analysis Situs" (1895)

- Emmy Noether's work on abstract algebra and homology theory in the 1920s and 1930s provided a solid foundation for the development of algebraic topology
- Solomon Lefschetz made significant contributions to algebraic topology, including the development of the Lefschetz fixed-point theorem and the study of manifolds

Further Developments and Contributions

- In the 1940s and 1950s, Norman Steenrod and Samuel Eilenberg further developed the theory of homology and cohomology, establishing them as powerful tools in algebraic topology
- Their work helped to systematize and generalize the concepts introduced by earlier mathematicians
- They introduced the notion of axiomatic homology and cohomology theories, which provided a unified framework for studying topological spaces
- J.H.C. Whitehead introduced the concept of homotopy groups in the 1930s, extending the ideas of the fundamental group to higher dimensions
- Homotopy groups capture information about the higher-dimensional holes and the different ways in which spheres can be mapped into a space
- Whitehead's work laid the foundation for the study of homotopy theory, which has become a central part of algebraic topology
- René Thom's work on cobordism theory in the 1950s and his development of catastrophe theory in the 1960s had a significant impact on algebraic topology and its applications
- Cobordism theory studies the relationships between manifolds of different dimensions and has important applications in differential topology and physics
- Catastrophe theory, which studies sudden changes in the behavior of dynamical systems, has found applications in a wide range of fields, including biology, economics, and social sciences

Motivations for Algebraic Topology

Solving Topological Problems with Algebraic Methods

- Algebraic topology arose from the desire to solve topological problems using algebraic methods, which often provide more computable and tractable solutions
- The study of knots and links, which has applications in physics and chemistry, motivated the development of invariants such as the fundamental group and polynomial invariants
- These invariants allow mathematicians to distinguish between different knots and links and to study their properties
- Examples of polynomial invariants include the Alexander polynomial and the Jones polynomial
- The classification of surfaces and higher-dimensional manifolds drove the development of homology and cohomology theories, which provide computable invariants for distinguishing topological spaces
- Homology groups capture information about the holes in a space and can be used to distinguish between spaces that are not homeomorphic (topologically equivalent)
- Cohomology groups, which are related to homology groups by duality, provide additional algebraic invariants and have important applications in physics and geometry

Understanding Global Properties and Dynamical Systems

- Algebraic topology has been motivated by the desire to understand the global properties of spaces, such as connectedness, compactness, and orientability, using algebraic structures
- Connectedness refers to the property of a space being in one piece, while compactness is a generalization of the notion of a space being closed and bounded
- Orientability is a property of manifolds that allows for a consistent choice of orientation (e.g., a consistent choice of clockwise or counterclockwise rotation)
- The study of vector fields and dynamical systems on manifolds led to the development of techniques such as the Lefschetz fixed-point theorem and the Poincaré-Hopf theorem
- The Lefschetz fixed-point theorem relates the fixed points of a continuous map on a manifold to the topology of the manifold, providing a powerful tool for studying dynamical systems
- The Poincaré-Hopf theorem relates the zeros of a vector field on a manifold to the Euler characteristic of the manifold, which is a topological invariant

Algebraic Topology and Other Mathematics

Interactions with Abstract Algebra and Differential Geometry

- The development of abstract algebra, particularly group theory and ring theory, provided the necessary algebraic tools for the growth of algebraic topology
- Groups, which capture the notion of symmetry, are fundamental objects in algebraic topology and are used to define invariants such as homology and homotopy groups
- Rings, which generalize the algebraic structure of integers, are used in the construction of cohomology theories and in the study of algebraic invariants of topological spaces
- Advances in differential geometry, such as the study of Riemannian manifolds and vector bundles, have been closely intertwined with the development of algebraic topology
- Riemannian manifolds, which are smooth manifolds equipped with a metric (a way of measuring distances and angles), provide a natural setting for studying the interplay between topology and geometry
- Vector bundles, which are spaces that locally look like the product of a manifold and a vector space, are important objects in both differential geometry and algebraic topology and have applications in physics and engineering

Applications and Influence on Other Fields

- The study of partial differential equations on manifolds has benefited from the tools and techniques of algebraic topology, such as the index theorem and the Atiyah-Singer theorem
- The index theorem relates the solutions of certain differential equations on a manifold to topological invariants of the manifold, providing a deep connection between analysis and topology
- The Atiyah-Singer theorem generalizes the index theorem to a wider class of differential operators and has important applications in physics, particularly in quantum field theory
- Algebraic topology has had significant applications in mathematical physics, particularly in the study of gauge theories and string theory

- Gauge theories, which describe the fundamental interactions of particles in terms of connections on vector bundles, rely heavily on the tools of algebraic topology, such as characteristic classes and Chern-Simons theory
- String theory, which is a candidate for a unified theory of physics, makes extensive use of algebraic topology, particularly in the study of Calabi-Yau manifolds and mirror symmetry
- The development of category theory and homological algebra has provided a unifying framework for many constructions in algebraic topology, such as the study of functors and spectral sequences
- Category theory, which studies the abstract relationships between mathematical objects, has become an essential language for modern algebraic topology
- Homological algebra, which studies the algebraic properties of chain complexes and their homology groups, provides a general framework for constructing and studying algebraic invariants in topology
- Algebraic topology has influenced the development of other areas of mathematics, such as algebraic geometry and representation theory, through the use of common tools and analogies
- Algebraic geometry, which studies geometric objects defined by polynomial equations, has benefited from the tools of algebraic topology, such as cohomology theories and K-theory
- Representation theory, which studies the ways in which abstract algebraic structures (such as groups and algebras) can be represented as linear transformations of vector spaces, has been influenced by the ideas of algebraic topology, particularly in the study of Lie groups and their representations

1.3 Basic concepts and terminology

Algebraic topology studies spaces through algebraic structures. It introduces topological spaces, continuous functions, and homeomorphisms as foundational concepts. These ideas help classify spaces based on their topological properties and lay the groundwork for more advanced topics.

The field uses algebraic invariants like homology groups and the fundamental group to distinguish spaces. These tools capture information about holes and loops in spaces, allowing mathematicians to classify and analyze complex topological structures beyond visual intuition.

Topological Spaces and Examples

Topological Spaces and their Axioms

- A topological space is a set X together with a collection of subsets of X , called open sets, satisfying certain axioms
- The empty set and X are open
- Arbitrary unions of open sets are open
- Finite intersections of open sets are open
- Examples of topological spaces
- Metric spaces (the real line, Euclidean space)
- Discrete spaces
- Indiscrete spaces
- Finite topological spaces

Continuous Functions and Topological Properties

- Continuous functions between topological spaces are maps that preserve the topological structure
- The preimage of an open set under a continuous function is open
- Homeomorphisms are continuous bijections with continuous inverses, capturing the notion of topological equivalence between spaces
- Topological properties are those preserved by homeomorphisms
- Connectedness
- Compactness
- Hausdorffness

Homeomorphism vs Homotopy Equivalence

Homeomorphisms and Topological Equivalence

- Two topological spaces X and Y are homeomorphic if there exists a homeomorphism between them
- A homeomorphism is a continuous bijection with a continuous inverse
- Homeomorphic spaces have the same topological properties and are considered topologically equivalent
- Example: The unit interval $[0, 1]$ and the unit circle S^1 are homeomorphic via the exponential map
- Example: The open interval $(0, 1)$ and the real line $\mathbb{R}$ are homeomorphic

Homotopy and Homotopy Equivalence

- Homotopy is a continuous deformation of one continuous function into another
- It captures the idea of continuously transforming one map into another
- Two continuous functions $f, g: X \rightarrow Y$ are homotopic if there exists a continuous function $H: X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$
- Homotopy equivalence is a weaker notion than homeomorphism
- Two spaces X and Y are homotopy equivalent if there exist continuous maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f$ is homotopic to the identity on X and $f \circ g$ is homotopic to the identity on Y
- Contractible spaces, such as convex subsets of Euclidean space, are homotopy equivalent to a point
- Example: The unit disk D^2 is contractible and homotopy equivalent to a point
- Example: The real line $\mathbb{R}$ is contractible

Algebraic Invariants for Classification

Algebraic Invariants and their Properties

- Algebraic invariants are algebraic objects associated with a topological space that remain unchanged under homeomorphisms or homotopy equivalences

- Examples: homology groups, cohomology groups, homotopy groups, fundamental group
- Algebraic invariants can be used to distinguish between non-homeomorphic or non-homotopy equivalent spaces
- If two spaces have different algebraic invariants, they cannot be homeomorphic or homotopy equivalent
- The fundamental group captures information about the loops in a topological space and their homotopy classes
- Example: The fundamental group of a circle S^1 is isomorphic to the integers $\mathbb{Z}$
- Example: The fundamental group of a torus T^2 is isomorphic to $\mathbb{Z} \times \mathbb{Z}$

Homology and Cohomology

- Homology and cohomology groups provide a measure of the "holes" in a topological space
- They are computed using chain complexes and boundary operators
- Homology groups are denoted by $H_n(X)$ for a topological space X and a non-negative integer n
- $H_0(X)$ measures the connected components of X
- $H_1(X)$ measures the 1-dimensional holes (loops) in X
- Higher homology groups measure higher-dimensional holes
- Cohomology groups are denoted by $H^n(X)$ and are dual to homology groups
- They capture similar information but with additional algebraic structure (e.g., cup product)

Terminology in Algebraic Topology

Simplices and Simplicial Complexes

- A simplex is a generalization of a triangle or tetrahedron to higher dimensions
- A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and so on
- Simplices serve as building blocks for more complex spaces
- Simplicial complexes are spaces constructed by gluing simplices together along their faces
- They provide a combinatorial description of a topological space
- Example: A simplicial complex representing a torus can be constructed using 18 triangles
- Example: A simplicial complex representing a Klein bottle can be constructed using 18 triangles with certain identifications

CW Complexes and Chain Complexes

- CW complexes are a generalization of simplicial complexes, allowing for more flexible cell attachments
- They provide a convenient framework for computations in algebraic topology
- A CW complex is built by inductively attaching cells of increasing dimension

- 0-cells are points, 1-cells are intervals, 2-cells are disks, and so on
- Chain complexes are algebraic objects that encode the boundary information of a topological space
- They are used in the computation of homology and cohomology groups
- A chain complex consists of a sequence of abelian groups (or modules) connected by boundary operators
- The boundary operators satisfy the property that the composition of any two consecutive boundary operators is zero

Exact Sequences and Homotopy Groups

- Exact sequences are a powerful tool in algebraic topology, relating various algebraic invariants
- They allow for the computation of one invariant in terms of others
- A sequence of abelian groups (or modules) and homomorphisms is exact if the image of each homomorphism equals the kernel of the next
- Example: The long exact sequence of a pair relates the homology groups of a space X , a subspace A , and the quotient space X/A
- Homotopy groups generalize the fundamental group to higher dimensions
- They capture information about higher-dimensional holes and the homotopy classes of maps between spaces
- The n -th homotopy group of a space X , denoted $\pi_n(X)$, is the set of homotopy classes of maps from the n -dimensional sphere S^n to X
- $\pi_0(X)$ is the set of path-connected components of X
- $\pi_1(X)$ is the fundamental group of X
- Higher homotopy groups ($\pi_n(X)$ for $n \geq 2$) are abelian and measure higher-dimensional homotopy information

Unit 2 – Simplicial Complexes and Homology

2.1 Simplicial complexes and simplicial maps

Simplicial complexes are building blocks for studying shapes in algebraic topology. They're made of simple pieces called simplices that fit together in specific ways, allowing us to model complex spaces using basic geometric shapes.

Simplicial maps help us understand relationships between different simplicial complexes. These maps preserve the structure of the complexes, letting us compare and analyze various shapes while maintaining their key features.

Simplicial Complexes and Properties

Definition and Structure

- A simplicial complex is a collection of simplices that fit together in a specific way to form a topological space
- A simplex is a generalization of a triangle or tetrahedron to arbitrary dimensions
- An n -simplex is determined by $n+1$ vertices (point, line segment, triangle, tetrahedron)
- The faces of a simplex are the simplices formed by subsets of its vertices
- A simplicial complex requires that any face of a simplex in the complex also belongs to the complex
- Example: In a 2D simplicial complex, if a triangle is present, its edges and vertices must also be in the complex

Dimension and Intersection Properties

- The dimension of a simplicial complex is the maximum dimension of its simplices
- Example: A complex with triangles (2-simplices) as its highest-dimensional simplices has dimension 2
- A simplicial complex is called pure if all its maximal simplices have the same dimension
- Maximal simplices are not faces of any other simplex in the complex
- Two simplices in a simplicial complex can intersect only along a common face
- This ensures that the simplicial complex has a well-defined topology
- Example: Two triangles can intersect along a shared edge or vertex, but not along a portion of an edge

Simplicial Maps between Complexes

Definition and Properties

- A simplicial map is a continuous function between two simplicial complexes that maps simplices to simplices

- It is determined by its action on the vertices of the domain complex
- Each vertex is mapped to a vertex of the codomain complex
- A simplicial map must respect the structure of the simplicial complexes
- If a set of vertices forms a simplex in the domain complex, their images must form a simplex in the codomain complex
- Simplicial maps compose: if $f : K \rightarrow L$ and $g : L \rightarrow M$ are simplicial maps, then $g \circ f : K \rightarrow M$ is also a simplicial map

Isomorphisms

- An isomorphism of simplicial complexes is a bijective simplicial map whose inverse is also simplicial
- Isomorphic simplicial complexes have the same combinatorial structure
- Example: Two triangulations of a surface that can be obtained from each other by relabeling vertices are isomorphic

Geometric Realization of a Complex

Construction and Properties

- The geometric realization of a simplicial complex is a topological space obtained by gluing together geometric simplices according to the combinatorial data of the complex
- Each n -simplex in the complex is realized as a standard geometric n -simplex
- A standard n -simplex is the convex hull of $n+1$ affinely independent points in Euclidean space
- The geometric simplices are glued together along their faces as specified by the combinatorial structure of the simplicial complex
- The geometric realization provides a way to visualize abstract simplicial complexes and to study their topological properties
- Example: The geometric realization of a simplicial complex representing a torus will be a torus-shaped surface

Relationship with Simplicial Maps

- Simplicial maps between simplicial complexes induce continuous maps between their geometric realizations
- The geometric realization is a functor from the category of simplicial complexes to the category of topological spaces
- Example: A simplicial map that collapses a simplex to a point will induce a continuous map that collapses the corresponding geometric simplex to a point

Structure of Abstract Simplicial Complexes

Combinatorial Definition

- An abstract simplicial complex is a purely combinatorial description of a simplicial complex

- It is defined as a collection of finite sets (called abstract simplices) that is closed under taking subsets
- Example: $\{\{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1\}, \{2\}, \{3\}\}$ is an abstract simplicial complex representing a triangle
- Every geometric simplicial complex has an associated abstract simplicial complex, obtained by taking the vertex sets of its simplices
- Conversely, every abstract simplicial complex has a geometric realization

Combinatorial Invariants

- The f-vector of an abstract simplicial complex records the number of simplices of each dimension
- It provides a basic combinatorial invariant of the complex
- Example: The f-vector of a triangle is (3,3,1), indicating 3 vertices, 3 edges, and 1 triangle
- The Euler characteristic of a simplicial complex can be computed from its f-vector
- It is a topological invariant that captures information about the number of holes and components of the complex
- Formula:

$$\chi = \sum_{i=0}^n (-1)^i f_i$$

, where f_i is the number of i-dimensional simplices

Simplicial Homology

- Simplicial homology is a powerful algebraic tool for studying the topology of simplicial complexes
- It assigns to each simplicial complex a sequence of abelian groups (the homology groups)
- These groups measure the presence of "holes" of different dimensions in the complex
- Example: The torus has a 1-dimensional hole (the tunnel) and a 2-dimensional hole (the void), which are captured by its homology groups

2.2 Simplicial homology groups

Simplicial homology groups are a powerful tool for understanding the topological features of spaces. They use chain complexes and boundary operators to capture information about connected components, holes, and voids in simplicial complexes.

By computing homology groups and their ranks (Betti numbers), we can quantify and compare the topological structures of different spaces. This approach provides a systematic way to analyze and classify complex geometric objects.

Chain Complexes and Boundary Operators

Definition and Notation

- A chain complex is a sequence of abelian groups and homomorphisms between them, denoted as (C_n, ∂_n) , where C_n is the n-th chain group and ∂_n is the n-th boundary operator

- The boundary operator ∂_n maps elements from C_n to C_{n-1} , satisfying the condition $\partial_{n-1} \circ \partial_n = 0$ for all n
- The n -th chain group C_n is the free abelian group generated by the n -simplices of a simplicial complex
- Example: In a simplicial complex with vertices $\{a, b, c\}$ and edges $\{[a, b], [b, c]\}$, $C_0 = \langle a, b, c \rangle$ and $C_1 = \langle [a, b], [b, c] \rangle$

Boundary Operator Definition and Properties

- The boundary operator ∂_n is defined on the generators of C_n (the n -simplices) and extended linearly to all elements of C_n
- For an n -simplex $[v_0, v_1, \dots, v_n]$, the boundary operator is defined as

$$\partial_n([v_0, v_1, \dots, v_n]) = \sum_{i=0}^n (-1)^i [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$$

- Example: For a 2-simplex $[a, b, c]$, $\partial_2([a, b, c]) = [b, c] - [a, c] + [a, b]$
- The boundary of a boundary is always zero: $\partial_{n-1} \circ \partial_n = 0$ for all n
- Example: $\partial_1(\partial_2([a, b, c])) = \partial_1([b, c] - [a, c] + [a, b]) = (c - b) - (c - a) + (b - a) = 0$

Simplicial Homology Groups

Definition and Computation

- The n -th homology group H_n is defined as the quotient group $\text{Ker}(\partial_n)/\text{Im}(\partial_{n+1})$, where $\text{Ker}(\partial_n)$ is the kernel of the n -th boundary operator and $\text{Im}(\partial_{n+1})$ is the image of the $(n+1)$ -th boundary operator
- Elements of $\text{Ker}(\partial_n)$ are called n -cycles, and elements of $\text{Im}(\partial_{n+1})$ are called n -boundaries
- To compute the n -th homology group, first find a basis for the n -cycles (elements in $\text{Ker}(\partial_n)$) and a basis for the n -boundaries (elements in $\text{Im}(\partial_{n+1})$)
- Express the n -boundaries in terms of the basis for the n -cycles to determine the quotient group $H_n = \text{Ker}(\partial_n)/\text{Im}(\partial_{n+1})$

Betti Numbers and Homological Features

- The rank of the n -th homology group H_n is called the n -th Betti number, denoted as β_n , which counts the number of " n -dimensional holes" in the simplicial complex
- Example: In a simplicial complex representing a circle, $\beta_0 = 1$ (one connected component) and $\beta_1 = 1$ (one 1-dimensional hole)
- Betti numbers provide a way to quantify the topological features of a simplicial complex
- Example: In a simplicial complex representing a torus, $\beta_0 = 1$ (one connected component), $\beta_1 = 2$ (two 1-dimensional holes), and $\beta_2 = 1$ (one 2-dimensional void)

Geometric Interpretation of Homology

Low-Dimensional Homology Groups

- The 0-th homology group H_0 represents the connected components of the simplicial complex
- The 1-st homology group H_1 represents the "1-dimensional holes" or non-contractible loops in the simplicial complex
- The 2-nd homology group H_2 represents the "2-dimensional voids" or non-contractible cavities in the simplicial complex
- Example: In a simplicial complex representing a sphere, $H_0 \cong \mathbb{Z}$ (one connected component), $H_1 \cong 0$ (no 1-dimensional holes), and $H_2 \cong \mathbb{Z}$ (one 2-dimensional void)

Higher-Dimensional Homology Groups

- Higher-dimensional homology groups H_n ($n > 2$) represent higher-dimensional "holes" or non-contractible subspaces in the simplicial complex
- Example: In a simplicial complex representing a 3-torus, $H_3 \cong \mathbb{Z}$ (one 3-dimensional void)
- Higher-dimensional homology groups capture more intricate topological features of the simplicial complex
- The geometric interpretation of homology groups provides insight into the shape and connectivity of the underlying space

Functoriality of Simplicial Homology

Simplicial Maps and Induced Homomorphisms

- A simplicial map f between two simplicial complexes K and L is a function that maps vertices of K to vertices of L , preserving the simplicial structure (i.e., if $[v_0, \dots, v_n]$ is a simplex in K , then $[f(v_0), \dots, f(v_n)]$ is a simplex in L)
- A simplicial map f induces a chain map $f_\#$ between the chain complexes of K and L , which is a sequence of homomorphisms $f_n : C_n(K) \rightarrow C_n(L)$ that commute with the boundary operators (i.e., $\partial_n \circ f_n = f_{n-1} \circ \partial_n$)
- The induced chain map $f_\#$ further induces a homomorphism $f_* : H_n(K) \rightarrow H_n(L)$ between the homology groups of K and L for each dimension n

Functoriality Property and Its Consequences

- The induced homomorphism f_* satisfies the functoriality property: if $g : L \rightarrow M$ is another simplicial map, then $(g \circ f)_* = g_* \circ f_*$
- Functoriality allows for the study of simplicial complexes and their homology groups through simplicial maps, enabling the comparison of topological properties between different spaces
- Example: If $f : K \rightarrow L$ and $g : L \rightarrow M$ are simplicial maps inducing isomorphisms $f_* : H_n(K) \rightarrow H_n(L)$ and $g_* : H_n(L) \rightarrow H_n(M)$ for all n , then K , L , and M have the same homology groups and are thus homologically equivalent

- Functoriality is a powerful tool for studying the relationships between simplicial complexes and their homological invariants, allowing for the classification of spaces up to homological equivalence

2.3 Computation of simplicial homology

Simplicial homology groups are powerful tools for understanding the structure of spaces. They capture different types of "holes" in simplicial complexes, with each dimension corresponding to a specific kind of hole. Computing these groups involves analyzing chain groups and boundary maps.

The process of computing simplicial homology groups requires determining chain groups, boundary maps, and their kernels and images. For basic spaces, this can be done through inspection. More complex spaces may require advanced techniques like the Mayer-Vietoris sequence or simplicial collapses.

Simplicial Homology Groups

Definition and Interpretation

- Simplicial homology groups are algebraic invariants that capture the "holes" in a simplicial complex, with different dimensions corresponding to different types of holes
- H_0 captures connected components
- H_1 captures loops or 1-dimensional holes
- H_2 captures voids or 2-dimensional holes
- The simplicial homology group $H_n(X)$ is defined as the quotient group $\ker(\partial_n)/\text{im}(\partial_{n+1})$
- ∂_n is the boundary map from the n -th chain group $C_n(X)$ to the $(n-1)$ -th chain group $C_{n-1}(X)$
- $\ker(\partial_n)$ is the kernel of the boundary map, consisting of n -chains whose boundary is zero (n -cycles)
- $\text{im}(\partial_{n+1})$ is the image of the boundary map, consisting of n -chains that are boundaries of $(n+1)$ -chains (n -boundaries)

Computation for Basic Spaces

- To calculate the simplicial homology groups, one needs to:
 1. Determine the chain groups $C_n(X)$
 2. Determine the boundary maps ∂_n
 3. Compute the kernel $\ker(\partial_n)$ and image $\text{im}(\partial_{n+1})$ of each boundary map
 4. Calculate the quotient group $\ker(\partial_n)/\text{im}(\partial_{n+1})$
- For basic spaces like simplicial complexes consisting of a single point, a single edge, or a single triangle, the homology groups can be easily determined by inspecting the simplicial structure
- Example: A simplicial complex consisting of a single point has $H_0 \cong \mathbb{Z}$ and $H_n = 0$ for all $n > 0$
- Example: A simplicial complex consisting of a single edge has $H_0 \cong \mathbb{Z}$ and $H_n = 0$ for all $n > 0$
- The homology groups of a contractible space are trivial
- A contractible space is a space that can be continuously deformed to a point (examples: simplex, simplicial complex)

- For a contractible space X , $H_n(X) = 0$ for all $n > 0$ and $H_0(X) \cong \mathbb{Z}$

Chain Complex Structure

Definition and Properties

- A chain complex is a sequence of abelian groups (chain groups) connected by boundary maps
- The composition of any two consecutive boundary maps is zero: $\partial_n \circ \partial_{n+1} = 0$
- This property ensures that the image of each boundary map is contained in the kernel of the next boundary map
- The structure of a chain complex allows for the decomposition of the homology computation into smaller, more manageable parts
- One can compute the kernel and image of each boundary map separately
- The homology groups are determined by the relationship between the kernel and image of consecutive boundary maps

Algebraic Tools and Techniques

- The rank-nullity theorem from linear algebra can be applied to the boundary maps to relate the dimensions of the chain groups, kernel, and image
- $\text{rank}(\partial_n) + \text{nullity}(\partial_n) = \dim(C_n(X))$
- This relationship can be used to simplify computations or deduce information about the homology groups
- Exact sequences, which arise from the structure of chain complexes, can be used to:
- Relate the homology groups of different spaces
- Compute homology groups by breaking down the chain complex into simpler pieces
- Example: The Mayer-Vietoris sequence relates the homology of a space to the homology of its subspaces

Euler Characteristic Application

Definition and Formulas

- The Euler characteristic of a simplicial complex X is defined as $\chi(X) = \sum_n (-1)^n \cdot \alpha_n$
- α_n is the number of n -simplices in X
- The alternating sum of the number of simplices in each dimension
- The Euler-Poincaré formula relates the Euler characteristic to the ranks of the homology groups
- $\chi(X) = \sum_n (-1)^n \cdot \text{rank}(H_n(X))$
- $\text{rank}(H_n(X))$ is the dimension of the n -th homology group as a vector space over a field
- The alternating sum of the dimensions of the chain groups equals the Euler characteristic

$$\chi(X) = \sum_n (-1)^n \cdot \dim(C_n(X))$$

- This relationship follows from the rank-nullity theorem and the properties of chain complexes

Applications and Insights

- In some cases, the Euler characteristic can be easily computed from the simplex count
- This allows one to deduce information about the homology groups without explicitly computing them
- Example: If a simplicial complex has Euler characteristic 0 and no simplices in dimensions greater than 2, then $\text{rank}(H_0) = \text{rank}(H_2)$ and $\text{rank}(H_1) = \text{rank}(H_0) + \text{rank}(H_2)$
- The Euler characteristic is a topological invariant
- It remains constant under continuous deformations (homeomorphisms) of the space
- Spaces with different Euler characteristics cannot be homeomorphic

Computational Strategies for Complexes

Simplification Techniques

- Mayer-Vietoris sequence: relates the homology of a space to the homology of its subspaces
- Allows breaking down the computation into smaller parts
- Useful for more complex simplicial complexes
- Simplicial collapses: remove certain pairs of simplices without changing the homotopy type and homology of the complex
- Helps to simplify the complex before computing its homology
- Example: If a complex has a free face (a simplex that is the face of only one other simplex), it can be collapsed along with the simplex it is a face of
- Discrete Morse theory: provides a way to further simplify a simplicial complex
- "Cancels" pairs of cells that do not contribute to the homology
- Leads to a smaller complex with the same homology as the original one
- Example: A discrete Morse function assigns a unique number to each simplex, and pairs of simplices can be canceled if they satisfy certain conditions

Algorithmic and Computational Tools

- Smith normal form: an algorithmic tool for computing the homology groups of a simplicial complex
- Represents the boundary matrices of the complex over a principal ideal domain (PID)
- Diagonalizes the matrices, revealing the structure of the homology groups
- Can be computed efficiently using algorithms from linear algebra
- Efficient data structures: facilitate the representation and manipulation of simplicial complexes
- Simplex tree: a hierarchical data structure that represents the simplicial complex and its faces

- Hasse diagram: a graph that captures the face relations between simplices
- These data structures can be used to efficiently implement homology computation algorithms
- Computational software: there are various software packages and libraries that provide tools for computing simplicial homology
- Examples: GAP (Groups, Algorithms, Programming), PHAT (Persistent Homology Algorithms Toolbox), GUDHI (Geometry Understanding in Higher Dimensions)
- These tools can handle large and complex simplicial complexes and automate many of the computational steps involved in homology calculations

2.4 Mayer-Vietoris sequence

The Mayer-Vietoris sequence is a powerful tool in algebraic topology. It connects the homology groups of a space to those of its subspaces, helping us understand complex structures by breaking them down into simpler parts.

This sequence is particularly useful when dealing with spaces that can be split into two simpler subspaces. By analyzing the relationships between these subspaces, we can uncover important information about the original space's homology groups.

Mayer-Vietoris Sequence for Homology

Definition and Structure

- The Mayer-Vietoris sequence relates the homology groups of a space X to the homology groups of two subspaces A and B whose union is X
-

Given a space X and two subspaces A and B such that $X = A \cup B$, the Mayer-Vietoris sequence is:

$$\dots \rightarrow H_n(A \cap B) \rightarrow H_n(A) \oplus H_n(B) \rightarrow H_n(X) \rightarrow H_{n-1}(A \cap B) \rightarrow \dots$$

•

The maps in the sequence are induced by the inclusions of $A \cap B$ into A and B , and the inclusions of A and B into X

- The sequence is infinite in both directions, with the homology groups repeating with a shift in dimension (e.g., $H_n(X)$ is followed by $H_{n-1}(A \cap B)$)

Properties and Naturality

- The Mayer-Vietoris sequence is an exact sequence, meaning that the kernel of each map is equal to the image of the preceding map
- The sequence is natural, meaning it is compatible with maps between spaces that preserve the subspace structure
- If $f: X \rightarrow Y$ is a continuous map and $A, B \subset X, C, D \subset Y$ are subspaces such that $f(A) \subset C$ and $f(B) \subset D$, then f induces a map between the corresponding Mayer-Vietoris sequences
- The Mayer-Vietoris sequence is functorial, meaning it respects the composition of maps between spaces

Applying the Mayer-Vietoris Sequence

Computing Homology Groups

- The Mayer-Vietoris sequence can be used to compute the homology groups of a space X by splitting it into two subspaces A and B and using the known homology groups of A , B , and $A \cap B$
- The sequence relates the homology groups of X to the homology groups of A , B , and $A \cap B$ through a series of maps, which can be analyzed to determine the homology groups of X
- To apply the Mayer-Vietoris sequence:
 1. Start with the known homology groups of A , B , and $A \cap B$
 2. Use the exactness of the sequence to determine the unknown homology groups of X
 3. Analyze the maps in the sequence to derive relationships between the homology groups
- The Mayer-Vietoris sequence is particularly useful when the space X has a complicated structure but can be decomposed into simpler subspaces A and B (e.g., a torus can be decomposed into two cylinders)

Simplifying the Computation

- In some cases, the Mayer-Vietoris sequence may split into short exact sequences, which can simplify the computation of the homology groups
- A short exact sequence is a sequence of the form $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, where the map $A \rightarrow B$ is injective, the map $B \rightarrow C$ is surjective, and the image of $A \rightarrow B$ is equal to the kernel of $B \rightarrow C$
- When the Mayer-Vietoris sequence splits into short exact sequences, the homology groups of X can be determined by studying the short exact sequences and using algebraic techniques (e.g., the splitting lemma)
- Splitting of the Mayer-Vietoris sequence often occurs when the space X and its subspaces A and B have a particularly nice structure (e.g., when they are all contractible spaces)

Exactness of the Mayer-Vietoris Sequence

Proving Exactness

- The proof of the exactness of the Mayer-Vietoris sequence involves showing that the kernel of each map in the sequence is equal to the image of the preceding map
- The proof typically uses the snake lemma, a result in homological algebra that relates the kernels and cokernels of maps between long exact sequences
- To prove the exactness of the Mayer-Vietoris sequence:
 1. Construct a short exact sequence of chain complexes from the inclusions of A , B , and $A \cap B$ into X
 2. Apply the snake lemma to the resulting long exact sequence in homology
 3. Identify the connecting homomorphism in the Mayer-Vietoris sequence with the boundary map in the long exact sequence obtained from the snake lemma

- The proof requires careful tracking of the maps involved and their compatibility with the boundary maps in the chain complexes

Key Steps and Techniques

- The short exact sequence of chain complexes is constructed using the inclusions $i_A : A \rightarrow X$, $i_B : B \rightarrow X$, and $i_{A \cap B} : A \cap B \rightarrow A, B$
- The short exact sequence is of the form $0 \rightarrow C_*(A \cap B) \rightarrow C_*(A) \oplus C_*(B) \rightarrow C_*(X) \rightarrow 0$, where C_* denotes the chain complex functor
- Applying the snake lemma to the short exact sequence of chain complexes yields a long exact sequence in homology involving the homology groups of $A, B, A \cap B$, and X
- The connecting homomorphism in the long exact sequence obtained from the snake lemma is identified with the boundary map in the Mayer-Vietoris sequence
- The exactness of the Mayer-Vietoris sequence follows from the exactness of the long exact sequence obtained from the snake lemma

Generalizing the Mayer-Vietoris Sequence

Multiple Subspaces

- The Mayer-Vietoris sequence can be generalized to a space X that is the union of more than two subspaces
- For a space X that is the union of three subspaces A, B , and C , the generalized Mayer-Vietoris sequence involves the homology groups of the pairwise intersections $A \cap B, A \cap C$, and $B \cap C$, as well as the triple intersection $A \cap B \cap C$

•

The generalized Mayer-Vietoris sequence for three subspaces is:

$$\dots \rightarrow H_n(A \cap B \cap C) \rightarrow H_n(A \cap B) \oplus H_n(A \cap C) \oplus H_n(B \cap C) \rightarrow H_n(A) \oplus H_n(B) \oplus H_n(C) \rightarrow H_n(X) \rightarrow \dots$$

•

The maps in the generalized sequence are induced by the appropriate inclusions of the intersections into the subspaces and the subspaces into X

Arbitrary Finite Number of Subspaces

- The generalized Mayer-Vietoris sequence can be extended to any finite number of subspaces
- For a space X that is the union of subspaces $A_1, A_2, \dots, A_n$, the generalized Mayer-Vietoris sequence involves the homology groups of all possible intersections of the subspaces
- The sequence includes terms for the homology groups of the subspaces $H_*(A_i)$, the pairwise intersections $H_*(A_i \cap A_j)$, the triple intersections $H_*(A_i \cap A_j \cap A_k)$, and so on, up to the intersection of all subspaces $H_*(A_1 \cap A_2 \cap \dots \cap A_n)$
- The maps in the sequence are induced by the appropriate inclusions of the intersections into the subspaces and the subspaces into X

- The generalized Mayer-Vietoris sequence for n subspaces can be constructed inductively by applying the Mayer-Vietoris sequence for two subspaces to the union of the first $n - 1$ subspaces and the n -th subspace

Unit 3 – CW Complexes and Cellular Homology

3.1 CW complexes and cellular maps

CW complexes are building blocks in topology, constructed by attaching cells of increasing dimension. They start with points, then add edges, faces, and higher-dimensional cells. This structure allows for a systematic study of topological spaces.

Cellular maps between CW complexes preserve the skeletal structure, making them powerful tools for analyzing topological relationships. These maps enable us to compare spaces, compute homology groups, and understand homotopy equivalences between different topological structures.

CW Complexes: Definition and Construction

Definition and Basic Structure

- A CW complex is a topological space constructed by attaching cells of increasing dimension to a discrete set of points
- The construction begins with a discrete set X^0 , called the 0-skeleton
- Inductively builds the n -skeleton X^n from X^{n-1} by attaching n -cells via continuous maps from their boundary to X^{n-1}
- The resulting space has the weak topology with respect to the closed cells, meaning a subset is open if and only if its intersection with each closed cell is open

Attaching Maps and Closure-Finiteness

- The attaching maps for n -cells are defined on the closed n -ball, with the interior of the n -ball mapped homeomorphically onto an open subset of the CW complex
- For example, when constructing a 2-dimensional CW complex, the attaching maps for 2-cells are defined on the closed disk D^2 , with the interior of the disk mapped homeomorphically onto an open subset of the CW complex
- The closure-finite condition requires that each closed cell intersects only finitely many other cells
- This ensures the CW complex is Hausdorff, meaning any two distinct points have disjoint open neighborhoods
- For instance, in a finite CW complex, such as a torus or a sphere, each cell intersects only finitely many other cells

Characteristics of CW Complexes

Topological Properties

- CW complexes are Hausdorff spaces, which means that any two distinct points have disjoint open neighborhoods

- CW complexes are locally contractible, implying that every point has a neighborhood that is homotopy equivalent to a point
- For example, in a CW complex structure of a sphere, every point has a neighborhood that can be continuously deformed to a single point
- The topology of a CW complex is completely determined by its cellular structure, as it has the weak topology with respect to the closed cells

Skeletal Structure and Homotopy Type

- The skeleta of a CW complex form a filtration, with each skeleton being a closed subspace of the next
- For instance, in a 3-dimensional CW complex, the 1-skeleton is a closed subspace of the 2-skeleton, which is a closed subspace of the 3-skeleton
- CW complexes have a well-defined dimension, given by the highest dimension of the cells in the complex
- A CW complex consisting of only 0-cells and 1-cells (edges) has dimension 1, while a CW complex with 2-cells (faces) has dimension 2
- The homotopy type of a CW complex is determined by its cellular structure, and homotopy equivalent CW complexes have isomorphic cellular chain complexes
- Two CW complexes with the same cellular structure up to homotopy equivalence will have the same homology groups

Maps Between CW Complexes

Cellular Maps and Their Properties

- A continuous map between CW complexes is called a cellular map if it maps the n -skeleton of the domain into the n -skeleton of the codomain for all n
- Cellular maps induce homomorphisms between the cellular chain complexes of the CW complexes, providing a means to study the map algebraically
- For example, a cellular map $f: X \rightarrow Y$ induces a chain map $f_*: C_*(X) \rightarrow C_*(Y)$ between the cellular chain complexes
- A cellular map that is a homeomorphism is called a cellular homeomorphism, and it provides an isomorphism between the cellular structures of the CW complexes

Cellular Approximation and Homotopy

- The cellular approximation theorem states that any continuous map between CW complexes is homotopic to a cellular map
- This allows for the study of homotopy classes of maps via cellular techniques
- For instance, if two continuous maps $f, g: X \rightarrow Y$ are homotopic, then their cellular approximations $f', g': X \rightarrow Y$ are also homotopic
- Cellular maps can be used to construct new CW complexes, such as the mapping cylinder and the mapping cone, which encode information about the map

- The mapping cylinder of a cellular map $f: X \rightarrow Y$ is a CW complex M_f that contains X and Y as subcomplexes, with X and Y connected by a "cylinder" of cells corresponding to the map f

CW Complexes vs Simplicial Complexes

Relationship Between CW and Simplicial Complexes

- Every simplicial complex is a CW complex, with the simplices serving as the cells and the attaching maps being the inclusion of the boundaries of the simplices
- For example, a triangle (2-simplex) can be viewed as a 2-cell attached to its boundary (1-simplices)
- The geometric realization of a simplicial complex is a CW complex, obtained by realizing each simplex as a topological simplex and gluing them together according to the face relations
- The geometric realization of a simplicial complex is a topological space that "fills in" the simplices, making it a CW complex

Differences and Similarities

- While every simplicial complex is a CW complex, not every CW complex is a simplicial complex, as CW complexes allow for more general cell shapes and attaching maps
- CW complexes can have cells that are not simplices, such as cubes or octahedra, and the attaching maps can be more general continuous maps
- The barycentric subdivision of a CW complex produces a simplicial complex that is homotopy equivalent to the original CW complex
- Barycentric subdivision involves iteratively subdividing cells by connecting the barycenters (centroids) of the cells, resulting in a simplicial complex
- The simplicial approximation theorem states that any continuous map between the geometric realizations of simplicial complexes is homotopic to a simplicial map
- This provides a connection between continuous and simplicial maps, allowing for the study of maps between simplicial complexes using combinatorial methods

3.2 Cellular homology groups

Cellular homology groups are a powerful tool for understanding the topological structure of CW complexes. They provide a way to compute and interpret the "holes" and "twists" in spaces, using the cellular structure to simplify calculations compared to simplicial homology.

The cellular boundary formula is key to computing these groups. It expresses boundary maps in terms of attaching map degrees, allowing us to determine kernels and images. This process reveals the free and torsion components of homology groups, giving insights into a space's topology.

Cellular chain complexes

Definition and properties

- A cellular chain complex is a sequence of abelian groups and homomorphisms between them, denoted as $C_n(X)$, associated with the cellular structure of a CW complex X

- The boundary maps, denoted as $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$, connect the chain groups and are defined based on the attaching maps of the cells in the CW complex
- The composition of two consecutive boundary maps is always zero, i.e., $\partial_{n-1} \circ \partial_n = 0$, which is a crucial property for defining homology groups
- The cellular chain groups $C_n(X)$ are free abelian groups generated by the n -cells of the CW complex X (e.g., 0-cells, 1-cells, 2-cells)
- The boundary map ∂_n applied to an n -cell gives a linear combination of $(n-1)$ -cells in the cellular structure of X

Boundary maps and attaching maps

- The boundary maps ∂_n are determined by the attaching maps of the cells in the CW complex
- For an n -cell e_α^n , the boundary $\partial_n(e_\alpha^n)$ is given by the sum of $(n-1)$ -cells e_β^{n-1} multiplied by the degrees of the attaching maps $d_{\alpha\beta}$
- The degree $d_{\alpha\beta}$ is the number of times the attaching map of the n -cell e_α^n wraps around the $(n-1)$ -cell e_β^{n-1} , counted with signs depending on the orientation
- The cellular boundary formula expresses the boundary map ∂_n in terms of the degrees of the attaching maps: $\partial_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} \cdot e_\beta^{n-1}$
- Calculating the boundary maps using the cellular boundary formula requires determining the degrees of the attaching maps for each pair of cells in the CW complex

Cellular homology groups

Definition and computation

- The n -th cellular homology group, denoted as $H_n(X)$, is defined as the quotient group $\ker(\partial_n)/\text{im}(\partial_{n+1})$, where $\ker(\partial_n)$ is the kernel of the boundary map ∂_n and $\text{im}(\partial_{n+1})$ is the image of the boundary map ∂_{n+1}
- To compute $H_n(X)$, one needs to find the kernel of ∂_n , which consists of the n -cycles (elements in $C_n(X)$ that are mapped to zero by ∂_n), and the image of ∂_{n+1} , which consists of the n -boundaries (elements in $C_n(X)$ that are the image of elements in $C_{n+1}(X)$ under ∂_{n+1})
- The rank of the free part of $H_n(X)$ is equal to the dimension of $\ker(\partial_n)$ minus the dimension of $\text{im}(\partial_{n+1})$ as vector spaces
- The torsion coefficients of $H_n(X)$ can be determined by analyzing the quotient group structure and identifying the orders of the torsion elements (e.g., $\mathbb{Z}_2, \mathbb{Z}_3$)

Interpreting homology groups

- The rank of the free part of $H_n(X)$ corresponds to the number of n -dimensional "holes" in the CW complex X that are not boundaries of $(n+1)$ -dimensional cells
- Torsion coefficients in $H_n(X)$ represent the presence of "twists" or "torsion" in the n -dimensional structure of the CW complex X
- The homology groups provide a way to distinguish between topological spaces based on their "hole" structure and torsion

- Examples of spaces with non-trivial homology groups include the circle S^1 ($H_1(S^1) = \mathbb{Z}$), the torus ($H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$), and the real projective plane ($H_1(\mathbb{RP}^2) = \mathbb{Z}_2$)

Cellular vs simplicial homology

Relationship between cellular and simplicial homology

- For a CW complex X , the cellular homology groups $H_n(X)$ are isomorphic to the simplicial homology groups of any simplicial complex that is a subdivision of X
- The cellular boundary maps and the simplicial boundary maps are compatible under this isomorphism, meaning that the homology groups computed using either method will be the same
- The relationship between cellular and simplicial homology is an example of the functoriality of homology, which states that homology is independent of the choice of the chain complex used to compute it

Advantages of cellular homology

- Cellular homology provides a more efficient way to compute homology groups compared to simplicial homology, as it often requires fewer generators and relations
- The cellular chain complex is typically smaller than the simplicial chain complex for the same topological space, leading to simpler computations
- Cellular homology can be applied to a wider range of topological spaces, including those that do not have a natural simplicial structure (e.g., CW complexes obtained by attaching cells)

Cellular boundary formula

Formulation and application

- The cellular boundary formula expresses the boundary map ∂_n in terms of the degrees of the attaching maps of the cells in the CW complex X
- For an n -cell e_α^n , the boundary $\partial_n(e_\alpha^n)$ is given by the sum of $(n-1)$ -cells e_β^{n-1} multiplied by the degrees of the attaching maps $d_{\alpha\beta}$: $\partial_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} \cdot e_\beta^{n-1}$
- The degree $d_{\alpha\beta}$ is the number of times the attaching map of the n -cell e_α^n wraps around the $(n-1)$ -cell e_β^{n-1} , counted with signs depending on the orientation
- To calculate the boundary maps using the cellular boundary formula, one needs to determine the degrees of the attaching maps for each pair of cells in the CW complex

Examples and calculations

- Consider the CW complex structure of the torus T , which consists of one 0-cell, two 1-cells (a and b), and one 2-cell (e) attached via the map $aba^{-1}b^{-1}$
- The cellular boundary formula yields: $\partial_1(a) = \partial_1(b) = 0$, and $\partial_2(e) = a + b - a - b = 0$
- The resulting cellular chain complex is $0 \rightarrow \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \rightarrow 0$, which leads to the homology groups $H_0(T) = \mathbb{Z}$, $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$, and $H_2(T) = \mathbb{Z}$
- Similar calculations can be performed for other CW complexes, such as the real projective plane $\mathbb{RP}^2$ or the Klein bottle, using the cellular boundary formula to determine the boundary maps and compute

the homology groups

3.3 Computation of cellular homology

Cellular homology is a powerful tool for understanding the topological structure of spaces. It uses the CW complex structure to compute homology groups, which measure "holes" in different dimensions. This approach is often more efficient than simplicial homology.

The computation of cellular homology involves defining chain complexes based on cell attachments. By analyzing these complexes and applying key axioms, we can determine the homology groups of a space, revealing its fundamental topological features.

Cellular Homology Groups

Cellular Chain Complex and Homology

- The cellular chain complex $C(X)$ is defined using the cellular boundary maps $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$, which connect the n -dimensional cells to the $(n-1)$ -dimensional cells
- The n -th cellular homology group $H_n(X)$ is defined as the quotient $\ker(\partial_n)/\text{im}(\partial_{n+1})$, where $\ker(\partial_n)$ is the kernel of ∂_n and $\text{im}(\partial_{n+1})$ is the image of ∂_{n+1}
- The homology groups $H_n(X)$ measure the "holes" in the complex X that are not filled by the $(n+1)$ -cells, modulo the boundaries of the $(n+1)$ -cells
- For example, $H_1(X)$ measures 1-dimensional holes (loops) that are not boundaries of 2-cells

Cellular Chain Groups and Boundary Maps

- For a CW complex X , the cellular chain groups $C_n(X)$ are free abelian groups with basis in one-to-one correspondence with the n -cells of X
- For instance, if X has three 2-cells, then $C_2(X)$ is isomorphic to $\mathbb{Z}^3$
- The cellular boundary map ∂_n is determined by the degrees of the attaching maps of the n -cells, which can be computed using the cellular boundary formula
- The degree of an attaching map $\varphi : S^{n-1} \rightarrow X^{(n-1)}$ counts the number of times the image of φ wraps around each $(n-1)$ -cell in $X^{(n-1)}$
- The boundary maps ∂_n form a chain complex, meaning that $\partial_{n-1} \circ \partial_n = 0$ for all n , which is crucial for the definition of homology groups

Cellular Homology Axioms

Dimension and Degree Axioms

- The dimension axiom states that if X is a single point, then $H_n(X) = 0$ for all $n \neq 0$, and $H_0(X) \cong \mathbb{Z}$
- This axiom sets the base case for homology computations
- The degree axiom relates the degree of a map between spheres to the induced homomorphism on homology groups
- For a map $f : S^n \rightarrow S^n$ of degree d , the induced homomorphism $f_* : H_n(S^n) \rightarrow H_n(S^n)$ is multiplication by d

Excision and Additivity Axioms

- The excision axiom allows the computation of homology groups of a space by decomposing it into simpler pieces
- If (X, A) is a CW pair and U is a subcomplex of A such that the closure of U is contained in the interior of A , then the inclusion $(X - U, A - U) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - U, A - U) \cong H_n(X, A)$ for all n
- The additivity axiom states that the homology of a disjoint union of spaces is the direct sum of the homology of each space
- For a collection of spaces $\{X_i\}_{i \in I}$, there is an isomorphism $H_n(\bigsqcup_i X_i) \cong \bigoplus_i H_n(X_i)$ for all n , where $\bigsqcup$ denotes disjoint union and $\bigoplus$ denotes direct sum

Homotopy Invariance and Exactness Axioms

- The homotopy invariance axiom implies that homotopy equivalent spaces have isomorphic homology groups
- If $f, g : X \rightarrow Y$ are homotopic maps, then the induced homomorphisms $f_*, g_* : H_n(X) \rightarrow H_n(Y)$ are equal for all n
- The exactness axiom relates the homology of a space, a subspace, and the quotient space through a long exact sequence
- For a CW pair (X, A) , there is a long exact sequence of homology groups:
 $\dots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X/A) \rightarrow H_{n-1}(A) \rightarrow \dots$ where the maps are induced by the inclusions $A \hookrightarrow X$ and $X \hookrightarrow X/A$

Cellular vs Simplicial Homology

Efficiency of Cellular Homology

- Cellular homology is often more efficient than simplicial homology because CW complexes typically have fewer cells than simplices in a triangulation
- For example, a torus can be constructed with just one 0-cell, two 1-cells, and one 2-cell, while a simplicial complex requires many more simplices
- The cellular chain complex is usually smaller than the simplicial chain complex, leading to simpler computations of homology groups
- The number of generators in each dimension is determined by the number of cells, which is often fewer than the number of simplices

Flexibility of Cellular Homology

- Cellular homology allows for more flexibility in the choice of cell structure, which can be adapted to the specific space being studied
- Different CW structures on the same space can lead to different cellular chain complexes, but the homology groups will be isomorphic
- The cellular boundary maps are determined by the attaching maps of the cells, which can be easier to compute than the simplicial boundary maps

- Attaching maps are defined on spheres, which have a simpler structure than the boundaries of simplices

Simplifying Homology Calculations

Exploiting CW Structure

- Identify the n -cells of the CW complex and their attaching maps to determine the cellular chain groups $C_n(X)$ and the cellular boundary maps ∂_n
- The chain groups are generated by the cells, and the boundary maps are determined by the degrees of the attaching maps
- Use the dimension of the cells and the connectivity of the attaching maps to infer properties of the kernel and image of the boundary maps
- For example, if there are no $(n+1)$ -cells, then $\text{im}(\partial_{n+1}) = 0$, simplifying the computation of $H_n(X)$

Decomposition and Excision

- Exploit the presence of contractible subcomplexes or subspaces with known homology to simplify the computation using the excision axiom
- If A is a contractible subcomplex of X , then $H_n(X, A) \cong H_n(X/A)$ for all n , reducing the computation to the quotient space
- Decompose the CW complex into simpler pieces, such as wedge sums or product spaces, and use the additivity and Künneth formula to compute the homology of the whole space
- For a wedge sum $X \vee Y$, there is an isomorphism $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for all $n > 0$, and $H_0(X \vee Y) \cong H_0(X) \cong H_0(Y) \cong \mathbb{Z}$

Strategic Application of Axioms

- Apply the cellular homology axioms strategically to minimize the number of direct computations needed, leveraging the relationships between different spaces and their homology groups
- Use the dimension axiom for base cases, the degree axiom for maps between spheres, and the homotopy invariance axiom for homotopy equivalent spaces
- Combine the excision and additivity axioms to break down a complex space into simpler components, compute their homology groups separately, and then reassemble the results using the Mayer-Vietoris sequence or other algebraic tools
- The Mayer-Vietoris sequence relates the homology of a space X to the homology of two subspaces A and B such that $X = A \cup B$, providing a powerful tool for computing homology by decomposition

3.4 Comparison of simplicial and cellular homology

Simplicial and cellular homology are two ways to measure a space's "holes" and topological features. While they use different building blocks (simplices vs. cells), they give the same results. This equivalence lets us choose the most efficient method for each situation.

Cellular homology often wins in efficiency, using fewer components to describe a space. However, finding the right cell structure can be tricky. The choice between methods depends on the space and available tools.

Simplicial vs Cellular Homology

Definition and Topological Invariants

- Simplicial homology is defined using simplicial complexes, while cellular homology is defined using CW complexes
- Both simplicial and cellular homology capture topological invariants of a space, specifically the "holes" in different dimensions (Betti numbers, Euler characteristic)
- The simplicial chain complex is generated by oriented simplices (vertices, edges, triangles, tetrahedra), while the cellular chain complex is generated by oriented cells (0-cells, 1-cells, 2-cells)
- The boundary operators in simplicial and cellular homology are defined differently but serve an analogous purpose of encoding the incidence relations between simplices or cells of consecutive dimensions

Isomorphism of Homology Groups

- Simplicial and cellular homology groups are isomorphic for a given space, meaning they capture the same topological information
- The isomorphism arises from the fact that a CW complex can be constructed from a simplicial complex by taking its geometric realization and giving it a CW structure
- Conversely, a CW complex can be subdivided into a simplicial complex without changing its topology
- This equivalence allows for the choice of the most convenient or efficient homology computation method based on the available representation of the space

Computational Efficiency: Simplicial vs Cellular

Advantages of Cellular Homology

- Cellular homology is often more computationally efficient than simplicial homology, especially for spaces with a small CW complex structure
- The number of cells in a CW complex is typically much smaller than the number of simplices in a triangulation, leading to smaller chain complexes in cellular homology (fewer generators and boundary relations)
- Computing boundary operators in cellular homology involves fewer calculations compared to simplicial homology due to the reduced number of cells
- The sparsity of the cellular boundary matrices can be exploited for efficient computation, whereas simplicial boundary matrices tend to be denser

Challenges and Trade-offs

- In some cases, finding an optimal CW complex structure for a space may be challenging, potentially offsetting the computational advantage of cellular homology
- The process of converting a simplicial complex to a CW complex or vice versa may introduce additional computational overhead

- The choice between simplicial and cellular homology depends on the specific characteristics of the space, the available representation, and the computational resources at hand
- In practice, a combination of both methods may be used, leveraging their strengths in different stages of the homology computation pipeline

Isomorphism of Homology Groups

Chain Map Construction

- The proof relies on constructing a chain map between the simplicial and cellular chain complexes that induces an isomorphism on homology
- Given a simplicial complex K , one can construct a CW complex X by taking the geometric realization of K and giving it a CW complex structure
- A chain map $f: C(K) \rightarrow C(X)$ is defined by sending each simplex to the sum of the cells it intersects, with appropriate orientations (e.g., a triangle is sent to the sum of the 2-cells it overlaps)
- The map f commutes with the boundary operators, i.e., $f \circ \partial_{\text{simplicial}} = \partial_{\text{cellular}} \circ f$, making it a chain map

Induced Homomorphism and Isomorphism

- By the algebraic theorem that chain maps induce homomorphisms on homology, f induces a homomorphism $f_*: H_n(K) \rightarrow H_n(X)$ for each dimension n
- Using the geometric properties of the map f and the construction of X from K , one can show that f_* is an isomorphism (bijective homomorphism)
- The injectivity of f_* follows from the fact that if a cycle in K is mapped to a boundary in X , then it must have been a boundary in K
- The surjectivity of f_* is proven by showing that every cycle in X can be approximated by a cycle in K , using the cellular approximation theorem
- Consequently, the simplicial homology groups of K are isomorphic to the cellular homology groups of X

Homology Comparisons in Different Settings

Comparison Theorem

- The comparison theorem states that if two chain complexes are connected by a chain map inducing an isomorphism on homology, then their homology groups are isomorphic
- This theorem allows for the comparison of homology computations performed using different methods or in different settings (simplicial, cellular, singular)
- When computing homology, one can choose the most convenient or efficient method based on the available information about the space

Flexibility in Homology Computations

- If a space is given as a simplicial complex, one can compute its simplicial homology directly or convert it to a CW complex and compute cellular homology

- Conversely, if a space is given as a CW complex, one can compute its cellular homology directly or subdivide it into a simplicial complex and compute simplicial homology
- The comparison theorem ensures that the resulting homology groups will be isomorphic, regardless of the chosen method
- This flexibility allows for the selection of the most suitable approach based on the specific characteristics of the space and the available computational tools (software packages, algorithms)

Applications and Examples

- In topological data analysis, simplicial complexes are often used to represent high-dimensional datasets, and simplicial homology is computed to extract topological features (persistent homology)
- In algebraic topology, CW complexes are frequently used to construct spaces with desired properties, and cellular homology is employed to study their topological invariants (homotopy groups, cohomology rings)
- The comparison theorem enables the transfer of results between different homology theories and facilitates the study of topological spaces from multiple perspectives (Morse theory, sheaf theory, spectral sequences)

Unit 4 – Homotopy and the Fundamental Group

4.1 Homotopy of maps and homotopy equivalence

Homotopy of maps and homotopy equivalence are key concepts in algebraic topology. They provide a way to compare continuous functions and topological spaces, allowing us to group similar objects and study their shared properties.

These ideas form the foundation for understanding the fundamental group and other homotopy groups. By focusing on continuous deformations, we can explore the essential structure of spaces and maps, leading to powerful classification tools in topology.

Homotopy of Maps

Definition and Notation

- A homotopy between two continuous maps $f, g : X \rightarrow Y$ is a continuous map $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$
- The interval $[0, 1]$ represents the parameter space for the homotopy
- The homotopy continuously deforms the map f into the map g as the parameter varies from 0 to 1
- If a homotopy exists between f and g , then f and g are said to be homotopic, denoted by $f \simeq g$
- The set of homotopy classes of maps from X to Y is denoted by $[X, Y]$

Properties of Homotopy

- The homotopy relation is an equivalence relation on the set of continuous maps from X to Y
- Reflexivity: Every map is homotopic to itself via the constant homotopy $H(x, t) = f(x)$
- Symmetry: If $f \simeq g$, then $g \simeq f$ by reversing the homotopy $H(x, t) = H(x, 1 - t)$
- Transitivity: If $f \simeq g$ and $g \simeq h$, then $f \simeq h$ by concatenating the homotopies
- Homotopic maps induce the same homomorphisms on homotopy groups and homology groups
- If $f \simeq g$, then $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ and $g_* : \pi_n(X) \rightarrow \pi_n(Y)$ are equal for all $n \geq 0$
- If $f \simeq g$, then $f_* : H_n(X) \rightarrow H_n(Y)$ and $g_* : H_n(X) \rightarrow H_n(Y)$ are equal for all $n \geq 0$

Homotopic Maps and Equivalences

Homotopic Maps

- Two continuous maps $f, g : X \rightarrow Y$ are homotopic if there exists a homotopy $H : X \times [0, 1] \rightarrow Y$ between them
- Example: The identity map $id_X : X \rightarrow X$ and any constant map $c : X \rightarrow X$ are homotopic via $H(x, t) = (1 - t)x + tc(x)$

- Homotopic maps share many topological properties, such as inducing the same homomorphisms on homotopy and homology groups

Homotopy Equivalences

- A homotopy equivalence between two topological spaces X and Y is a pair of continuous maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$, where id_X and id_Y are the identity maps on X and Y , respectively
- The maps f and g are called homotopy equivalences or homotopy inverses of each other
- If a homotopy equivalence exists between X and Y , then X and Y are said to be homotopy equivalent or have the same homotopy type, denoted by $X \simeq Y$
- Example: The unit interval $[0, 1]$ and the unit circle S^1 are homotopy equivalent via the maps $f(t) = e^{2\pi it}$ and $g(z) = \frac{1}{2\pi i} \log z$
- Homotopy equivalent spaces share many topological properties, such as the same fundamental group, homology groups, and cohomology rings

Equivalence Relation of Homotopy

Reflexivity

- For any topological space X , the identity map $id_X: X \rightarrow X$ is a homotopy equivalence, as $id_X \circ id_X = id_X \simeq id_X$
- The constant homotopy $H(x, t) = id_X(x)$ demonstrates that id_X is homotopic to itself

Symmetry

- If $f: X \rightarrow Y$ is a homotopy equivalence with homotopy inverse $g: Y \rightarrow X$, then g is also a homotopy equivalence with homotopy inverse f , as $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$
- The homotopies demonstrating $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$ can be reversed to show $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$, respectively

Transitivity

- If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are homotopy equivalences with homotopy inverses $f': Y \rightarrow X$ and $g': Z \rightarrow Y$, respectively, then $g \circ f: X \rightarrow Z$ is a homotopy equivalence with homotopy inverse $f' \circ g': Z \rightarrow X$
- The composition of homotopies and the homotopy inverses demonstrate that $(g \circ f) \circ (f' \circ g') \simeq id_Z$ and $(f' \circ g') \circ (g \circ f) \simeq id_X$
- Example: If $X \simeq Y$ and $Y \simeq Z$, then $X \simeq Z$ by composing the homotopy equivalences

Homotopy in Topological Classification

Classifying Spaces by Homotopy Type

- Homotopy provides a way to classify topological spaces by grouping together spaces that have the same homotopy type

- Spaces that are homotopy equivalent share many important topological invariants, such as the fundamental group, homology groups, and cohomology rings
- Example: All contractible spaces (spaces homotopy equivalent to a point) form a single homotopy equivalence class
- Homotopy equivalence is a weaker notion than homeomorphism, as homeomorphic spaces are always homotopy equivalent, but homotopy equivalent spaces may not be homeomorphic
- Example: The punctured plane $\mathbb{R}^2 \setminus \{0\}$ and the unit circle S^1 are homotopy equivalent but not homeomorphic

Homotopy Groups and Distinguishing Spaces

- The study of homotopy groups, particularly the fundamental group, helps distinguish between spaces that are not homotopy equivalent and provides insight into the structure of topological spaces
- The fundamental group $\pi_1(X)$ is a homotopy invariant that captures information about the loops in a space X
- Higher homotopy groups $\pi_n(X)$ capture information about higher-dimensional spheres in X
- Example: The fundamental group distinguishes between the torus T^2 and the sphere S^2 , as $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(S^2) = 0$
- Homotopy theory is a powerful tool in algebraic topology, allowing for the development of invariants and techniques to study and classify topological spaces

4.2 The fundamental group

The fundamental group is a powerful tool in algebraic topology, capturing the essence of a space's "loopiness." It measures how loops in a space can be continuously deformed into each other, revealing crucial information about the space's structure and holes.

By studying the fundamental group, we gain insights into a space's topology that go beyond simple connectivity. This concept forms the foundation for understanding higher-dimensional topological features and is essential for classifying spaces up to homotopy equivalence.

Fundamental Group of a Space

Definition and Notation

- Define the fundamental group $\pi_1(X, x_0)$ of a topological space X at a basepoint x_0 as the set of homotopy equivalence classes of loops based at x_0
- Denote a loop in X based at x_0 as a continuous function $f: [0, 1] \rightarrow X$ such that $f(0) = f(1) = x_0$
- Consider two loops f and g homotopic if there exists a continuous function $H: [0, 1] \times [0, 1] \rightarrow X$ such that:
 - $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ for all $s \in [0, 1]$
 - $H(0, t) = H(1, t) = x_0$ for all $t \in [0, 1]$

Group Structure

- Define the group operation in $\pi_1(X, x_0)$ by concatenation of loops

- The inverse of a loop is the same loop traversed in the opposite direction
- Identify the identity element in $\pi_1(X, x_0)$ as the constant loop at x_0
- Recognize that the fundamental group is a homotopy invariant
- If X and Y are homotopy equivalent spaces, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ for any choice of basepoints x_0 and y_0
- Understand that if X is a path-connected space, the fundamental groups at different basepoints are isomorphic
- The isomorphism is induced by conjugation with paths connecting the basepoints

Geometric Interpretation of the Fundamental Group

Loops and Holes

- Interpret the fundamental group as capturing the notion of loops in a space that cannot be continuously deformed to a point within the space
- Recognize that elements of the fundamental group represent different ways to loop around holes or obstacles in the space
- Non-trivial elements correspond to loops that encircle holes or obstacles and cannot be contracted to a point
- Conclude that if $\pi_1(X, x_0)$ is trivial (consists only of the identity element), every loop in X based at x_0 can be continuously shrunk to the point x_0 without leaving the space

Covering Spaces

- Relate the fundamental group of a covering space to the fundamental group of the base space by the lifting criterion
- A loop in the base space lifts to a loop in the covering space if and only if it represents an element of the fundamental group that lies in the image of the induced homomorphism from the covering space
- Understand that the fundamental group of a covering space is a subgroup of the fundamental group of the base space
- The index of this subgroup is equal to the number of sheets in the covering space

Computing the Fundamental Group

Simple Spaces

- Recognize that the fundamental group of a convex subset of Euclidean space is trivial
- Any loop can be continuously shrunk to a point within the space
- Compute the fundamental group of a circle S^1 as isomorphic to the integers under addition, $\pi_1(S^1) \cong (\mathbb{Z}, +)$
- Each integer represents the number of times a loop winds around the circle (winding number)
- Determine that the fundamental group of a punctured plane $\mathbb{R}^2 \setminus \{0\}$ is also isomorphic to $(\mathbb{Z}, +)$

- Loops can wind around the removed point

Product Spaces and Wedge Sums

- Prove that the fundamental group of a product space $X \times Y$ is isomorphic to the direct product of the fundamental groups of X and Y
- $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$
- Calculate the fundamental group of a figure-eight space (two circles joined at a single point) as the free group on two generators
- The generators correspond to loops around each circle
- Compute the fundamental group of a torus as isomorphic to $\mathbb{Z} \times \mathbb{Z}$
- The generators are loops that wind around the two independent directions of the torus (meridian and longitude)

Properties of the Fundamental Group

Homotopy Invariance

- Understand that the fundamental group is a homotopy invariant
- If two spaces X and Y are homotopy equivalent, then their fundamental groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are isomorphic for any choice of basepoints x_0 and y_0
- Apply homotopy invariance to simplify the computation of fundamental groups
- If a space is homotopy equivalent to a simpler space with a known fundamental group, the original space will have the same fundamental group (deformation retract)

Basepoint Independence

- Recognize that for path-connected spaces, the fundamental groups at different basepoints are isomorphic
- Understand that the isomorphism between fundamental groups at different basepoints is induced by conjugation with paths connecting the basepoints
- If γ is a path from x_0 to x_1 in X , then the isomorphism $\pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ is given by $[\alpha] \mapsto [\gamma^{-1} * \alpha * \gamma]$, where $*$ denotes path concatenation
- Apply basepoint independence to simplify the computation of fundamental groups
- Choose a convenient basepoint for calculations, knowing that the resulting fundamental group will be isomorphic to the fundamental group at any other basepoint in the same path component

4.3 Calculation of fundamental groups

Calculating fundamental groups is a key skill in algebraic topology. It involves analyzing loops in spaces and understanding how they relate to each other. This topic builds on earlier concepts of homotopy and introduces tools for computing these groups.

The Seifert-van Kampen theorem is a powerful method for calculating fundamental groups of complex spaces. By breaking spaces into simpler parts, we can piece together their fundamental groups. This

approach connects to broader themes of decomposition and reconstruction in topology.

Calculating Fundamental Groups

Fundamental Group Definition and Properties

- The fundamental group of a space X , denoted $\pi_1(X)$, describes the different ways of looping a path in X , starting and ending at the same point, up to homotopy equivalence
- The fundamental group is a homotopy invariant
- If two spaces are homotopy equivalent, their fundamental groups are isomorphic
- The fundamental group of a convex set in Euclidean space is trivial, consisting of only the identity element

Fundamental Groups of Common Spaces

- The fundamental group of a circle, $\pi_1(S^1)$, is isomorphic to the integers under addition, denoted $(\mathbb{Z}, +)$
- The fundamental group of a figure eight (wedge sum of two circles) is the free group on two generators, denoted F_2
- The fundamental group of a torus, $\pi_1(T^2)$, is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, the direct product of two copies of the integers
- Higher-dimensional spheres, S^n for $n \geq 2$, have trivial fundamental groups
- Example: The fundamental group of a 2-sphere (surface of a ball) is trivial

Seifert-van Kampen Theorem for Fundamental Groups

Theorem Statement and Conditions

- The Seifert-van Kampen theorem computes the fundamental group of a space that can be decomposed into simpler, overlapping subspaces
- The theorem applies when:
 - X is the union of two open, path-connected subspaces U and V
 - $U \cap V$ is also path-connected
- Under these conditions, the fundamental group of X is isomorphic to the free product of the fundamental groups of U and V , amalgamated along the fundamental group of their intersection

Free Product with Amalgamation

- The free product with amalgamation, denoted $\pi_1(U) *_\{\pi_1(U \cap V)\} \pi_1(V)$, is a quotient group of the free product $\pi_1(U) * \pi_1(V)$
- It is quotiented by the normal subgroup generated by elements of the form $i_1(g)i_2(g)^{-1}$, where:
 - $g \in \pi_1(U \cap V)$
 - i_1 and i_2 are induced homomorphisms from the inclusions of $U \cap V$ into U and V , respectively

- The Seifert-van Kampen theorem can be applied iteratively to decompose a space into multiple overlapping subspaces
- This allows for the calculation of the fundamental group of more complex spaces

Fundamental Groups of Product Spaces

Fundamental Group of a Product

- The fundamental group of a product space $X \times Y$ is isomorphic to the direct product of the fundamental groups of X and Y
- $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$
- This result follows from the fact that loops in a product space can be projected onto loops in each factor space
- These projections uniquely determine the original loop up to homotopy

Application to n-Torus

- The fundamental group of an n -torus, the product of n circles, is isomorphic to the direct product of n copies of $\mathbb{Z}$
- $\pi_1(T^n) \cong \mathbb{Z}^n$
- Example: The fundamental group of a 3-torus (T^3) is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

Fundamental Groups vs Covering Spaces

Covering Spaces and Covering Maps

- A covering space of a topological space X is a space C together with a continuous surjective map $p: C \rightarrow X$, called the covering map
- Each point $x \in X$ has an open neighborhood U for which $p^{-1}(U)$ is a disjoint union of open sets in C
- Each open set in $p^{-1}(U)$ is mapped homeomorphically onto U by p
- The fundamental group of X acts on the fiber of a covering space $p: C \rightarrow X$ over a basepoint $x_0 \in X$
- The fiber is the set $p^{-1}(x_0)$
- The action is by lifting loops in X based at x_0 to paths in C

Relationship between Fundamental Groups and Covering Spaces

- The action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ is transitive if and only if the covering space is connected
- For a connected covering space $p: C \rightarrow X$, the subgroup $p_*\pi_1(C, c_0)$ of $\pi_1(X, x_0)$, where $c_0 \in p^{-1}(x_0)$, is isomorphic to the stabilizer of c_0 under the action of $\pi_1(X, x_0)$ on the fiber
- There is a one-to-one correspondence between:
 - Connected covering spaces of X (up to isomorphism)
 - Conjugacy classes of subgroups of $\pi_1(X)$

- The universal covering space of X is the unique simply connected covering space of X (up to isomorphism)
- Its group of deck transformations is isomorphic to the fundamental group of X

4.4 Covering spaces and lifting properties

Covering spaces are a powerful tool in algebraic topology, connecting the fundamental group to the structure of spaces. They allow us to lift paths and homotopies, providing insight into the topology of spaces and their relationships.

The lifting property of covering spaces is crucial for understanding homotopy classes and computing fundamental groups. It enables us to classify covering spaces and explore connections between different topological spaces, making it a key concept in this field.

Covering spaces and their properties

Definition and basic properties of covering spaces

- A covering space of a topological space X is a space C together with a continuous surjective map $p : C \rightarrow X$, called a covering map
- Each point x in X has an open neighborhood U for which $p^{-1}(U)$ is a union of disjoint open sets in C
- Each disjoint open set in C maps homeomorphically onto U under p
- The preimage $p^{-1}(x)$ of each point x in X is a discrete space called the fiber over x
- The cardinality of the fiber is the same for all points in X (number of sheets of the covering)

Universal covering spaces and their uniqueness

- A covering space is called a universal covering space if it is simply connected
- Every connected, locally path-connected, and semilocally simply connected space has a universal covering space, unique up to homeomorphism
- Examples of universal covering spaces: the real line $\mathbb{R}$ for the circle S^1 , the Riemann surface of the logarithm for the punctured plane $\mathbb{C} \setminus \{0\}$

Relationship between covering spaces and the fundamental group

- Covering spaces are used to study the fundamental group of a space
- There is a bijection between the set of basepoint-preserving isomorphism classes of connected coverings of X and the set of subgroups of the fundamental group of X
- Example: the fundamental group of the circle S^1 is isomorphic to $\mathbb{Z}$, and its connected covering spaces correspond to subgroups of $\mathbb{Z}$ (the n -fold covers for each $n \in \mathbb{N}$)

Lifting in covering spaces

Definition of lifts and the commutative diagram

- Given a covering map $p : C \rightarrow X$ and a continuous map $f : Y \rightarrow X$, a lift of f is a continuous map $\tilde{f} : Y \rightarrow C$ such that $p \circ \tilde{f} = f$

- The lifting property is represented by the following commutative diagram:

$$\begin{array}{ccc}
 Y & \xrightarrow{\tilde{f}} & C \\
 \searrow f & & \downarrow p \\
 & & X
 \end{array}$$

Homotopy lifting property

- Given a covering map $p : C \rightarrow X$, a homotopy $F : Y \times [0, 1] \rightarrow X$, and a lift $\tilde{f}_0 : Y \rightarrow C$ of $F_0 = F(\cdot, 0)$, there exists a unique homotopy $\tilde{F} : Y \times [0, 1] \rightarrow C$ that lifts F and satisfies $\tilde{F}(\cdot, 0) = \tilde{f}_0$
- The homotopy lifting property relates the homotopy classes of maps in the base space X to those in the covering space C

Relationship between lifts and the topology of covering spaces

- The existence and uniqueness of lifts depend on the properties of the covering space and the map f
- Lifts are used to study the relationship between the topology of the base space X and the covering space C
- Example: the lifting property can be used to show that the fundamental group of a covering space is isomorphic to a subgroup of the fundamental group of the base space

Existence and uniqueness of lifts

Lifting criterion

- Let $p : C \rightarrow X$ be a covering map, and let $f : Y \rightarrow X$ be a continuous map. If Y is connected and locally path-connected, then a lift $\tilde{f} : Y \rightarrow C$ of f exists if and only if $f_*(\pi_1(Y)) \subseteq p_*(\pi_1(C))$, where f_* and p_* are the induced homomorphisms on the fundamental groups
- The lifting criterion relates the existence of lifts to the fundamental groups of the spaces involved

Uniqueness of lifts

- If $p : C \rightarrow X$ is a covering map, Y is a connected space, and $\tilde{f}_1, \tilde{f}_2 : Y \rightarrow C$ are two lifts of a continuous map $f : Y \rightarrow X$ that agree at a single point, then $\tilde{f}_1 = \tilde{f}_2$
- The uniqueness of lifts is a consequence of the local homeomorphism property of covering maps

Proofs of existence and uniqueness

- The proofs of the lifting criterion and the uniqueness of lifts rely on the properties of covering spaces, such as:
- The local homeomorphism condition
- The path-lifting property, which states that any path in the base space X can be uniquely lifted to a path in the covering space C , given a starting point in the fiber over the initial point of the path
- These proofs involve techniques from point-set topology and the theory of fundamental groups

Applications of the lifting property

Computation of fundamental groups

- The lifting criterion can be used to compute the fundamental group of a space X by studying the covering spaces of X and their fundamental groups
- Example: if $p : C \rightarrow X$ is a universal covering space, then $\pi_1(X) \cong \text{Aut}(C/X)$, the group of deck transformations of the covering
- This method is particularly useful when the covering spaces have simpler fundamental groups than the base space

Classification of covering spaces

- The lifting property can be used to classify covering spaces of a given space X
- There is a one-to-one correspondence between the conjugacy classes of subgroups of $\pi_1(X)$ and the isomorphism classes of connected covering spaces of X
- Example: the connected covering spaces of the torus $S^1 \times S^1$ are classified by pairs of integers (m, n) , corresponding to the subgroups generated by (m, n) in $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$

Homotopy invariance of lifting

- If two continuous maps $f_0, f_1 : Y \rightarrow X$ are homotopic and $\tilde{f}_0$ is a lift of f_0 with respect to a covering map $p : C \rightarrow X$, then there exists a unique lift $\tilde{f}_1$ of f_1 such that $\tilde{f}_0$ and $\tilde{f}_1$ are homotopic in C
- This property is useful in studying the homotopy classes of maps between spaces
- Example: the homotopy invariance of lifting can be used to show that the induced homomorphism on fundamental groups is well-defined

Applications in algebraic topology

- The lifting property is a fundamental tool in algebraic topology and is used in various contexts, such as:
 - The study of fiber bundles
 - The Seifert-van Kampen theorem for computing fundamental groups
 - The classification of principal G -bundles over a space X in terms of homotopy classes of maps from X to the classifying space BG
- These applications demonstrate the wide-ranging utility of the lifting property in understanding the structure and properties of topological spaces

Unit 5 – Higher Homotopy Groups

5.1 Higher homotopy groups

Higher homotopy groups extend the fundamental group concept to higher dimensions. They measure "holes" in spaces using maps from n -dimensional spheres. This powerful tool helps distinguish and classify topological spaces beyond what the fundamental group alone can do.

Unlike the fundamental group, higher homotopy groups are always abelian. They play a crucial role in obstruction theory and space classification. Computing these groups often involves advanced techniques like spectral sequences and Postnikov towers.

Higher homotopy groups

Definition and role in algebraic topology

- The n -th homotopy group of a topological space X , denoted $\pi_n(X)$, consists of homotopy equivalence classes of based maps from the n -dimensional sphere S^n to X
- The group operation in $\pi_n(X)$ arises from the concatenation of based maps, making $\pi_n(X)$ an abelian group for $n \geq 2$
- Higher homotopy groups measure the number of "holes" or "voids" of different dimensions in a topological space (e.g., a 2-dimensional void in a 3-dimensional space)
- They provide a more refined way to distinguish topological spaces than the fundamental group alone
- Higher homotopy groups play a central role in obstruction theory and the classification of topological spaces
- Obstruction theory studies the existence and properties of continuous maps between topological spaces using higher homotopy groups
- Classification of topological spaces involves determining when two spaces are homotopy equivalent or homeomorphic based on their higher homotopy groups

Properties and examples

- The n -th homotopy group of a point is trivial for all $n \geq 1$, i.e., $\pi_n(\text{point}) = 0$
- This reflects the fact that a point has no higher-dimensional holes or voids
- The n -th homotopy group of the n -sphere is the integers for $n \geq 1$, i.e., $\pi_n(S^n) \cong \mathbb{Z}$, and $\pi_m(S^n) = 0$ for $m \neq n$
- For example, $\pi_2(S^2) \cong \mathbb{Z}$, while $\pi_1(S^2) = 0$ and $\pi_3(S^2) = 0$
- The generator of $\pi_n(S^n)$ corresponds to the identity map from S^n to itself
- The n -th homotopy group of a contractible space, such as the Euclidean space $\mathbb{R}^n$, is trivial for all $n \geq 1$
- Contractible spaces can be continuously deformed to a point, so they have no higher-dimensional holes or voids

Computing higher homotopy groups

Product spaces and wedge sums

- The higher homotopy groups of a product space can be computed using the product formula: $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$ for $n \geq 1$
- For example, $\pi_n(S^1 \times S^2) \cong \pi_n(S^1) \times \pi_n(S^2)$ for all $n \geq 1$
- This formula reflects the fact that the holes or voids in a product space arise from the holes or voids in its constituent spaces
- The higher homotopy groups of a wedge sum of spaces can be computed using the wedge sum formula: $\pi_n(\vee_i X_i) \cong \bigoplus_i \pi_n(X_i)$ for $n \geq 2$
- For example, $\pi_2(S^1 \vee S^2) \cong \pi_2(S^1) \oplus \pi_2(S^2) \cong 0 \oplus \mathbb{Z} \cong \mathbb{Z}$
- The wedge sum formula holds only for $n \geq 2$ because the fundamental group of a wedge sum is generally not the direct sum of the fundamental groups of its constituent spaces

Techniques and tools

- Spectral sequences, such as the Serre spectral sequence and the Atiyah-Hirzebruch spectral sequence, can be used to compute higher homotopy groups of fibrations and cell complexes
- These spectral sequences relate the homology groups of the base space and the fiber to the homotopy groups of the total space
- The Postnikov tower of a space X is a sequence of spaces X_n that approximate X up to the n -th homotopy group
- The homotopy groups of the Postnikov tower stages can be used to reconstruct the higher homotopy groups of the original space X
- The Whitehead tower of a connected space X is a sequence of spaces X^n that capture the higher homotopy groups of X
- The homotopy fibers of the maps in the Whitehead tower are Eilenberg-MacLane spaces, whose homotopy groups are related to the higher homotopy groups of X

Higher homotopy groups vs the fundamental group

Similarities and differences

- The fundamental group, $\pi_1(X)$, measures the number of 1-dimensional "loops" in a topological space, while higher homotopy groups, $\pi_n(X)$ for $n \geq 2$, measure higher-dimensional holes or voids
- Both the fundamental group and higher homotopy groups are topological invariants that distinguish spaces up to homotopy equivalence
- Higher homotopy groups, $\pi_n(X)$ for $n \geq 2$, are always abelian groups, while the fundamental group may be non-abelian
- The Eckmann-Hilton argument shows that the higher homotopy groups are commutative, i.e., the group operation is independent of the order of composition

- The non-commutativity of the fundamental group reflects the fact that loops in a space can be composed in different orders, yielding different results

Interactions and connections

- The action of the fundamental group on higher homotopy groups is given by the π_1 -action, which describes how loops in $\pi_1(X)$ act on elements of $\pi_n(X)$
- The π_1 -action is a group homomorphism $\pi_1(X) \rightarrow \text{Aut}(\pi_n(X))$, where $\text{Aut}(\pi_n(X))$ is the group of automorphisms of $\pi_n(X)$
- The π_1 -action measures the extent to which the fundamental group "twists" or "permutes" the higher homotopy groups
- The Whitehead products provide a connection between the fundamental group and higher homotopy groups, measuring the non-commutativity of the π_1 -action
- The Whitehead product is a bilinear map $[-, -]: \pi_m(X) \times \pi_n(X) \rightarrow \pi_{m+n-1}(X)$ that generalizes the commutator in the fundamental group
- Non-trivial Whitehead products indicate the presence of non-trivial interactions between the fundamental group and higher homotopy groups

Higher homotopy groups under maps

Induced homomorphisms and functoriality

- A continuous map $f: X \rightarrow Y$ induces group homomorphisms $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ for each $n \geq 1$, called the induced homomorphisms on homotopy groups
- The induced homomorphisms are defined by composing a representative map $S^n \rightarrow X$ with f to obtain a map $S^n \rightarrow Y$, and then taking the homotopy class of this composition
- The induced homomorphisms are functorial, meaning they respect the composition of continuous maps: $(g \circ f)_* = g_* \circ f_*$
- Functoriality reflects the idea that the induced homomorphisms are compatible with the categorical structure of topological spaces and continuous maps
- This property allows for the study of higher homotopy groups in the context of category theory and functorial constructions

Homotopy invariance and applications

- Homotopy equivalences induce isomorphisms on all homotopy groups, i.e., if $f: X \rightarrow Y$ is a homotopy equivalence, then $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ is an isomorphism for all $n \geq 1$
- This property reflects the fact that homotopy equivalent spaces have the same higher homotopy groups
- Homotopy invariance is a key tool in algebraic topology for comparing and classifying spaces using higher homotopy groups
- The behavior of induced homomorphisms under fibrations and cofibrations can be studied using the long exact sequence of homotopy groups
- For a fibration $F \rightarrow E \rightarrow B$, there is a long exact sequence $\dots \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \rightarrow \pi_{n-1}(F) \rightarrow \dots$

- For a cofibration $A \rightarrow X \rightarrow X/A$, there is a long exact sequence $\dots \rightarrow \pi_n(A) \rightarrow \pi_n(X) \rightarrow \pi_n(X/A) \rightarrow \pi_{n-1}(A) \rightarrow \dots$
- These long exact sequences relate the higher homotopy groups of the spaces involved in a fibration or cofibration, providing a powerful tool for computation and analysis
- The Hurewicz theorem relates the higher homotopy groups to the homology groups of a space, providing a connection between homotopy theory and homology theory
- The Hurewicz theorem states that for a simply connected space X , the first non-trivial homotopy group $\pi_n(X)$ is isomorphic to the n -th homology group $H_n(X)$
- This theorem allows for the computation of higher homotopy groups using homology groups, which are often easier to calculate

5.2 Properties of higher homotopy groups

Higher homotopy groups build on the fundamental group, extending the concept to higher dimensions. They capture more intricate topological information about spaces, allowing us to distinguish between spaces that may seem similar at first glance.

These groups are abelian for dimensions greater than one, making them easier to work with mathematically. They also have important properties like functoriality and homotopy invariance, which help us compare different spaces and understand their topological structure.

Higher Homotopy Groups for CW Complexes

Definition and Existence

- Define the n -th homotopy group of a space X , denoted $\pi_n(X)$, as the set of homotopy classes of maps from the n -dimensional sphere S^n to X , with a base point
- Prove that for a CW complex X , the n -th homotopy group $\pi_n(X)$ is well-defined for all $n \geq 1$, as the set of homotopy classes of based maps from the n -sphere to X
- Use the cellular approximation theorem, which states that any continuous map from a CW complex to another space is homotopic to a cellular map, to prove the existence of higher homotopy groups for CW complexes

Computation and Relation to Homology

- Compute the homotopy groups of a CW complex X using the skeletal filtration of X , where the n -th homotopy group depends only on the $(n+1)$ -skeleton of X
- Apply the Hurewicz theorem, which relates the first non-trivial homotopy group of a connected CW complex to its first non-trivial homology group, to establish a connection between homotopy and homology
- Example: For the n -dimensional sphere S^n , the homotopy group $\pi_k(S^n)$ is trivial for $k < n$ and isomorphic to the integers $\mathbb{Z}$ for $k = n$
- Example: The Hurewicz theorem implies that for a simply connected CW complex X , the first non-trivial homotopy group $\pi_n(X)$ is isomorphic to the first non-trivial homology group $H_n(X)$

Abelian Nature of Higher Homotopy Groups

Commutativity and the Eckmann-Hilton Argument

- Prove that for $n \geq 2$, the n -th homotopy group $\pi_n(X)$ of a space X is always an abelian group, meaning that the group operation (composition of homotopy classes of maps) is commutative
- Apply the higher-dimensional analogue of the Eckmann-Hilton argument, which uses the fact that the composition of maps from S_n to X can be performed in two different ways that are homotopic, to establish the abelian nature of higher homotopy groups

Group Operation and Inverses

- Define the group operation in $\pi_n(X)$ by the concatenation of loops (representing elements of the homotopy group) in the n -dimensional sphere S_n , followed by a rescaling of the parameter space
- Prove the existence of an inverse element in $\pi_n(X)$ for $n \geq 2$ by showing that a map from S_n to X can be composed with a reflection map on S_n , which reverses the orientation of the loop
- Example: In the fundamental group $\pi_1(X)$, the group operation is not necessarily commutative, as the composition of loops depends on the order in which they are traversed
- Example: For the n -th homotopy group $\pi_n(X)$ with $n \geq 2$, the inverse of a homotopy class $[f]$ is represented by the composition of f with a reflection map on S_n

Functorial Properties of Higher Homotopy Groups

Induced Homomorphisms and Functoriality

- Define homotopy groups as functorial, meaning that a continuous map $f: X \rightarrow Y$ between topological spaces induces group homomorphisms $f^*: \pi_n(X) \rightarrow \pi_n(Y)$ between their homotopy groups for all $n \geq 1$
- Construct the induced homomorphisms $f^*: \pi_n(X) \rightarrow \pi_n(Y)$ by composing the representative maps from S_n to X with the map $f: X \rightarrow Y$
- Prove that the composition of induced homomorphisms $(g \circ f): \pi_n(X) \rightarrow \pi_n(Z)$ is equal to the induced homomorphism of the composition $(g \circ f): \pi_n(X) \rightarrow \pi_n(Z)$ for continuous maps $f: X \rightarrow Y$ and $g: Y \rightarrow Z$

Homotopy Invariance and Comparison of Spaces

- Use the functoriality of homotopy groups to study homotopy-invariant properties of spaces and compare the homotopy groups of different spaces via continuous maps
- Example: If two spaces X and Y are homotopy equivalent, then their homotopy groups $\pi_n(X)$ and $\pi_n(Y)$ are isomorphic for all $n \geq 1$
- Example: The functoriality of homotopy groups allows for the construction of long exact sequences of homotopy groups associated with fibrations or cofibrations of spaces

Homotopy Excision Theorem for Higher Homotopy Groups

Statement and Relative Homotopy Groups

- State the homotopy excision theorem (Blakers-Massey theorem): if (X, A) is a CW pair and the inclusion $A \hookrightarrow X$ induces an isomorphism $\pi_k(A) \cong \pi_k(X)$ for all $k < n$, then the homotopy groups $\pi_k(X/A)$ of the quotient space X/A are isomorphic to $\pi_{k+1}(X, A)$, the relative homotopy groups of the pair (X, A) , for all $k < 2n-1$
- Define the relative homotopy group $\pi_k(X, A)$ as the set of homotopy classes of maps $(D_k, \partial D_k) \rightarrow (X, A)$, where D_k is the k -dimensional disk and ∂D_k is its boundary, the $(k-1)$ -dimensional sphere

Applications and Computations

- Apply the homotopy excision theorem to break down the computation of higher homotopy groups of a space into the computation of homotopy groups of simpler spaces and their relative homotopy groups
- Compute the homotopy groups of spheres, suspensions, and other spaces that can be decomposed into CW pairs satisfying the hypothesis of the homotopy excision theorem
- Example: Use the homotopy excision theorem to show that the homotopy groups of the n -sphere S_n are isomorphic to the homotopy groups of the suspension of the $(n-1)$ -sphere, $\pi_k(S_n) \cong \pi_{k+1}(\Sigma S_{n-1})$, for $k \geq 2$
- Example: Apply the homotopy excision theorem to compute the homotopy groups of the torus $S^1 \times S^1$ by decomposing it into a CW pair consisting of two cylinders glued along their boundaries

5.3 The Hurewicz theorem

The Hurewicz theorem connects homotopy and homology groups, revealing deep relationships between these fundamental concepts in algebraic topology. It shows that for simply connected spaces, the first non-trivial homotopy group matches the first non-trivial homology group.

This powerful result simplifies the computation of homotopy groups, which are often harder to calculate directly. By leveraging more accessible homology calculations, the Hurewicz theorem provides a crucial tool for understanding the topological structure of spaces.

Hurewicz Theorem: Homotopy vs Homology

Relationship between Homotopy and Homology Groups

- The Hurewicz theorem establishes a relationship between the homotopy groups and homology groups of a space, under certain connectedness conditions
- For a simply connected space X , the Hurewicz theorem states that the first non-trivial homotopy group is isomorphic to the first non-trivial homology group
- Mathematically, this is expressed as $\pi_n(X) \cong H_n(X)$ for the smallest $n \geq 2$ such that $\pi_n(X)$ is non-trivial
- Example: If X is a simply connected space with $\pi_2(X) \cong \mathbb{Z}$ and $\pi_n(X) = 0$ for $n > 2$, then $H_2(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for $n > 2$

Proof of the Hurewicz Theorem

- The proof of the Hurewicz theorem involves the construction of the Hurewicz homomorphism, which maps elements of the homotopy group to elements of the homology group
- The Hurewicz homomorphism is defined using the concept of singular homology, where a continuous map from the n -simplex to the space X is considered
- The proof shows that the Hurewicz homomorphism is an isomorphism under the given conditions, using techniques from algebraic topology
- Key steps in the proof include:
 - Constructing the Hurewicz homomorphism $h_n : \pi_n(X) \rightarrow H_n(X)$
 - Showing that h_n is a group homomorphism
 - Proving that h_n is an isomorphism using the long exact sequence of a pair and the relative Hurewicz theorem
 - Applying the connectivity conditions to ensure the isomorphism holds for the first non-trivial homotopy and homology groups

Computing Homotopy Groups with Hurewicz

Determining Connectivity and Homology Groups

- To compute the first non-trivial homotopy group of a space X using the Hurewicz theorem, one needs to determine the connectivity of the space and find the first non-trivial homology group
- If X is simply connected (i.e., $\pi_1(X) = 0$), then the Hurewicz theorem applies
- Example: The 2-dimensional sphere S^2 is simply connected, so the Hurewicz theorem can be used to compute its first non-trivial homotopy group
- The homology groups of a space can be computed using various techniques, such as cellular homology or singular homology, depending on the properties of the space
- Example: For a CW complex, cellular homology can be used to calculate the homology groups efficiently

Applying the Hurewicz Theorem

- Once the first non-trivial homology group $H_n(X)$ is determined, the Hurewicz theorem implies that $\pi_n(X) \cong H_n(X)$, thus providing the first non-trivial homotopy group
- If the homology groups are known, the Hurewicz theorem allows for the immediate computation of the corresponding homotopy group
- Example: If $H_3(X) \cong \mathbb{Z}_2$ is the first non-trivial homology group of a simply connected space X , then $\pi_3(X) \cong \mathbb{Z}_2$ by the Hurewicz theorem
- The Hurewicz theorem simplifies the computation of homotopy groups by reducing the problem to calculating homology groups, which are often more tractable

Hurewicz Theorem for Simply Connected Spaces

Isomorphism between Homotopy and Homology Groups

- For a simply connected space X , the theorem establishes an isomorphism between the first non-trivial homotopy group and the first non-trivial homology group, providing a link between the two fundamental invariants
- This isomorphism allows for the computation of homotopy groups using homology groups, which are often easier to calculate
- Example: For a simply connected CW complex X , the first non-trivial homotopy group can be determined by calculating the first non-trivial homology group using cellular homology

Implications for the Study of Simply Connected Spaces

- The Hurewicz theorem has significant implications for the study of simply connected spaces in algebraic topology
- The theorem implies that for a simply connected space, the first non-trivial homotopy group determines the first non-trivial homology group, and vice versa
- Example: If $\pi_4(X) \cong \mathbb{Z}$ for a simply connected space X , then $H_4(X) \cong \mathbb{Z}$, and all lower homotopy and homology groups are trivial
- The Hurewicz theorem provides a foundation for the study of higher-dimensional homotopy groups and their relationship to homology groups in simply connected spaces
- It allows for the extension of results and techniques from homology theory to the study of homotopy groups
- The theorem is a crucial tool in the computation and classification of homotopy groups of simply connected spaces

Relative Homotopy Groups with Hurewicz

Relative Hurewicz Theorem

- The relative Hurewicz theorem is an extension of the Hurewicz theorem that relates relative homotopy groups to relative homology groups
- For a pair of spaces (X, A) , where A is a subspace of X , the relative Hurewicz theorem states:
- If X and A are connected and X is obtained from A by attaching cells of dimension $\geq n$, then the relative homotopy group $\pi_n(X, A)$ is isomorphic to the relative homology group $H_n(X, A)$ for $n \geq 2$
- Mathematically, this is expressed as $\pi_n(X, A) \cong H_n(X, A)$ under the given conditions

Applying the Relative Hurewicz Theorem

- To apply the relative Hurewicz theorem, one needs to identify the connectivity of the spaces X and A , and determine the dimension of the cells attached to A to form X
- The relative homology groups $H_n(X, A)$ can be computed using techniques such as the long exact sequence of a pair or the excision theorem

- Example: If (X, A) is a CW pair, the relative homology groups can be computed using the cellular chain complex of the pair
- Once the relative homology group $H_n(X, A)$ is determined, the relative Hurewicz theorem implies that $\pi_n(X, A) \cong H_n(X, A)$, thus providing the relative homotopy group
- Example: If $H_3(X, A) \cong \mathbb{Z}$ and the conditions of the relative Hurewicz theorem are satisfied, then $\pi_3(X, A) \cong \mathbb{Z}$
- The relative Hurewicz theorem allows for the computation of relative homotopy groups by reducing the problem to calculating relative homology groups, which can be more accessible using algebraic techniques

5.4 Eilenberg-MacLane spaces

Eilenberg-MacLane spaces are key players in algebraic topology. They're special spaces with just one non-zero homotopy group, helping us link homotopy theory and group theory. These spaces are crucial for defining cohomology groups and classifying spaces up to homotopy equivalence.

These spaces have practical applications in computing cohomology groups for various spaces. Examples include infinite-dimensional real and complex projective spaces, which are Eilenberg-MacLane spaces for specific groups. They're also used in constructing Postnikov towers and in obstruction theory.

Eilenberg-MacLane Spaces

Definition and Role in Algebraic Topology

- An Eilenberg-MacLane space, denoted $K(G, n)$, is a topological space with a single non-trivial homotopy group, specifically the n th homotopy group isomorphic to the group G
- Eilenberg-MacLane spaces provide a way to study the relationship between homotopy theory and group theory in algebraic topology
- The Eilenberg-MacLane theorem guarantees the existence of Eilenberg-MacLane spaces for any abelian group G and any positive integer n
- Eilenberg-MacLane spaces define the cohomology groups of a space X , denoted $H^n(X; G)$, as the set of homotopy classes of maps from X to $K(G, n)$
- Studying Eilenberg-MacLane spaces allows for the classification of spaces up to homotopy equivalence based on their homotopy groups

Applications and Examples

- Eilenberg-MacLane spaces are used to compute the cohomology groups of various spaces, such as spheres, projective spaces, and lens spaces
- The infinite-dimensional real projective space $\mathbb{R}P^\infty$ is an example of an Eilenberg-MacLane space $K(\mathbb{Z}/2\mathbb{Z}, 1)$
- The infinite-dimensional complex projective space $\mathbb{C}P^\infty$ is an example of an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$
- The classifying space of a discrete group G , denoted BG , is an Eilenberg-MacLane space $K(G, 1)$

Construction of Eilenberg-MacLane Spaces

Trivial and Cyclic Groups

- For the trivial group $G = \{0\}$, the Eilenberg-MacLane space $K(\{0\}, n)$ is contractible for all $n \geq 0$
- For the integers $G = \mathbb{Z}$ and $n = 1$, the Eilenberg-MacLane space $K(\mathbb{Z}, 1)$ is homotopy equivalent to the circle S^1
- For the integers $G = \mathbb{Z}$ and $n > 1$, the Eilenberg-MacLane space $K(\mathbb{Z}, n)$ can be constructed as an infinite-dimensional CW complex with a single n -cell in each dimension $\geq n$
- For a finite cyclic group $G = \mathbb{Z}/m\mathbb{Z}$ and $n = 1$, the Eilenberg-MacLane space $K(\mathbb{Z}/m\mathbb{Z}, 1)$ is homotopy equivalent to the lens space $L(m, 1)$

Direct Sums and Products

- For a direct sum of abelian groups $G = H \oplus K$, the Eilenberg-MacLane space $K(G, n)$ is homotopy equivalent to the product $K(H, n) \times K(K, n)$
- The product of Eilenberg-MacLane spaces $K(G, n) \times K(H, m)$ is an Eilenberg-MacLane space $K(G \oplus H, n)$ if $n = m$ and is not an Eilenberg-MacLane space otherwise
- The loop space of an Eilenberg-MacLane space $\Omega K(G, n + 1)$ is homotopy equivalent to the Eilenberg-MacLane space $K(G, n)$

Eilenberg-MacLane Spaces and Cohomology

Cohomology Groups and Homotopy Classes of Maps

- The n th cohomology group of a space X with coefficients in an abelian group G , denoted $H^n(X; G)$, is isomorphic to the set of homotopy classes of maps from X to the Eilenberg-MacLane space $K(G, n)$
- The isomorphism between $H^n(X; G)$ and $[X, K(G, n)]$ is natural, meaning it is compatible with the induced homomorphisms of cohomology groups and the induced maps of homotopy classes
- The cohomology groups of an Eilenberg-MacLane space $K(G, n)$ are given by $H^i(K(G, n); H) = G$ for $i = n$ and $H^i(K(G, n); H) = 0$ for $i \neq n$, where H is any abelian group

Cup Product and Cohomology Ring Structure

- The cup product in cohomology corresponds to the composition of maps between Eilenberg-MacLane spaces, providing a geometric interpretation of the cohomology ring structure
- The cohomology ring $H^*(K(G, n); \mathbb{Z})$ of an Eilenberg-MacLane space $K(G, n)$ is isomorphic to the exterior algebra $\Lambda_{\mathbb{Z}}(G)$ if n is odd and to the divided power algebra $\Gamma_{\mathbb{Z}}(G)$ if n is even
- The cohomology ring $H^*(K(G, n); \mathbb{F}_p)$ of an Eilenberg-MacLane space $K(G, n)$ with coefficients in a field $\mathbb{F}_p$ is isomorphic to the polynomial algebra $\mathbb{F}_p[x]$ if n is even and $p \nmid |G|$, and to the truncated polynomial algebra $\mathbb{F}_p[x]/(x^p)$ if n is even and $p \mid |G|$

Classifying Homotopy Types with Eilenberg-MacLane Spaces

Homotopy Equivalence and Whitehead Theorem

- Two spaces X and Y are of the same homotopy type if and only if there exist maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that the compositions $f \circ g$ and $g \circ f$ are homotopic to the respective identity maps
- The Whitehead theorem states that the homotopy type of a connected CW complex X is determined by its homotopy groups $\pi_n(X)$ for all $n \geq 1$
- A connected CW complex X is homotopy equivalent to a product of Eilenberg-MacLane spaces $\prod_n K(\pi_n(X), n)$ if and only if X is a $K(\pi, 1)$ space, i.e., $\pi_i(X) = 0$ for $i > 1$

Postnikov Towers and Obstruction Theory

- The homotopy type of a simply connected space X is determined by its cohomology ring $H^*(X; \mathbb{Z})$ and its Postnikov tower, which consists of a sequence of fibrations involving Eilenberg-MacLane spaces
- The Postnikov tower of a space X is a sequence of spaces X_n and maps $p_n: X \rightarrow X_n$ such that $\pi_i(X_n) = \pi_i(X)$ for $i \leq n$ and $\pi_i(X_n) = 0$ for $i > n$, and the homotopy fiber of p_n is the Eilenberg-MacLane space $K(\pi_{n+1}(X), n+1)$
- The obstruction to the existence of a map between two spaces X and Y can be described using the cohomology groups of X with coefficients in the homotopy groups of Y , which are related to the Eilenberg-MacLane spaces $K(\pi_n(Y), n)$
- The obstruction to extending a map $f: X^{(n)} \rightarrow Y$ from the n -skeleton of a CW complex X to the $(n+1)$ -skeleton lies in the cohomology group $H^{n+1}(X; \pi_n(Y))$, which can be interpreted as the set of homotopy classes of maps from X to the Eilenberg-MacLane space $K(\pi_n(Y), n+1)$

Unit 6 – Homology and Cohomology Theories

6.1 Axioms for homology and cohomology theories

Homology and cohomology theories are powerful tools in algebraic topology. They use sequences of functors to study topological spaces, measuring "holes" and "obstructions" respectively. These theories provide complementary information about spaces' topology.

The axioms for homology and cohomology theories define their key properties. These include homotopy invariance, exactness, excision, and dimension axioms. Understanding these axioms is crucial for grasping how these theories work and applying them to various topological problems.

Axioms for Homology and Cohomology

Homology Theory Axioms

- A homology theory is a sequence of functors H_n from the category of topological pairs to the category of abelian groups for each integer n , satisfying the following axioms:
- Homotopy axiom: If two maps between pairs are homotopic, they induce the same homomorphism on homology groups
- Exactness axiom: For any pair (X, A) , there is a long exact sequence involving the homology groups of X , A , and the quotient space X/A
- Excision axiom: If (X, A) is a pair and $U \subset A$ such that $\overline{U} \subset \text{int}(A)$, then the inclusion $(X - U, A - U) \rightarrow (X, A)$ induces an isomorphism on homology groups
- Dimension axiom: The homology group of a one-point space is the coefficient group in degree 0 and the trivial group in all other degrees
- Examples of homology theories include singular homology, simplicial homology, and cellular homology

Cohomology Theory Axioms

- A cohomology theory is a sequence of contravariant functors H^n from the category of topological pairs to the category of abelian groups for each integer n , satisfying axioms analogous to those of a homology theory
- The axioms for a cohomology theory are dual to those of a homology theory:
- Homotopy axiom: If two maps between pairs are homotopic, they induce the same homomorphism on cohomology groups
- Exactness axiom: For any pair (X, A) , there is a long exact sequence involving the cohomology groups of X , A , and the quotient space X/A
- Excision axiom: If (X, A) is a pair and $U \subset A$ such that $\overline{U} \subset \text{int}(A)$, then the inclusion $(X - U, A - U) \rightarrow (X, A)$ induces an isomorphism on cohomology groups

- Dimension axiom: The cohomology group of a one-point space is the coefficient group in degree 0 and the trivial group in all other degrees
- Examples of cohomology theories include singular cohomology, de Rham cohomology, and Čech cohomology

Homology vs Cohomology

Duality and Complementary Information

- Homology and cohomology theories are dual to each other and provide complementary information about the topology of a space
- Homology groups $H_n(X)$ measure the "holes" in X of dimension n
- Cohomology groups $H^n(X)$ measure the "obstructions" to extending continuous functions from n -dimensional subspaces of X to the entire space
- The universal coefficient theorem establishes a relationship between homology and cohomology groups:
- For a space X with coefficients in a field, the cohomology group $H^n(X)$ is isomorphic to the dual vector space of the homology group $H_n(X)$
- This theorem allows for the computation of cohomology groups from homology groups and vice versa

Relating Homology and Cohomology Classes

- The cap product and cup product provide ways to relate homology and cohomology classes
- The cap product is a bilinear pairing $H^n(X) \times H_m(X) \rightarrow H_{m-n}(X)$, which associates a homology class to a cohomology class and a homology class
- The cup product is a bilinear pairing $H^n(X) \times H^m(X) \rightarrow H^{n+m}(X)$, which associates a cohomology class to two cohomology classes
- These products allow for the construction of new homology and cohomology classes from existing ones and provide a rich algebraic structure on the homology and cohomology groups of a space

Properties of Homology and Cohomology

Consequences of the Axioms

- The long exact sequence of a pair is a consequence of the exactness axiom
- For a pair (X, A) , it relates the homology (or cohomology) groups of X , A , and the quotient space X/A
- This sequence is a powerful tool for computing homology and cohomology groups
- The excision theorem follows from the excision axiom
- It states that the homology (or cohomology) groups of a space are unchanged by removing a subspace that has a closure contained in the interior of another subspace
- This theorem allows for the simplification of computations by decomposing a space into simpler pieces
- The homotopy invariance of homology and cohomology is a result of the homotopy axiom

- Homotopy equivalent spaces have isomorphic homology and cohomology groups
- This property allows for the study of topological spaces up to homotopy equivalence

Implications of the Dimension Axiom

- The dimension axiom implies that the homology and cohomology groups of a one-point space are concentrated in degree 0
- The homology and cohomology groups in degree 0 are isomorphic to the coefficient group
- All other homology and cohomology groups of a one-point space are trivial
- This axiom serves as a starting point for computing the homology and cohomology groups of more complex spaces using the other axioms and properties

Computing Homology and Cohomology Groups

Simple Spaces

- Contractible spaces (spaces that are homotopy equivalent to a point), such as a point or a convex subset of Euclidean space, have trivial homology and cohomology groups in all degrees except $H_0(X)$ and $H^0(X)$, which are isomorphic to the coefficient group
- For a wedge sum of circles $\bigvee_i S^1$, the homology groups are:
 - $H_0(\bigvee_i S^1) \cong$ coefficient group
 - $H_1(\bigvee_i S^1) \cong$ free abelian group with rank equal to the number of circles
 - $H_n(\bigvee_i S^1) \cong 0$ for $n \geq 2$
- The cohomology groups have the same ranks but with the free abelian groups appearing in degree 1
- For a sphere S^n , the homology and cohomology groups are:
 - $H_0(S^n) \cong H^0(S^n) \cong$ coefficient group
 - $H_n(S^n) \cong H^n(S^n) \cong$ coefficient group
 - $H_i(S^n) \cong H^i(S^n) \cong 0$ for $0 < i < n$

Computational Tools

- The Künneth formula expresses the homology and cohomology groups of a product space $X \times Y$ in terms of the tensor product and Tor functor of the homology and cohomology groups of X and Y
- This formula allows for the computation of homology and cohomology groups of product spaces from the groups of their factors
- The Mayer-Vietoris sequence relates the homology (or cohomology) groups of a space X that can be decomposed into two subspaces A and B with $X = A \cup B$
- The sequence involves the homology (or cohomology) groups of X , A , B , and $A \cap B$

- This tool is useful for computing the homology and cohomology groups of spaces that can be decomposed into simpler pieces with known homology and cohomology groups

6.2 Singular homology and cohomology

Singular homology and cohomology are powerful tools for studying topological spaces. They use continuous maps from simplices to capture a space's structure, measuring "holes" and "obstructions" in different dimensions.

These theories satisfy key axioms like homotopy invariance and exactness. They're connected by the universal coefficient theorem, allowing us to compute one from the other in many cases. Understanding these concepts is crucial for analyzing complex topological spaces.

Singular Simplices and Chain Complexes

Definition and Notation

- A singular n -simplex in a topological space X is a continuous map $\sigma : \Delta^n \rightarrow X$, where Δ^n is the standard n -simplex in $\mathbb{R}^{n+1}$
- The standard n -simplex Δ^n is the convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$ (e.g., the 1-simplex Δ^1 is a line segment, the 2-simplex Δ^2 is a triangle)
- The set of all singular n -simplices in X is denoted by $S_n(X)$
- The singular chain complex of X is the sequence of free abelian groups $C_n(X)$ generated by the singular n -simplices, with boundary maps $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$
- The boundary maps are defined by the alternating sum of the faces of each simplex, where the i -th face of a simplex σ is obtained by omitting the i -th vertex

Properties and Interpretation

- The boundary maps satisfy the property $\partial_{n-1} \circ \partial_n = 0$, making the singular chain complex a chain complex
- This property ensures that the composition of two consecutive boundary maps is always zero, which is crucial for defining homology groups
- The singular chain complex encodes the topological information of the space X through the continuous maps from simplices to X
- Intuitively, the singular chain complex captures how the space X can be "triangulated" or approximated by simplices of various dimensions
- The boundary maps describe how these simplices are connected to each other, with the boundary of an n -simplex being a combination of $(n-1)$ -simplices

Singular Homology and Cohomology Groups

Definition and Interpretation

- The singular homology groups $H_n(X)$ are defined as the homology groups of the singular chain complex, i.e., $H_n(X) = \ker(\partial_n) / \text{im}(\partial_{n+1})$

- Elements of $H_n(X)$ are equivalence classes of n -cycles (singular n -chains with zero boundary) modulo n -boundaries (singular n -chains that are the boundary of an $(n+1)$ -chain)
- Intuitively, homology groups measure the "holes" or "voids" in a space X that cannot be filled by simplices of the corresponding dimension (e.g., $H_0(X)$ measures the connected components of X , $H_1(X)$ measures the 1-dimensional holes or loops)
- The singular cohomology groups $H^n(X; G)$ are defined as the cohomology groups of the singular cochain complex, which is the dual of the singular chain complex
- The singular cochain complex is obtained by applying the Hom functor $\text{Hom}(-, G)$ to the singular chain complex, where G is an abelian group (usually $\mathbb{Z}$ or a field)
- The coboundary maps $\delta^n : C^n(X; G) \rightarrow C^{n+1}(X; G)$ are the dual maps of the boundary maps, satisfying $\delta^{n+1} \circ \delta^n = 0$
- Elements of $H^n(X; G)$ are equivalence classes of n -cocycles (singular n -cochains with zero coboundary) modulo n -coboundaries (singular n -cochains that are the coboundary of an $(n-1)$ -cochain)

Relationship between Homology and Cohomology

- Singular homology and cohomology are related by the universal coefficient theorem, which states that for a topological space X and an abelian group G , there is a short exact sequence:

$$0 \rightarrow \text{Ext}(H_{n-1}(X), G) \rightarrow H^n(X; G) \rightarrow \text{Hom}(H_n(X), G) \rightarrow 0$$
- This sequence splits (non-naturally), so $H^n(X; G) \cong \text{Ext}(H_{n-1}(X), G) \oplus \text{Hom}(H_n(X), G)$
- When G is a field or $H_{n-1}(X)$ is free, $\text{Ext}(H_{n-1}(X), G) = 0$, and $H^n(X; G) \cong \text{Hom}(H_n(X), G)$
- Intuitively, cohomology groups can be thought of as dual to homology groups, measuring the "co-holes" or "obstructions" in a space X
- While homology groups are computed using cycles and boundaries, cohomology groups are computed using cocycles and coboundaries, which are dual notions

Axioms for Homology and Cohomology Theories

Homotopy Invariance and Exactness

- Homotopy invariance: If $f, g : X \rightarrow Y$ are homotopic continuous maps, then the induced homomorphisms $f_*, g_* : H_n(X) \rightarrow H_n(Y)$ and $f^*, g^* : H^n(Y; G) \rightarrow H^n(X; G)$ are equal for all n
- This axiom states that homology and cohomology groups are invariant under homotopy equivalence, capturing the idea that they depend only on the "shape" of a space, not on its specific embedding or parametrization
- Exactness: For any pair of topological spaces $A \subset X$, there are long exact sequences in both homology and cohomology connecting the groups of A , X , and X/A :

$$\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow \cdots$$

$$\cdots \rightarrow H^n(X, A; G) \rightarrow H^n(X; G) \rightarrow H^n(A; G) \rightarrow H^{n+1}(X, A; G) \rightarrow \cdots$$
- These sequences relate the homology and cohomology groups of a space X , a subspace A , and the quotient space X/A , providing a powerful tool for computing these groups in terms of simpler spaces

Excision and Additivity

- Excision: If $U \subset A \subset X$ and the closure of U is contained in the interior of A , then the inclusion $(X - U, A - U) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - U, A - U) \cong H_n(X, A)$ and $H^n(X, A; G) \cong H^n(X - U, A - U; G)$ for all n
- This axiom allows the computation of relative homology and cohomology groups by "excising" a subset U that is contained in the interior of A , without changing the groups
- It is a crucial tool for proving the Mayer-Vietoris sequence, which relates the homology and cohomology groups of a space X to those of two subspaces A and B whose union is X
- Additivity: For any disjoint union of topological spaces $X = \bigsqcup_{\alpha} X_{\alpha}$, there are natural isomorphisms $H_n(X) \cong \bigoplus_{\alpha} H_n(X_{\alpha})$ and $H^n(X; G) \cong \prod_{\alpha} H^n(X_{\alpha}; G)$ for all n
- This axiom states that homology and cohomology groups of a disjoint union of spaces are the direct sum and direct product, respectively, of the groups of the individual spaces
- It allows the computation of homology and cohomology groups of a space by decomposing it into simpler pieces

Dimension Axiom

- Dimension axiom: For a one-point space $\{*\}$, the homology and cohomology groups are given by:

•

$$H_n(\{*\}) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

•

$$H^n(\{*\}; G) \cong \begin{cases} G, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

- This axiom specifies the homology and cohomology groups of a single point, which serve as the building blocks for the groups of more complex spaces via the other axioms
- It also ensures that the homology and cohomology theories are non-trivial, as they assign non-zero groups to the simplest possible space

Singular Homology and Cohomology Computations

Basic Topological Spaces

- For a contractible space X (e.g., a point, a ball, or a convex set in $\mathbb{R}^n$), $H_n(X) \cong H^n(X; G) \cong G$ for $n = 0$ and 0 for $n \neq 0$
- Contractible spaces have the same homology and cohomology groups as a single point, as they can be continuously deformed to a point without changing their topology
- For the n -sphere S^n , the homology and cohomology groups are given by:
- $H_0(S^n) \cong H_n(S^n) \cong \mathbb{Z}$, $H_0(S^n; G) \cong H_n(S^n; G) \cong G$, and $H_k(S^n) \cong H_k(S^n; G) \cong 0$ for $0 < k < n$
- The n -sphere has non-trivial homology and cohomology groups only in dimensions 0 and n , reflecting the fact that it is a connected space with a single "hole" or "void" in dimension n

Operations on Topological Spaces

- For a wedge sum of spaces $X \vee Y$, there are natural isomorphisms $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ and $H^n(X \vee Y; G) \cong H^n(X; G) \times H^n(Y; G)$ for all n , provided that the wedge point is the only common point of X and Y
- The wedge sum of two spaces is obtained by gluing them together at a single point, and its homology and cohomology groups are the direct sum and direct product, respectively, of the groups of the individual spaces
- For a product of spaces $X \times Y$, there are natural isomorphisms:
 - $H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$ (Künneth formula for homology)
 - $0 \rightarrow \bigoplus_{i+j=n} H^i(X; G) \otimes H^j(Y; G) \rightarrow H^n(X \times Y; G) \rightarrow \bigoplus_{i+j=n-1} \text{Tor}(H^i(X; G), H^j(Y; G)) \rightarrow 0$ (Künneth formula for cohomology)
- These formulas relate the homology and cohomology groups of a product space to the tensor products and torsion products of the groups of the individual spaces
- For a quotient space X/A , where A is a contractible subspace of X , there are natural isomorphisms $H_n(X/A) \cong H_n(X, A)$ and $H^n(X/A; G) \cong H^n(X, A; G)$ for all n (relative homology and cohomology)
- These isomorphisms allow the computation of the homology and cohomology groups of a quotient space by considering the relative groups of the original space and the subspace being collapsed
- They are particularly useful when the subspace A is contractible, as the relative groups (X, A) are easier to compute than the groups of the quotient space X/A directly

6.3 Čech homology and cohomology

Čech homology and cohomology are powerful tools for studying topological spaces. They use open covers and nerves to capture local information, making them ideal for analyzing complex structures. These theories complement singular homology, offering unique insights into spatial properties.

Čech theories satisfy important axioms like homotopy invariance and excision. They're great for computing topological invariants like Betti numbers and Euler characteristics. Čech cohomology is particularly useful for studying continuous functions and local properties of spaces.

Čech homology and cohomology

Definition using open covers and nerves

- An open cover of a topological space X is a collection of open sets $\{U_i\}$ whose union is the entire space X
- The nerve of an open cover $\mathcal{U} = \{U_i\}$ is a simplicial complex $N(\mathcal{U})$ where each k -simplex corresponds to a non-empty intersection of $k+1$ open sets from the cover
- The Čech complex $\check{C}(\mathcal{U})$ is the chain complex associated with the nerve $N(\mathcal{U})$
- Its homology groups are the Čech homology groups $\check{H}_n(X; G)$ with coefficients in an abelian group G
- Čech cohomology groups $\check{H}^n(X; G)$ are defined as the cohomology groups of the cochain complex $\check{C}^*(\mathcal{U}; G)$ associated with the nerve $N(\mathcal{U})$ and the coefficient group G

- The Čech (co)homology groups are independent of the choice of open cover, provided the open cover is sufficiently fine
- Each point in X should be contained in an open set of the cover

Properties of Čech (co)homology

- Homotopy invariance: If $f, g: X \rightarrow Y$ are homotopic maps, then the induced homomorphisms $f, g: \check{H}_n(X; G) \rightarrow \check{H}_n(Y; G)$ are equal
- Exactness: For any pair (X, A) of topological spaces, there is a long exact sequence relating the Čech homology groups of X , A , and the quotient space X/A
- Excision: If $U \subset A \subset X$ and the closure of U is contained in the interior of A , then the inclusion $(X - U, A - U) \hookrightarrow (X, A)$ induces isomorphisms $\check{H}_n(X - U, A - U; G) \cong \check{H}_n(X, A; G)$ for all n
- Dimension axiom: For a one-point space P , $\check{H}_0(P; G) \cong G$ and $\check{H}_n(P; G) = 0$ for $n \neq 0$

Čech vs Singular homology

Relationship between Čech and singular (co)homology

- For "nice" spaces (CW complexes, manifolds), Čech homology and cohomology are isomorphic to singular homology and cohomology, respectively
- The isomorphism is established using a direct limit construction over all open covers of the space, with refinement of covers as the directed set
- For general topological spaces, Čech (co)homology may differ from singular (co)homology
- Čech theory captures more local information about the space

Similarities and differences

- Both Čech and singular (co)homology satisfy the Eilenberg-Steenrod axioms for (co)homology theories
- These axioms include homotopy invariance, exactness, excision, and the dimension axiom
- Čech (co)homology is defined using open covers and nerves, while singular (co)homology is defined using singular simplices and their boundaries
- Čech (co)homology is better suited for studying local properties of spaces, while singular (co)homology is more straightforward for computations on simplicial complexes and CW complexes

Axioms for Čech homology

Eilenberg-Steenrod axioms

- Homotopy invariance: Homotopic maps induce equal homomorphisms on Čech homology groups
- Exactness: Long exact sequence relating the Čech homology groups of a pair (X, A) and the quotient space X/A
- Excision: Inclusion of a "nice" pair $(X - U, A - U)$ into (X, A) induces isomorphisms on Čech homology groups

- Dimension axiom: Čech homology of a one-point space P is isomorphic to the coefficient group G in dimension 0 and trivial in other dimensions

Proofs of the axioms

- The proofs involve using the properties of open covers, nerves, and the associated chain complexes
- Homotopy invariance follows from the fact that homotopic maps induce chain homotopic maps on the Čech complex
- Exactness is derived from the short exact sequence of chain complexes associated with a pair (X, A)
- Excision is proved by showing that the open covers of X and A can be chosen to satisfy the excision condition
- The dimension axiom is a direct consequence of the definition of Čech homology for a one-point space

Applications of Čech homology

Computing topological invariants

- The k th Betti number $\beta_k(X)$ is the rank of the k th Čech homology group $\check{H}_k(X; \mathbb{Z})$
- It counts the number of k -dimensional "holes" in the space X (genus for surfaces, number of connected components for $k=0$)
- The Euler characteristic $\chi(X)$ is the alternating sum of the Betti numbers: $\chi(X) = \sum_k (-1)^k \beta_k(X)$
- It is a topological invariant that can be used to distinguish non-homeomorphic spaces (orientable vs non-orientable surfaces)

Studying continuous functions

- Čech cohomology can be used to study the existence of certain continuous functions on a space
- Nowhere vanishing functions are related to the zeroth cohomology group (non-vanishing vector fields on manifolds)
- Functions with prescribed zeros and poles are related to higher cohomology groups (meromorphic functions on complex manifolds)
- The cup product in Čech cohomology provides a way to study the multiplicative structure of cohomology and its relation to the topology of the space
- It can detect the non-triviality of certain cohomology classes (powers of the fundamental class of a manifold)

Local properties of spaces

- Čech (co)homology can be applied to study the local properties of spaces
- Local homology groups at a point capture the (co)homological behavior of the space near that point (local structure of singularities)
- Čech cohomology can be used to define sheaf cohomology, which is a powerful tool for studying the global properties of sheaves on a space (algebraic geometry, complex analysis)

6.4 de Rham cohomology

De Rham cohomology bridges differential geometry and algebraic topology by studying smooth manifolds through differential forms. It provides a way to measure "holes" in manifolds and understand their topological structure using calculus-like tools.

This approach connects to the broader themes of homology and cohomology theories in the chapter. De Rham cohomology offers a concrete realization of cohomology using differential forms, illustrating how these abstract concepts apply to smooth manifolds.

Differential Forms on Manifolds

Definition and Notation

- A differential k -form on a smooth n -manifold M is a smooth section of the k th exterior power of the cotangent bundle of M
- The set of all differential k -forms on M is denoted by $\Omega^k(M)$

Exterior Derivative and de Rham Complex

- The exterior derivative d is a linear map from $\Omega^k(M)$ to $\Omega^{k+1}(M)$ satisfying:
- $d^2 = 0$
- Leibniz rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$
- The de Rham complex is the cochain complex $(\Omega^*(M), d)$ consisting of the differential forms on M with the exterior derivative as the coboundary map

Properties of Differential Forms

- Closed forms are differential forms α satisfying $d\alpha = 0$, while exact forms are differential forms α such that $\alpha = d\beta$ for some differential form β
- The wedge product $\wedge$ is a bilinear map from $\Omega^k(M) \times \Omega^l(M)$ to $\Omega^{k+l}(M)$ that is graded commutative:
- $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$
- Examples: $dx \wedge dy = -dy \wedge dx$, $dx \wedge dx = 0$

De Rham Cohomology Groups

Construction using the de Rham Complex

- The k th de Rham cohomology group $H_{dR}^k(M)$ is defined as the quotient of the kernel of $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ by the image of $d : \Omega^{k-1}(M) \rightarrow \Omega^k(M)$
- $Z^k(M) = \{\alpha \in \Omega^k(M) \mid d\alpha = 0\}$ is the space of closed k -forms (kernel of d)
- $B^k(M) = \{\alpha \in \Omega^k(M) \mid \alpha = d\beta \text{ for some } \beta \in \Omega^{k-1}(M)\}$ is the space of exact k -forms (image of d)
- $H_{dR}^k(M) = Z^k(M)/B^k(M)$
- Elements of $H_{dR}^k(M)$ are equivalence classes $[\alpha]$ of closed k -forms α , where two closed k -forms α and β are equivalent if their difference $\alpha - \beta$ is exact

Induced Maps and Graded Algebra Structure

- The exterior derivative d induces a well-defined linear map from $H_{dR}^k(M)$ to $H_{dR}^{k+1}(M)$, which is the zero map due to $d^2 = 0$
- The wedge product of differential forms induces a well-defined graded commutative product on the de Rham cohomology groups, making $H_{dR}^*(M)$ a graded commutative algebra
- Examples: $[dx] \wedge [dy] = -[dy] \wedge [dx]$, $[dx] \wedge [dx] = 0$ in $H_{dR}^*(\mathbb{R}^2)$

Cohomology Theory Axioms

Homotopy Invariance

- If $f_0, f_1 : M \rightarrow N$ are homotopic smooth maps, then the induced maps $f_0^*, f_1^* : H_{dR}^*(N) \rightarrow H_{dR}^*(M)$ on de Rham cohomology are equal
- The proof involves constructing a chain homotopy between the pullback maps f_0^* and f_1^* using the homotopy operator and the pullback of differential forms
- Example: The identity map and a constant map on a manifold induce the same map on de Rham cohomology

Exactness (Mayer-Vietoris Sequence)

- For any pair of smooth manifolds M and N , the Mayer-Vietoris sequence in de Rham cohomology is exact
- The proof involves constructing a short exact sequence of de Rham complexes using a partition of unity subordinate to an open cover of $M \cup N$ and applying the zig-zag lemma
- Example: For $M = S^1$ and $N = (0, 2\pi)$, the Mayer-Vietoris sequence relates the cohomology of the circle to that of the interval

Excision

- If U is an open subset of M and K is a compact subset of U , then the inclusion map $i : (M - K, U - K) \rightarrow (M, U)$ induces an isomorphism $i^* : H_{dR}^*(M, U) \rightarrow H_{dR}^*(M - K, U - K)$ on relative de Rham cohomology
- The proof involves showing that the restriction map $\Omega^*(M, U) \rightarrow \Omega^*(M - K, U - K)$ is a quasi-isomorphism using a smooth bump function supported in U
- Example: For $M = \mathbb{R}^2$, $U = \mathbb{R}^2 - \{0\}$, and $K = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$, the relative cohomology $H_{dR}^*(\mathbb{R}^2, \mathbb{R}^2 - \{0\})$ is isomorphic to $H_{dR}^*(\mathbb{R}^2 - K, \mathbb{R}^2 - \{0\})$

Topology of Smooth Manifolds

Topological Invariance and Betti Numbers

- The de Rham cohomology groups $H_{dR}^k(M)$ are topological invariants of the smooth manifold M , independent of the choice of Riemannian metric or orientation
- The dimension of $H_{dR}^k(M)$ is equal to the k th Betti number of M , which counts the number of independent k -dimensional "holes" in M

- Examples: For the torus T^2 , $\dim H_{dR}^0(T^2) = 1$ (connected), $\dim H_{dR}^1(T^2) = 2$ (two independent loops), and $\dim H_{dR}^2(T^2) = 1$ (one cavity)

Cup Product and Intersection Theory

- The cup product on de Rham cohomology corresponds to the intersection of submanifolds in M , providing a way to study the intersection theory of M using differential forms
- Example: For two closed 1-forms α and β on a surface, the cup product $[\alpha] \smile [\beta]$ represents the intersection of the corresponding curves

Integration and Poincaré Duality

- The integration of closed forms over cycles in M defines a non-degenerate pairing between de Rham cohomology and singular homology, which can be used to detect non-trivial topological features of M
- The de Rham cohomology of a compact oriented manifold satisfies Poincaré duality, which relates the k th de Rham cohomology group to the $(n-k)$ th de Rham cohomology group via the wedge product and integration
- Example: For a compact oriented surface Σ , $H_{dR}^0(\Sigma) \cong H_{dR}^2(\Sigma)$ and $H_{dR}^1(\Sigma) \cong H_{dR}^1(\Sigma)$ via the Poincaré duality pairing

De Rham vs Singular Cohomology

De Rham Theorem

- The de Rham theorem states that for any smooth manifold M , there is a natural isomorphism between the de Rham cohomology groups $H_{dR}^k(M)$ and the singular cohomology groups $H^k(M; \mathbb{R})$ with real coefficients
- The isomorphism is given by the de Rham map, which sends a closed k -form α to the real cohomology class represented by the cocycle that evaluates a singular k -simplex σ by integrating α over σ

Proof Steps

- The proof of the de Rham theorem involves several steps:
- Constructing a cochain map from the de Rham complex to the singular cochain complex using the de Rham map
- Showing that the de Rham map induces an isomorphism on cohomology by constructing a homotopy inverse using smooth singular cochains and the Whitney approximation theorem
- Proving that the de Rham map is a ring isomorphism with respect to the wedge product on de Rham cohomology and the cup product on singular cohomology

Consequences and Interpretations

- The de Rham theorem implies that the algebraic and topological properties of the de Rham cohomology groups, such as functoriality, homotopy invariance, and Poincaré duality, also hold for the singular cohomology groups with real coefficients
- As a consequence of the de Rham theorem, the integration of closed forms over cycles in M can be interpreted as the evaluation of real cohomology classes on homology classes, providing a link between

differential geometry and algebraic topology

- Example: For a closed 1-form α on a circle S^1 and a loop $\gamma : [0, 1] \rightarrow S^1$, the integral $\int_\gamma \alpha$ computes the evaluation of the cohomology class $[\alpha] \in H_{dR}^1(S^1) \cong H^1(S^1; \mathbb{R})$ on the homology class $[\gamma] \in H_1(S^1; \mathbb{R})$

Unit 7 – Cohomology and Cup Product

7.1 Cup product in cohomology

The cup product is a powerful tool in cohomology that combines two classes to create a new one of higher degree. It's like a mathematical recipe that mixes ingredients to create something new and exciting. This operation helps us understand the structure of spaces and how they relate to each other.

By using the cup product, we can compute cohomology rings and find important invariants of spaces. It's like a secret code that reveals hidden properties of shapes and surfaces. This knowledge helps us solve puzzles about maps between spaces and discover geometric structures we couldn't see before.

Cup Product Operation

Definition and Construction

- The cup product is a binary operation that takes two cohomology classes and produces a new cohomology class of higher degree
- For cohomology classes $\alpha \in H^n(X; R)$ and $\beta \in H^m(X; R)$, their cup product is a class $\alpha \smile \beta \in H^{n+m}(X; R)$
- The cup product is induced by the diagonal map $\Delta : X \rightarrow X \times X$ and the cross product in cohomology
- The cup product can be defined using singular cohomology, simplicial cohomology, or cellular cohomology (singular, simplicial, cellular cohomology)

Naturality and Maps

- The cup product is natural with respect to continuous maps between topological spaces
- If $f : X \rightarrow Y$ is a continuous map and $\alpha \in H^n(Y; R)$ and $\beta \in H^m(Y; R)$ are cohomology classes, then $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$, where f^* is the induced homomorphism on cohomology
- This naturality property allows the cup product to be used in functorial constructions and to study maps between spaces
- Example: For the inclusion map $i : S^n \rightarrow \mathbb{R}^{n+1}$, the induced homomorphism $i^* : H^*(\mathbb{R}^{n+1}; R) \rightarrow H^*(S^n; R)$ preserves the cup product structure

Computing Cup Products

Cellular Cohomology

- In cellular cohomology, the cup product is computed using the cellular cochain complex
- For cellular cohomology classes represented by cocycles $\alpha \in C^n(X; R)$ and $\beta \in C^m(X; R)$, their cup product $\alpha \smile \beta$ is represented by the cocycle $(\alpha \smile \beta)(e^{n+m}) = \alpha(e^n)\beta(e^m)$ for each $(n+m)$ -cell e^{n+m}
- The cup product of cellular cohomology classes is computed by multiplying the values of the cocycles on the cells of the CW complex

- The cellular boundary formula $\partial(e^{n+m}) = \sum_i [e^{n+m} : e_i^n] e_i^n + \sum_j [e^{n+m} : e_j^m] e_j^m$ is used to determine the cup product on the cellular cochain level

Examples and Calculations

- Example: Let $X = S^2 \vee S^4$ be the wedge sum of spheres. The cellular cohomology groups are $H^0(X; R) = R$, $H^2(X; R) = R$, $H^4(X; R) = R$, and $H^i(X; R) = 0$ for $i \neq 0, 2, 4$. Let $\alpha \in H^2(X; R)$ and $\beta \in H^4(X; R)$ be the generators. Then $\alpha \smile \alpha = 0$ and $\alpha \smile \beta = 0$ because there are no cells of dimension 4 or 6 in X
- Example: For the torus $T = S^1 \times S^1$, let $a, b \in H^1(T; R)$ be the generators corresponding to the two circles. Then $a \smile b = -b \smile a$ generates $H^2(T; R)$, and $a \smile a = b \smile b = 0$. The cohomology ring $H^*(T; R)$ is isomorphic to the exterior algebra $\Lambda_R[a, b]$

Properties of Cup Products

Algebraic Properties

- The cup product is associative: $(\alpha \smile \beta) \smile \gamma = \alpha \smile (\beta \smile \gamma)$ for cohomology classes α, β , and γ
- The cup product is graded commutative: $\alpha \smile \beta = (-1)^{nm}(\beta \smile \alpha)$ for $\alpha \in H^n(X; R)$ and $\beta \in H^m(X; R)$
- This graded commutativity reflects the sign convention in the definition of the cup product and the orientation of cells
- Example: For the real projective plane $\mathbb{R}P^2$, let $\alpha \in H^1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ be the generator. Then $\alpha \smile \alpha \neq 0$, but $\alpha \smile \alpha = -(\alpha \smile \alpha)$, so $2(\alpha \smile \alpha) = 0$, showing that $H^*(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ is not commutative
- The proofs of associativity and graded commutativity rely on the properties of the diagonal map and the cross product in cohomology

Interaction with Other Operations

- The cup product is distributive over addition: $(\alpha + \beta) \smile \gamma = \alpha \smile \gamma + \beta \smile \gamma$ and $\alpha \smile (\beta + \gamma) = \alpha \smile \beta + \alpha \smile \gamma$
- The identity element for the cup product is the cohomology class $1 \in H^0(X; R)$, satisfying $1 \smile \alpha = \alpha \smile 1 = \alpha$ for any cohomology class α
- The identity element corresponds to the constant map $X \rightarrow R$ with value 1
- Example: For any space X , the cup product with the identity element induces isomorphisms $H^n(X; R) \cong H^0(X; R) \otimes H^n(X; R)$ and $H^n(X; R) \cong H^n(X; R) \otimes H^0(X; R)$

Applications of Cup Products

Cohomology Rings and Invariants

- The cup product can be used to compute the cohomology rings of various spaces, such as projective spaces, lens spaces, and Eilenberg-MacLane spaces
- The cohomology ring $H^*(\mathbb{C}P^n; \mathbb{Z})$ of the complex projective space $\mathbb{C}P^n$ is isomorphic to $\mathbb{Z}[x]/(x^{n+1})$, where x is the generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$ and the cup product corresponds to polynomial multiplication

multiplication

- Example: In $\mathbb{C}P^2$, the generator $x \in H^2(\mathbb{C}P^2; \mathbb{Z})$ satisfies $x \smile x = x^2 \neq 0$ but $x \smile x \smile x = x^3 = 0$
- The cup product provides a way to define and compute cohomological invariants of spaces, such as the cohomology ring, Betti numbers, and Poincaré polynomials
- Example: The Poincaré polynomial $P_X(t) = \sum_i \dim H^i(X; \mathbb{R}) t^i$ encodes the dimensions of the cohomology groups and can be used to distinguish spaces up to homotopy equivalence

Obstructions and Non-Existence of Maps

- The cup product can detect the non-existence of certain continuous maps between spaces by comparing their cohomology rings
- If $f: X \rightarrow Y$ is a continuous map and the induced homomorphism $f^*: H^*(Y; \mathbb{R}) \rightarrow H^*(X; \mathbb{R})$ does not preserve the cup product structure, then f cannot be a homotopy equivalence
- Example: There is no continuous map $f: \mathbb{C}P^2 \rightarrow S^4$ that is a homotopy equivalence because $H^*(\mathbb{C}P^2; \mathbb{Z})$ is not isomorphic to $H^*(S^4; \mathbb{Z})$ as rings
- The cup product can be used to define cohomological obstructions to the existence of certain geometric structures, such as complex structures, almost complex structures, and symplectic structures
- Example: The Chern classes $c_i(E) \in H^{2i}(X; \mathbb{Z})$ of a complex vector bundle $E \rightarrow X$ are defined using the cup product and provide obstructions to the existence of nowhere-zero sections and triviality of the bundle

Higher Operations and Duality

- The cup product can be used to define higher cohomology operations, such as Steenrod squares and Massey products, which provide additional tools for studying the topology of spaces
- Steenrod squares $Sq^i: H^n(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{n+i}(X; \mathbb{Z}/2\mathbb{Z})$ are cohomology operations that generalize the cup product and capture additional information about the cohomology of a space with $\mathbb{Z}/2\mathbb{Z}$ coefficients
- Massey products are higher-order cohomology operations that generalize the cup product and provide obstructions to the formality of spaces and the realization of cohomology classes by geometric constructions
- The cup product is a key ingredient in the definition of the cup product pairing between cohomology and homology, which is used in Poincaré duality and the study of manifolds
- For a closed orientable n -dimensional manifold M , the cup product pairing $H^k(M; \mathbb{R}) \otimes H^{n-k}(M; \mathbb{R}) \rightarrow H^n(M; \mathbb{R}) \cong \mathbb{R}$ is non-degenerate and induces isomorphisms $H^k(M; \mathbb{R}) \cong H_{n-k}(M; \mathbb{R})$
- Example: For the torus $T = S^1 \times S^1$, the cup product pairing $H^1(T; \mathbb{Z}) \otimes H^1(T; \mathbb{Z}) \rightarrow H^2(T; \mathbb{Z}) \cong \mathbb{Z}$ is given by $(a, b) \mapsto a \smile b$ and induces the isomorphisms $H^1(T; \mathbb{Z}) \cong H_1(T; \mathbb{Z})$ and $H^0(T; \mathbb{Z}) \cong H_2(T; \mathbb{Z})$

7.2 Cohomology rings

Cohomology rings take cohomology groups to the next level. They use the cup product to multiply cohomology classes, revealing deeper connections between different degrees. This structure helps us see topological features that individual groups might miss.

Cohomology rings are powerful tools for telling spaces apart. Even when two spaces have the same cohomology groups, their rings might be different. This extra info helps us spot subtle differences and understand spaces better than homology alone can.

Cohomology ring of a space

Definition and properties

- The cohomology ring of a topological space X is a graded ring, denoted $H^*(X)$, with the cup product as the multiplication operation
- The cup product is a bilinear map $\smile: H^p(X) \times H^q(X) \rightarrow H^{p+q}(X)$ takes two cohomology classes and produces a new cohomology class in a higher degree
- The cup product is associative, meaning $(a \smile b) \smile c = a \smile (b \smile c)$ for cohomology classes a , b , and c
- The cup product is distributive over addition, meaning $a \smile (b + c) = (a \smile b) + (a \smile c)$ for cohomology classes a , b , and c
- The cohomology ring encodes important topological information about the space X
- Multiplicative structure reveals relationships between cohomology classes in different degrees
- Ring structure can detect higher-order topological features not apparent from individual cohomology groups

Significance and applications

- The cohomology ring is a powerful invariant of a topological space
- Spaces with non-isomorphic cohomology rings cannot be homotopy equivalent
- Cohomology ring can distinguish spaces with isomorphic cohomology groups ($\mathbb{CP}^2$ and $S^2 \times S^2$)
- Cohomology ring computations provide insight into the structure of a space
- Generators and relations of the ring reveal essential topological features
- Ring structure can be used to compute other invariants (characteristic classes, Massey products)
- Applications in algebraic topology and related fields
- Cohomology rings are used to study vector bundles, fiber bundles, and principal G -bundles
- Cohomology rings play a central role in obstruction theory and the classification of manifolds
- Cohomology ring computations are essential in algebraic geometry and representation theory

Cohomology ring computations

Basic spaces and techniques

- The cohomology ring of the n -sphere, $H^*(S^n)$, is isomorphic to $\mathbb{Z}[\alpha]/\langle \alpha^2 \rangle$
- α is a generator of degree n
- $\langle \alpha^2 \rangle$ denotes the ideal generated by α^2 , meaning $\alpha \smile \alpha = 0$
- The cohomology ring of real projective n -space, $H^*(\mathbb{R}P^n)$, is isomorphic to $\mathbb{Z}[\alpha]/\langle 2\alpha, \alpha^{n+1} \rangle$
- α is a generator of degree 1
- Relations $2\alpha = 0$ and $\alpha^{n+1} = 0$ hold in the ring
- The cohomology ring of complex projective n -space, $H^*(\mathbb{C}P^n)$, is isomorphic to $\mathbb{Z}[\alpha]/\langle \alpha^{n+1} \rangle$
- α is a generator of degree 2
- Relation $\alpha^{n+1} = 0$ holds in the ring
- To compute the cohomology ring of a product space $X \times Y$, use the Künneth formula
- $H(X \times Y) \cong H(X) \otimes H(Y)$ as graded rings
- The cup product in $H(X \times Y)$ is determined by $(a \otimes b) \smile (c \otimes d) = (a \smile c) \otimes (b \smile d)$ for cohomology classes a, c in $H(X)$ and b, d in $H(Y)$

Advanced techniques and computations

- Spectral sequences can be used to compute cohomology rings of more complex spaces
- Serre spectral sequence for fibrations relates the cohomology of the base, fiber, and total space
- Eilenberg-Moore spectral sequence computes the cohomology of a pullback or pushout square
- Poincaré duality relates the cup product in cohomology to the intersection product in homology
- For a closed, oriented n -manifold M , Poincaré duality states $H^k(M) \cong H_{n-k}(M)$
- The cup product in cohomology corresponds to the intersection product in homology under this isomorphism
- Cohomology ring computations can be simplified using algebraic techniques
- Graded ring isomorphisms $\mathbb{Z}[\alpha]/\langle \alpha^2 \rangle \cong \mathbb{Z}[\beta]$ with $|\beta| = 2|\alpha|$ can make computations more manageable
- Graded ring invariants (Krull dimension, depth, regularity) provide information about the structure of the ring

Cohomology ring vs homology

Relationship between cohomology and homology

- The cohomology ring $H^*(X)$ is the graded ring obtained from the cohomology groups $H^n(X)$ with the cup product as multiplication

- The homology groups $H_n(X)$ are the duals of the cohomology groups $H^n(X)$
- $H_n(X) \cong \text{Hom}(H^n(X), \mathbb{Z})$, where Hom denotes the group of homomorphisms
- Elements of $H_n(X)$ can be viewed as functions that assign integers to cohomology classes in $H^n(X)$
- The cup product in cohomology is dual to the intersection product in homology
- For an n -dimensional manifold X , the intersection product is a bilinear map $H_p(X) \times H_q(X) \rightarrow H_{p+q-n}(X)$
- The intersection product computes the homology class of the intersection of two submanifolds representing homology classes

Advantages of the cohomology ring

- The cohomology ring has additional structure due to its multiplicative properties
- The cup product provides a way to multiply cohomology classes, while homology lacks a natural multiplication
- The ring structure encodes higher-order relationships between cohomology classes that are not apparent from homology groups
- The cohomology ring is a finer invariant than homology groups
- Spaces with isomorphic homology groups may have non-isomorphic cohomology rings ($\mathbb{CP}^2$ and $S^2 \times S^4$)
- The cohomology ring can detect subtle topological differences that homology alone cannot
- Cohomology rings are more naturally related to other algebraic structures
- Cohomology rings are graded commutative, meaning $a \smile b = (-1)^{|a||b|} b \smile a$ for cohomology classes a and b
- Cohomology rings are often isomorphic to well-known algebraic objects (polynomial rings, exterior algebras)

Distinguishing spaces with cohomology

Non-homotopy equivalent spaces

- If two spaces have non-isomorphic cohomology rings, then they cannot be homotopy equivalent
- Homotopy equivalence implies isomorphic cohomology rings, so non-isomorphic rings prove spaces are not homotopy equivalent
- Cohomology ring isomorphism is a necessary, but not sufficient, condition for homotopy equivalence
- The cohomology ring can detect differences in the multiplicative structure of cohomology classes
- Spaces with isomorphic cohomology groups may have different cup product structures
- Examples include $\mathbb{CP}^2$ and $S^2 \times S^4$, which have isomorphic cohomology groups but non-isomorphic rings

Higher-order cohomology operations

- Massey products are higher-order cohomology operations that can distinguish spaces with isomorphic cohomology rings
- Massey products are defined using the cup product and provide additional structure on the cohomology ring
- Spaces with isomorphic cohomology rings may have different Massey product structures, indicating different homotopy types
- Steenrod squares are cohomology operations that provide additional information beyond the cup product
- Steenrod squares are stable operations that map $H^n(X; \mathbb{Z}/2\mathbb{Z})$ to $H^{n+i}(X; \mathbb{Z}/2\mathbb{Z})$ for $i \geq 0$
- The action of Steenrod squares on the cohomology ring can distinguish spaces with isomorphic rings (real projective spaces)
- Higher-order cohomology operations are an active area of research in algebraic topology
- New operations and structures are being discovered and applied to distinguish spaces and study their properties
- The relationship between higher-order operations and other invariants (homotopy groups, K-theory) is an important topic of investigation

7.3 Poincaré duality

Poincaré duality is a powerful tool in algebraic topology, linking cohomology and homology groups of oriented closed manifolds. It establishes an isomorphism between these groups, allowing us to translate problems and results between them.

This duality provides deep insights into manifold structure and topology. It's crucial for computing cohomology groups, understanding intersection theory, and proving important results in manifold theory. The concept of orientation plays a key role in its application.

Poincaré Duality Theorem

Statement and Isomorphism

- The Poincaré duality theorem states that for an oriented closed manifold M of dimension n , there is an isomorphism between the k -th cohomology group $H^k(M)$ and the $(n - k)$ -th homology group $H_{n-k}(M)$ for all k
- The isomorphism is given by the cap product with a fundamental class $[M]$ in $H_n(M)$, which is a generator of the top homology group
- The cap product $\frown : H^k(M) \times H_n(M) \rightarrow H_{n-k}(M)$ is a bilinear pairing between cohomology and homology classes
- Poincaré duality holds with any coefficient ring, provided that the homology and cohomology groups are taken with the same coefficients (e.g., $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Z}_p$)

Naturality and Generalizations

- The Poincaré duality isomorphism is a natural isomorphism, meaning it is compatible with the induced maps on homology and cohomology by continuous maps between manifolds
- If $f: M \rightarrow N$ is a continuous map between oriented closed manifolds, then the following diagram commutes: $H^k(N) \xrightarrow{f^*} H^k(M); @V\cap V @VV\cap V; H_{(n-k)}(N) \xrightarrow{f_*} H_{(n-k)}(M)$
- The Poincaré duality theorem can be generalized to manifolds with boundary by considering relative homology and cohomology groups
- For an oriented compact manifold M with boundary ∂M , there is an isomorphism $H^k(M, \partial M) \cong H_{n-k}(M)$

Duality Between Cohomology and Homology

Interpretation of the Isomorphism

- Poincaré duality establishes a deep connection between the homology and cohomology groups of an oriented closed manifold
- The isomorphism $H^k(M) \cong H_{n-k}(M)$ relates the k -dimensional cochains (captured by cohomology) to the $(n - k)$ -dimensional chains (captured by homology)
- Intuitively, cohomology measures the "holes" in the manifold, while homology measures the "cycles" around those holes
- The Poincaré duality isomorphism can be understood in terms of the intersection of chains and cochains: the evaluation of a k -dimensional cochain on an $(n - k)$ -dimensional chain gives a scalar value
- This intersection pairing is realized by the cap product with the fundamental class $[M]$

Applications and Consequences

- Poincaré duality allows for the translation of problems and results between homology and cohomology, providing a powerful tool for studying the topology of manifolds
- Many properties of manifolds can be formulated in terms of either homology or cohomology, and Poincaré duality ensures that these formulations are equivalent
- The isomorphism between homology and cohomology has important consequences for the structure of the cohomology ring of a manifold
- The cup product in cohomology corresponds to the intersection product in homology via Poincaré duality, giving rise to the Poincaré intersection pairing
- Poincaré duality is a key ingredient in the proof of the Poincaré conjecture, which states that any simply connected closed 3-manifold is homeomorphic to the 3-sphere

Computing Manifold Cohomology

Using Poincaré Duality

- Poincaré duality can be used to compute the cohomology groups of a manifold from its homology groups, and vice versa

- For an oriented closed manifold M of dimension n , the k -th cohomology group $H^k(M)$ is isomorphic to the $(n - k)$ -th homology group $H_{n-k}(M)$
- For example, if M is a closed oriented surface of genus g , then $H^0(M) \cong H_2(M) \cong \mathbb{Z}$ and $H^1(M) \cong H_1(M) \cong \mathbb{Z}^{2g}$
- When computing the cohomology of a manifold using Poincaré duality, it is essential to consider the orientation of the manifold and the coefficient ring being used

Combining with Other Tools

- Poincaré duality can simplify computations by reducing the problem to computing homology groups, which are often easier to calculate than cohomology groups
- In some cases, Poincaré duality can be combined with other tools, such as the Mayer-Vietoris sequence or the Künneth formula, to compute the cohomology of more complex manifolds
- For example, if M and N are oriented closed manifolds, then the cohomology of their product $M \times N$ can be computed using the Künneth formula and Poincaré duality: $H^k(M \times N) \cong \bigoplus_{i+j=k} H^i(M) \otimes H^j(N)$
- Poincaré duality can also be used in conjunction with the long exact sequence of a pair to compute the relative cohomology groups of a manifold with boundary

Orientation in Poincaré Duality

Concept of Orientation

- Orientation is a crucial concept in the statement and application of Poincaré duality
- An oriented manifold is a manifold together with a consistent choice of orientation for each tangent space, which can be thought of as a "twisting" of the manifold
- For a connected manifold, an orientation is equivalent to a consistent choice of ordered basis for each tangent space
- The orientation of a manifold determines the sign of the fundamental class $[M]$, which is used to define the Poincaré duality isomorphism
- Reversing the orientation of a manifold changes the sign of the fundamental class and the Poincaré duality isomorphism

Role in Poincaré Duality

- Poincaré duality holds only for oriented manifolds; non-orientable manifolds, such as the Klein bottle or the real projective plane, do not satisfy Poincaré duality
- The choice of orientation affects the sign of the Poincaré duality isomorphism: reversing the orientation of a manifold changes the sign of the isomorphism
- This is because the cap product with the fundamental class $[M]$ depends on the orientation of M
- In some cases, the orientation of a manifold can be determined by the orientation of its boundary, as in the case of oriented cobordisms between manifolds
- An oriented cobordism between two oriented closed manifolds M and N is an oriented compact manifold W with boundary $\partial W = M \sqcup (-N)$, where $-N$ denotes N with the opposite orientation

- The orientation of a manifold plays a crucial role in the functoriality of Poincaré duality with respect to continuous maps between manifolds

7.4 The Künneth formula

The Künneth formula is a powerful tool for understanding the cohomology of product spaces. It connects the cohomology of a product to the cohomology of its factors, allowing us to break down complex spaces into simpler components.

This formula is closely tied to the cup product, which we've been studying in this chapter. Together, they provide a comprehensive view of the cohomology ring structure for product spaces, helping us compute and analyze more intricate topological spaces.

Künneth Formula for Cohomology

Statement and Exact Sequence

- The Künneth formula relates the cohomology of a product space to the cohomology of its factors for topological spaces X and Y

• It states that there is a short exact sequence:

$$0 \rightarrow \bigoplus_{p+q=n} H^p(X; R) \otimes H^q(Y; R) \rightarrow H^n(X \times Y; R) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}(H^p(X; R), H^q(Y; R)) \rightarrow 0$$

- R is a principal ideal domain (integers or a field)
- $\otimes$ denotes the tensor product
- The Künneth formula holds for any coefficient ring R that is a principal ideal domain
- The Tor term measures the twisting or torsion between the cohomology groups of X and Y
- It is trivial when either $H^*(X; R)$ or $H^*(Y; R)$ is a free R -module

Naturality and Induced Maps

- The Künneth formula is natural with respect to maps between spaces
- If $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ are continuous maps, then the induced map $(f \times g)^*: H^*(X' \times Y'; R) \rightarrow H^*(X \times Y; R)$ is compatible with the Künneth formula
- This allows studying the behavior of cohomology under maps
- Induced maps in cohomology can be computed using the Künneth formula

Computing Cohomology with Künneth Formula

Procedure and Simplifications

- To compute the cohomology of a product space $X \times Y$ using the Künneth formula: 1. Know the cohomology groups of the individual spaces X and Y 2. Compute the tensor product part $\bigoplus_{p+q=n} H^p(X; R) \otimes H^q(Y; R)$ using known cohomology groups and tensor product properties 3. Determine if the Tor term $\bigoplus_{p+q=n-1} \text{Tor}(H^p(X; R), H^q(Y; R))$ vanishes (often trivial with field coefficients)

- If the Tor term vanishes (cohomology groups are free R -modules), the Künneth formula simplifies to an isomorphism:
$$H^n(X \times Y; R) \cong \bigoplus_{p+q=n} H^p(X; R) \otimes H^q(Y; R)$$
- When the Tor term is non-trivial, compute it using the definition of Tor as the derived functor of the tensor product

Iterative Application and Examples

- The Künneth formula can be applied iteratively to compute the cohomology of a product of more than two spaces
- Example: The n -torus T^n is the product of n circles, so its cohomology ring can be computed using the Künneth formula and the known cohomology of the circle
- Other examples where the Künneth formula is useful:
- Computing the cohomology rings of product manifolds
- Proving results about cohomological dimension and Lusternik-Schnirelmann category of product spaces

Künneth Formula and Cup Product

Cross Product and Ring Structure

- The Künneth formula and the cup product are closely related in the cohomology of product spaces
- The cross product map $\times : H^p(X; R) \otimes H^q(Y; R) \rightarrow H^{p+q}(X \times Y; R)$, defined by $(a \times b)(x, y) = a(x) \cdot b(y)$, relates the tensor product to the cup product structure
- It is natural with respect to maps between spaces
- Satisfies the Leibniz rule: $(a \times b) \smile (a' \times b') = (-1)^{pq'}(a \smile a') \times (b \smile b')$
- When the Tor term vanishes, the cross product induces an isomorphism of graded rings:
$$H^*(X; R) \otimes H^*(Y; R) \cong H^*(X \times Y; R)$$
- The tensor product is equipped with the multiplication
$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{pq'}(a \smile a') \otimes (b \smile b')$$
- In this case, the cup product structure of $H^*(X \times Y; R)$ is completely determined by the cup product structures of $H^*(X; R)$ and $H^*(Y; R)$ via the Künneth isomorphism and the cross product

Relationship and Computational Implications

- The Künneth formula and the cup product are interconnected in the cohomology of product spaces
- Understanding their relationship helps in computing and understanding the ring structure of the cohomology of product spaces
- The cross product provides a way to relate the tensor product in the Künneth formula to the cup product structure
- When the Tor term vanishes, the cup product structure is fully determined by the factors' cup product structures via the Künneth isomorphism and cross product

Applying the Künneth Formula

Problem-Solving Strategies

- The Künneth formula is a powerful tool for computing the cohomology of product spaces in terms of the cohomology of the factors
- When solving problems, consider:
 - The coefficient ring R and whether the Tor term vanishes (simplifies computations)
 - Combining with other tools like the Mayer-Vietoris sequence and spectral sequences for more complicated spaces
 - The naturality of the Künneth formula allows studying the behavior of cohomology under maps and computing induced maps in cohomology

Applications and Examples

- Computing cohomology rings of tori and other product manifolds
- Example: The n -torus T^n cohomology ring via the Künneth formula and the known cohomology of the circle
- Proving general results about cohomology of product spaces
- Cohomological dimension
- Lusternik-Schnirelmann category
- Decomposing complicated spaces into products or fibrations
- Combining the Künneth formula with the Mayer-Vietoris sequence and the spectral sequence of a fibration to compute cohomology
- Studying the behavior of cohomology under maps and computing induced maps in cohomology using the naturality of the Künneth formula

Unit 8 – Fibrations and Homotopy Fiber Sequences

8.1 Fibrations and fiber bundles

Fibrations and fiber bundles are key concepts in algebraic topology. They generalize covering spaces, allowing for more complex fibers and global structures. These tools help us understand how spaces are built from simpler pieces.

The homotopy lifting property of fibrations connects the topology of different spaces. Fiber bundles, with their local product structure, offer insights into global twisting. Both concepts are crucial for studying geometric and topological relationships.

Fibrations and Fiber Bundles

Definition and Key Properties

- A fibration is a continuous surjective map $p : E \rightarrow B$ between topological spaces E and B that satisfies the homotopy lifting property for all spaces
- The homotopy lifting property states that given a homotopy $f : X \times I \rightarrow B$ and a lift $g : X \times \{0\} \rightarrow E$ of $f|_{X \times \{0\}}$, there exists a homotopy $\tilde{f} : X \times I \rightarrow E$ lifting f such that $\tilde{f}|_{X \times \{0\}} = g$
- A fiber bundle is a continuous surjective map $p : E \rightarrow B$ between topological spaces E and B , where each point in B has a neighborhood U such that $p^{-1}(U)$ is homeomorphic to the product space $U \times F$ for some fixed topological space F called the fiber
- The total space E of a fiber bundle locally looks like the product of the base space B and the fiber F , but globally it may have a different structure due to twisting or nontrivial gluing of the local product structures
- Fibrations and fiber bundles generalize the concept of a covering space, allowing for more flexibility in the fibers and the global structure (Möbius strip, Hopf fibration)

Examples and Constructions

- The projection map $p : E \rightarrow B$ from a product space $E = B \times F$ to its first factor B is a trivial fibration and a trivial fiber bundle with fiber F
- The Möbius strip is a nontrivial fiber bundle over the circle S^1 with fiber the unit interval $[0, 1]$, obtained by gluing the ends of $[0, 1] \times [0, 1]$ with a half-twist
- The Hopf fibration is a nontrivial fibration from the 3-sphere S^3 to the 2-sphere S^2 , with each fiber being a circle S^1
- It can be constructed using the complex numbers and quaternions
- The tangent bundle of a smooth manifold M is a vector bundle over M , with each fiber being the tangent space at a point of M
- Principal G -bundles are fiber bundles with fiber a Lie group G , where G acts freely and transitively on each fiber

- They play a crucial role in the theory of gauge fields and characteristic classes

Constructing Fibrations and Fiber Bundles

Local Trivializations and Transition Functions

- Locally, a fiber bundle looks like a product space $U \times F$, where U is an open set in the base space B and F is the fiber
- This local trivialization allows for the study of local properties of the bundle
- The global structure of a fiber bundle is determined by the gluing maps between the local trivializations, called transition functions
- These functions encode how the local product structures are twisted or identified when moving around the base space
- Constructing a fiber bundle involves specifying the local trivializations and the transition functions that glue them together consistently (cocycle condition)

Pullbacks and Associated Bundles

- Given a fibration $p : E \rightarrow B$ and a continuous map $f : X \rightarrow B$, the pullback of p along f is a fibration $f^*p : f^*E \rightarrow X$ over X that captures the idea of "restricting" the fibration p to the space X via the map f
- The pullback is a useful construction for studying the behavior of fibrations under base change and for defining induced bundles
- Associated bundles are fiber bundles constructed from a principal G -bundle $P \rightarrow B$ and a space F with a G -action
- The associated bundle $P \times_G F$ has fiber F and is obtained by taking the quotient of $P \times F$ by the diagonal G -action
- Many important bundles in geometry and physics, such as vector bundles and spinor bundles, can be constructed as associated bundles to a principal bundle

Structure of Fibrations and Fiber Bundles

Homotopy Lifting Property and Long Exact Sequences

- The global structure of a fibration is captured by the homotopy lifting property, which relates the topology of the base space to the topology of the total space and the fibers
- The homotopy lifting property allows for the construction of lifts of homotopies and the study of the homotopy theory of the spaces involved
- A fibration $p : E \rightarrow B$ with fiber F gives rise to a long exact sequence of homotopy groups:

$$\cdots \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \rightarrow \pi_{n-1}(F) \rightarrow \cdots$$
- This sequence relates the homotopy groups of the total space, base space, and fiber, and provides a powerful tool for computing and understanding their relationships

Characteristic Classes and Obstructions

- The global structure of a fiber bundle can be classified by its characteristic classes, which are cohomology classes that measure the nontriviality of the bundle
- Characteristic classes provide obstructions to the existence of certain cross-sections or lifts (Euler class, Chern classes, Pontryagin classes)
- The vanishing of certain characteristic classes is often a necessary condition for a bundle to admit a specific structure or property (parallelizability, existence of a connection with prescribed curvature)
- Characteristic classes can be used to distinguish between different bundles over the same base space and to study the topology of the base space itself (Chern character, Stiefel-Whitney classes)

Fibrations vs Other Topological Structures

Comparison with Covering Spaces and Vector Bundles

- Fibrations and fiber bundles generalize the concept of a covering space, allowing for more general fibers and a more flexible global structure
- Covering spaces are fibrations and fiber bundles with discrete fibers
- Vector bundles are a special case of fiber bundles where the fibers are vector spaces, and the transition functions are linear maps
- They play a fundamental role in differential geometry and topology (tangent bundle, normal bundle)
- The theory of fibrations and fiber bundles provides a unifying framework for studying covering spaces, vector bundles, and other geometric structures

Sheaves and Local-to-Global Principles

- Sheaves are another generalization of fiber bundles, where the fibers are allowed to vary over the base space and are equipped with a suitable topology and algebraic structure
- Sheaves provide a framework for studying local-to-global properties in geometry and analysis (sheaf cohomology, sheaf of sections)
- The local-to-global principles embodied by sheaves and fibrations allow for the study of global properties of a space by understanding its local behavior and how it is glued together
- Sheaf theory and the theory of fibrations and fiber bundles provide powerful tools for understanding the interplay between local and global phenomena in mathematics (Čech cohomology, Leray spectral sequence)

8.2 The long exact sequence of a fibration

Fibrations link spaces through a special map, and their long exact sequence reveals how these spaces relate. This sequence connects homotopy groups, showing how the fiber, total space, and base space interact topologically.

The long exact sequence is a powerful tool for computing homotopy groups and understanding fibrations. It helps uncover hidden relationships between spaces and can be used to detect interesting topological phenomena in various mathematical structures.

Long Exact Sequence of a Fibration

Definition and Properties

- Given a fibration $F \rightarrow E \xrightarrow{p} B$, there exists a long exact sequence of homotopy groups:
 $\cdots \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \rightarrow \pi_{n-1}(F) \rightarrow \cdots \rightarrow \pi_0(E) \rightarrow \pi_0(B)$
- The maps in the long exact sequence are induced by:
 - The inclusion of the fiber F into the total space E
 - The projection map $p : E \rightarrow B$
- Exactness means that at each stage, the image of the incoming map equals the kernel of the outgoing map
- The sequence is infinite in both directions, connecting homotopy groups of all dimensions
- The long exact sequence is natural with respect to morphisms of fibrations

Relationship to Fibration Topology

- The long exact sequence encodes the relationship between the topology of the fiber, total space, and base space of a fibration
- If the base space is contractible, the long exact sequence reveals that the total space has the same homotopy groups as the fiber
- When the fiber is discrete (has trivial homotopy groups above dimension 0), the long exact sequence shows that the total space is a covering space of the base space
- The connecting homomorphism in the long exact sequence measures the obstruction to lifting homotopy classes from the base space to the total space
- The long exact sequence can be used to detect the existence of interesting topological phenomena, such as:
 - Non-trivial bundles
 - Twisted products

Computing Homotopy Groups

Using the Long Exact Sequence

- The long exact sequence can be used to compute unknown homotopy groups of the total space, fiber, or base space when the other two are known
- If two out of three consecutive terms in the sequence are known to be zero, the third term must be isomorphic to the image or kernel of the adjacent maps
- The long exact sequence can be combined with other tools to compute homotopy groups, such as:
 - The Hurewicz theorem
 - The universal coefficients theorem

- In some cases, the long exact sequence may split into shorter exact sequences, simplifying computations

Examples and Applications

- Computing the homotopy groups of the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$
- Using the long exact sequence and the known homotopy groups of S^1 and S^2
- Determining the homotopy groups of the unit tangent bundle of a sphere $S^{n-1} \rightarrow UTB(S^n) \rightarrow S^n$
- Applying the long exact sequence and the homotopy groups of spheres
- Calculating the homotopy groups of loop spaces, such as ΩS^n , using the path-loop fibration
- $\Omega S^n \rightarrow PS^n \rightarrow S^n$, where PS^n is the path space of S^n

Existence and Exactness of the Sequence

Proof Outline

- The proof relies on the homotopy lifting property of fibrations, which allows the construction of lifts of homotopies from the base space to the total space
- The maps in the long exact sequence are defined using the boundary maps in the long exact sequence of homotopy groups for a pair (E, F)
- Exactness at $\pi_n(E)$ is proved by showing that the image of the map from $\pi_n(F)$ equals the kernel of the map to $\pi_n(B)$, using the homotopy lifting property
- Exactness at $\pi_n(B)$ is proved by showing that the image of the map from $\pi_n(E)$ equals the kernel of the connecting homomorphism to $\pi_{n-1}(F)$, using:
 - The homotopy lifting property
 - The definition of the boundary map
- Naturality of the long exact sequence is proved by showing that morphisms of fibrations induce commutative diagrams connecting the corresponding long exact sequences

Key Steps and Techniques

- Constructing lifts of homotopies using the homotopy lifting property
- Defining the maps in the long exact sequence using the boundary maps in the long exact sequence of homotopy groups for a pair
- Proving exactness by showing the equality of images and kernels of adjacent maps
- Utilizing the properties of the boundary map and the homotopy lifting property to establish exactness
- Demonstrating the naturality of the long exact sequence with respect to morphisms of fibrations

8.3 Homotopy fiber sequences

Homotopy fiber sequences are a powerful tool in algebraic topology. They link the fiber, total space, and base space of a fibration, revealing connections between their homotopy groups. This allows us to compute complex homotopy groups using simpler spaces in the sequence.

The long exact sequence of homotopy groups is key to understanding fiber sequences. It relates the homotopy groups of the spaces involved, letting us calculate one in terms of the others. This "zig-zag" method is crucial for unraveling the topology of spaces.

Homotopy Fiber Sequences

Definition and Role

- A homotopy fiber sequence is a sequence of topological spaces and continuous maps that arises from a fibration
- The sequence consists of the fiber, the total space, and the base space of the fibration, along with connecting maps between them
- Homotopy fiber sequences provide a powerful tool for studying the relationship between the homotopy groups of the involved spaces
- They allow for the computation of homotopy groups of a space by relating them to the homotopy groups of simpler spaces in the sequence (long exact sequence of homotopy groups)
- Example: Computing the homotopy groups of a complex projective space using the Hopf fibration
- Homotopy fiber sequences play a crucial role in understanding the structure and properties of topological spaces from a homotopy-theoretic perspective
- They reveal important connections between spaces and their homotopy invariants
- Example: Studying the topology of sphere bundles using homotopy fiber sequences

Long Exact Sequence of Homotopy Groups

- Associated with every homotopy fiber sequence is a long exact sequence of homotopy groups
- The long exact sequence relates the homotopy groups of the fiber, total space, and base space
- It provides a systematic way to compute the homotopy groups of one space in terms of the others
- The connecting homomorphisms in the sequence encode important information about the relationships between the spaces
- The exactness of the sequence means that the image of each map is equal to the kernel of the next map
- This property allows for the "zig-zag" computation of homotopy groups
- Example: Using the long exact sequence to compute the fundamental group of the circle

Constructing Fiber Sequences

Fibrations and Fibers

- Given a fibration $p : E \rightarrow B$, the homotopy fiber sequence is constructed by considering the fiber F of the fibration, which is the preimage of a point in the base space B
- The fiber F captures the "vertical" structure sitting above each point in the base space
- The total space E is obtained by "gluing" these fibers together in a continuous way

- The sequence is written as $F \rightarrow E \rightarrow B$, where the first map is the inclusion of the fiber into the total space, and the second map is the fibration P
- The maps in the sequence are continuous and satisfy certain properties related to homotopy lifting
- Example: The Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$ is a prototypical example of a homotopy fiber sequence

Constructing the Sequence

- The construction of homotopy fiber sequences relies on the lifting properties of fibrations and the ability to relate the homotopy groups of the involved spaces
- The connecting maps in the sequence are obtained by considering the long exact sequence of homotopy groups associated with the fibration
- These maps relate the homotopy groups of the fiber, total space, and base space
- The connecting homomorphisms provide a means to compute one homotopy group in terms of the others
- In some cases, the homotopy fiber sequence can be extended to the left or right by considering the homotopy groups of the base space and the fiber, respectively
- This extension allows for a more comprehensive understanding of the relationships between the spaces
- Example: Extending the Hopf fibration sequence to the left to include the higher homotopy groups of the sphere

Fiber Sequences vs Fibrations

Relationship between Fiber Sequences and Fibrations

- Homotopy fiber sequences arise naturally from fibrations, which are continuous maps satisfying the homotopy lifting property
- Fibrations capture the idea of a "parameterized family" of spaces over a base space
- The homotopy lifting property ensures that paths in the base space can be lifted to paths in the total space
- The fiber of a fibration can be viewed as the "vertical" structure sitting above each point in the base space, and the total space is obtained by "gluing" these fibers together
- The fibers may vary continuously over the base space, forming a fiber bundle
- Example: The Möbius strip as a non-trivial bundle over the circle
- Understanding the interplay between homotopy fiber sequences and fibrations is essential for studying the topology of fiber bundles and other related structures
- Fibrations provide a geometric perspective on homotopy fiber sequences
- Homotopy fiber sequences capture the algebraic and homotopy-theoretic aspects of fibrations

Long Exact Sequence and Fibrations

- The long exact sequence of homotopy groups associated with a fibration provides a powerful tool for studying the relationship between the homotopy groups of the fiber, total space, and base space
- It relates the homotopy groups of the spaces involved in the fibration
- The connecting homomorphisms in the sequence encode important information about the fibration
- The connecting homomorphisms in the long exact sequence relate the homotopy groups of the fiber and the base space, allowing for the computation of one in terms of the other
- This relationship is crucial for understanding the topology of the spaces involved in the fibration
- Example: Using the long exact sequence to study the topology of principal bundles

Applications of Fiber Sequences

Computing Homotopy Groups

- Homotopy fiber sequences can be used to compute the homotopy groups of a space by breaking it down into simpler spaces whose homotopy groups are known or easier to calculate
- The long exact sequence of homotopy groups associated with a homotopy fiber sequence provides a systematic way to relate the homotopy groups of the involved spaces
- By studying the properties of the maps in the homotopy fiber sequence, one can deduce information about the homotopy groups of the spaces and their relationships
- Example: Computing the homotopy groups of a complex projective space using the Hopf fibration
- The Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$ gives rise to a homotopy fiber sequence
- By exploiting the long exact sequence and the known homotopy groups of spheres, one can compute the homotopy groups of complex projective spaces

Proving Theorems and Studying Manifolds

- Homotopy fiber sequences can be used to prove important results in algebraic topology, such as the Hurewicz theorem and the Whitehead theorem
- These theorems relate the homotopy groups of a space to its homology groups and provide criteria for homotopy equivalence
- Homotopy fiber sequences offer a powerful framework for proving such results
- In applications, homotopy fiber sequences can be employed to study the topology of manifolds, classify principal bundles, and investigate the structure of classifying spaces
- Manifolds can be studied by considering their associated fiber bundles and the corresponding homotopy fiber sequences
- Principal bundles can be classified using the homotopy groups of their classifying spaces, which can be analyzed using homotopy fiber sequences
- Example: Studying the topology of loop spaces and their relationship to classifying spaces using homotopy fiber sequences

- The loop space of a topological space has a rich homotopy-theoretic structure
- Homotopy fiber sequences provide a means to relate the topology of a space to its loop space and classifying space

8.4 The Serre spectral sequence

The Serre spectral sequence is a powerful tool in algebraic topology that links the homology of a fibration's base space, fiber, and total space. It's like a mathematical recipe that helps us understand how these spaces fit together.

This spectral sequence starts with the homology of the base and fiber, then uses a series of steps to calculate the total space's homology. It's super useful for figuring out complex spaces by breaking them down into simpler parts.

The Serre spectral sequence

Definition and main properties

- The Serre spectral sequence is a powerful tool in algebraic topology that establishes a relationship between the homology of the base space, fiber, and total space of a fibration
- Given a fibration $F \rightarrow E \rightarrow B$, the Serre spectral sequence consists of a sequence of pages (E_r) , each containing a two-dimensional array of abelian groups $E_{p,q}^r$, where p and q are the indices of the array
- The pages are connected by differentials $d_r : E_{p,q}^r \rightarrow E_{p+r,q-r+1}^r$, satisfying $d_r \circ d_r = 0$, which gives rise to the next page E_{r+1} as the homology of E_r with respect to d_r
- The Serre spectral sequence converges to the homology of the total space E , with the limit term E_∞ being the associated graded of a filtration on $H_*(E)$

Structure of the E_2 page

- The E_2 page of the Serre spectral sequence has terms $E_{p,q}^2 \cong H_p(B; H_q(F))$, where $H_q(F)$ is the local coefficient system on B obtained from the homology of the fiber F
- This means that the E_2 page encodes the homology of the base space with coefficients in the homology of the fiber
- The local coefficient system captures the twisting or variation of the homology of the fiber as it moves along the base space
- The E_2 page provides a starting point for computing the homology of the total space by successively applying the differentials and taking homology

Applications of the Serre spectral sequence

Computing homology and cohomology of fibrations

- To compute the homology or cohomology of the total space E using the Serre spectral sequence:
 1. Identify the E_2 page using the homology of the base B and the homology of the fiber F
 2. Determine the differentials on each page of the spectral sequence using the structure of the fibration and the convergence properties

3. Compute the homology or cohomology groups on each page by taking the kernel of the outgoing differential modulo the image of the incoming differential
 4. Continue this process until the spectral sequence collapses (i.e., when all differentials are zero) and the terms stabilize
- The limit term E_∞ gives the associated graded of the homology or cohomology of the total space E , which can be used to reconstruct the actual homology or cohomology groups
 - In some cases, the spectral sequence may collapse at an early stage, allowing for a complete computation of the homology or cohomology of E

Examples and applications

- The Serre spectral sequence can be applied to compute the homology and cohomology of various fibrations, such as:
- The Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$, which is a principal S^1 -bundle over S^2
- The path space fibration $\Omega X \rightarrow PX \rightarrow X$, where PX is the space of paths in X and ΩX is the loop space of X
- The Borel construction $EG \times_G F \rightarrow BG$ for a G -space F , where EG is a contractible space with a free G -action and BG is the classifying space of G
- The Serre spectral sequence can also be used to derive important results in algebraic topology, such as the cohomology of Eilenberg-MacLane spaces and the Hurewicz theorem relating homotopy groups to homology groups

Structure and convergence of the Serre spectral sequence

Relationship between base, fiber, and total space

- The structure of the Serre spectral sequence reflects the intricate relationship between the homology of the base, fiber, and total space of a fibration
- The E_2 page captures the homology of the base with coefficients in the homology of the fiber, incorporating the twisting or local coefficient system induced by the fibration
- The differentials in the spectral sequence arise from the long exact sequence of the fibration and the multiplicative structure of the spectral sequence, encoding the interactions between the homology of the base and the fiber

Convergence and the associated graded

- Convergence of the Serre spectral sequence means that the limit term E_∞ is the associated graded of a filtration on the homology of the total space E
- The filtration on $H_*(E)$ is induced by the skeletal filtration of the base space B , with the associated graded pieces corresponding to the homology of the fibers over each skeleton
- This filtration provides a decomposition of the homology of the total space in terms of the contributions from the base and the fiber
- The edge homomorphisms of the spectral sequence, which are maps between the homology of the base, the homology of the total space, and the homology of the fiber, provide additional information about their relationships and can be used to extract specific homology groups or maps between them

Serre spectral sequence vs other spectral sequences

Comparison with other spectral sequences

- The Serre spectral sequence is one of several spectral sequences used in algebraic topology to compute homology and cohomology of spaces related by certain constructions, such as fibrations, cofibrations, or group actions
- The Atiyah-Hirzebruch spectral sequence relates the ordinary homology of a space to its generalized homology theories (K-theory, cobordism)
- Its E_2 term is given by $E_{p,q}^2 \cong H_p(X; h_q(pt))$, where $h^*(pt)$ is the coefficient ring of the generalized homology theory
- The Eilenberg-Moore spectral sequence converges to the homology of the total space of a fibration, with the E_2 term given by the tor groups of the homology of the base and the homology of the fiber

Generalizations and variants

- The Leray-Serre spectral sequence generalizes the Serre spectral sequence to fibrations with non-simply connected base spaces
- It involves local coefficient systems and the action of the fundamental group of the base on the homology of the fiber
- Variants of the Serre spectral sequence exist for cohomology, with the E_2 page given by $E_2^{p,q} \cong H^p(B; H^q(F))$, and for fibrations of spectra, which are used in stable homotopy theory
- Comparing the different spectral sequences provides insights into the relationships between various homology and cohomology theories and helps in choosing the most effective tool for computing the invariants of a given space or map

Unit 9 – Vector Bundles & Characteristic Classes

9.1 Vector bundles and classifying spaces

Vector bundles are like fancy spaces that combine a base space with vector spaces. They're crucial in algebraic topology, letting us study complex structures by breaking them into simpler parts. Think of them as mathematical Lego sets for building intricate geometric objects.

Classifying spaces are the ultimate organizers for vector bundles. They help us sort and understand different types of bundles, much like a library catalog for mathematical structures. This concept is key to grasping how vector bundles fit into the bigger picture of topology.

Vector bundles and their properties

Definition and local structure

- A vector bundle is a topological space that locally resembles the product of a base space and a vector space, equipped with a continuous projection map onto the base space
- The fibers of a vector bundle are vector spaces of the same dimension, known as the rank of the bundle
- A vector bundle has a local trivialization, meaning it can be locally described as the product of an open set in the base space and a vector space (Euclidean space)

Morphisms and operations

- Vector bundle morphisms are continuous maps between vector bundles that preserve the vector space structure of the fibers and commute with the projection maps
- Example: A map between the tangent bundles of two smooth manifolds induced by a smooth map between the manifolds
- The Whitney sum of two vector bundles is a new vector bundle whose fibers are the direct sums of the fibers of the original bundles
- Example: The Whitney sum of the tangent bundle and the normal bundle of a submanifold yields the restriction of the tangent bundle of the ambient manifold

Classifying spaces for vector bundles

Grassmannian manifolds

- Classifying spaces are topological spaces that parametrize isomorphism classes of vector bundles over a given base space
- The classifying space for rank- n vector bundles is the Grassmannian manifold $Gr(n, \infty)$, which represents the space of n -dimensional subspaces of an infinite-dimensional vector space
- Example: $Gr(1, \infty)$ is the infinite-dimensional projective space $\mathbb{R}P^\infty$

Universal bundles and pullbacks

- There exists a universal vector bundle over the classifying space, such that any rank- n vector bundle over a base space X is the pullback of the universal bundle along a continuous map from X to the classifying space
- The set of isomorphism classes of rank- n vector bundles over X is in one-to-one correspondence with the set of homotopy classes of maps from X to the classifying space $Gr(n, \infty)$
- Example: The tautological line bundle over $\mathbb{R}P^n$ is the pullback of the universal bundle over $\mathbb{R}P^\infty$ along the inclusion map

Constructing vector bundles with transition functions

Transition functions and cocycle conditions

- Transition functions are a way to construct vector bundles by specifying how the local trivializations are related on the overlaps of their domains
- Given an open cover of the base space and a vector space V , a vector bundle can be constructed by assigning a continuous function from each double overlap of the cover to the general linear group $GL(V)$
- Example: The Möbius band can be constructed using two transition functions on the circle, one of which is the identity and the other is a reflection
- The transition functions must satisfy the cocycle condition on triple overlaps, ensuring the consistency of the vector bundle structure

Isomorphism classes and cohomology

- Isomorphic vector bundles have transition functions that differ by a coboundary, i.e., they can be related by a continuous transformation of the local trivializations
- The set of isomorphism classes of vector bundles over a base space X is in one-to-one correspondence with the first Čech cohomology group of X with coefficients in the sheaf of continuous functions to $GL(n, \mathbb{R})$
- Example: The isomorphism classes of line bundles over a manifold M are classified by $H^1(M; \mathbb{Z}_2)$, as $GL(1, \mathbb{R})$ is homotopy equivalent to $\mathbb{Z}_2$

Vector bundles vs principal bundles

Principal bundles and group actions

- A principal G -bundle is a fiber bundle with a continuous right action of a topological group G that preserves the fibers and acts freely and transitively on each fiber
- Given a vector bundle E with rank n , one can construct the frame bundle $F(E)$, which is a principal $GL(n, \mathbb{R})$ -bundle whose fibers consist of ordered bases for the fibers of E
- Example: The frame bundle of the tangent bundle of a smooth manifold is a principal $GL(n, \mathbb{R})$ -bundle

Associated vector bundles

- The vector bundle E can be recovered from its frame bundle $F(E)$ as the associated bundle $F(E) \times_{GL(n, \mathbb{R})} \mathbb{R}^n$, where $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication
- The classifying space for principal G -bundles, denoted by BG , is related to the classifying space for vector bundles by the fact that $Gr(n, \infty)$ is homotopy equivalent to $BGL(n, \mathbb{R})$
- Example: The classifying space for oriented vector bundles is $BSO(n)$, which is homotopy equivalent to the oriented Grassmannian manifold

Classification of principal bundles

- The set of isomorphism classes of principal G -bundles over a base space X is in one-to-one correspondence with the set of homotopy classes of maps from X to the classifying space BG
- The classification of principal bundles is closely related to the classification of vector bundles, as every vector bundle is associated with a principal bundle
- Example: The isomorphism classes of principal $SO(n)$ -bundles over a manifold M are classified by $H^1(M; \mathbb{Z}_2)$ for $n = 1$ and $H^2(M; \mathbb{Z})$ for $n \geq 3$

9.2 Chern classes and Stiefel-Whitney classes

Chern classes and Stiefel-Whitney classes are key tools for understanding vector bundles. They measure how twisted or non-trivial a bundle is, providing crucial information about its structure and properties.

These characteristic classes have applications in geometry and topology. They help determine when bundles are trivial and provide obstructions to certain structures on manifolds, connecting abstract algebra to concrete geometric insights.

Chern Classes for Complex Bundles

Definition and Properties

- Chern classes are characteristic classes associated with complex vector bundles, providing a way to measure the twisting and non-triviality of the bundle
- The i -th Chern class $c_i(E)$ of a complex vector bundle E is an element of the $2i$ -th cohomology group $H^{2i}(B; \mathbb{Z})$, where B is the base space of the bundle
- The total Chern class $c(E)$ is defined as the formal sum of all Chern classes:
 $c(E) = 1 + c_1(E) + c_2(E) + \dots + c_n(E)$, where n is the rank of the vector bundle
- Chern classes are natural under pullbacks, meaning that if $f: X \rightarrow B$ is a continuous map and E is a complex vector bundle over B , then $f^*(c_i(E)) = c_i(f^*(E))$, where $f^*(E)$ is the pullback bundle
- The first Chern class $c_1(E)$ is also known as the Euler class when E is a complex line bundle (rank 1)

Computation and Applications

- Chern classes can be computed using the splitting principle, which states that every complex vector bundle can be pulled back to a sum of line bundles, and the Chern classes of the original bundle are the elementary symmetric polynomials in the first Chern classes of the line bundles

- The vanishing of all Chern classes is a necessary and sufficient condition for a complex vector bundle to be trivial
- Chern classes provide obstructions to the existence of certain geometric structures on manifolds, such as almost complex structures and complex structures
- The Chern character is a ring homomorphism from the K-theory of a space to its rational cohomology, defined in terms of Chern classes, which provides a connection between K-theory and ordinary cohomology
- Example: The complex projective space $\mathbb{C}P^n$ has a tautological line bundle γ^1 whose first Chern class generates the cohomology ring of $\mathbb{C}P^n$

Properties of Chern Classes

Invariance and Whitney Sum Formula

- Chern classes are invariant under bundle isomorphisms, meaning that isomorphic bundles have the same Chern classes
- The Whitney sum formula relates the Chern classes of a direct sum of vector bundles to the Chern classes of the individual bundles: $c(E \oplus F) = c(E) \smile c(F)$, where $\smile$ denotes the cup product
- Example: If E and F are complex line bundles, then $c_1(E \oplus F) = c_1(E) + c_1(F)$

Relation to Euler Class and Stiefel-Whitney Classes

- The top Chern class $c_n(E)$ of a rank n complex vector bundle E is equal to the Euler class of the underlying oriented real vector bundle of rank $2n$
- For a complex vector bundle E , the mod 2 reduction of the total Chern class $c(E)$ is equal to the total Stiefel-Whitney class of the underlying real vector bundle: $\rho_2(c(E)) = w(E_R)$, where ρ_2 is the mod 2 reduction homomorphism and E_R is the underlying real vector bundle of E
- The odd Stiefel-Whitney classes of the underlying real vector bundle of a complex vector bundle are all zero: $w_{2i+1}(E_R) = 0$ for all i
- The even Stiefel-Whitney classes of the underlying real vector bundle of a complex vector bundle are the mod 2 reductions of the Chern classes: $w_{2i}(E_R) = \rho_2(c_i(E))$ for all i

Stiefel-Whitney Classes for Real Bundles

Definition and Properties

- Stiefel-Whitney classes are characteristic classes associated with real vector bundles, analogous to Chern classes for complex vector bundles
- The i -th Stiefel-Whitney class $w_i(E)$ of a real vector bundle E is an element of the i -th cohomology group $H^i(B; \mathbb{Z}/2\mathbb{Z})$, where B is the base space of the bundle and $\mathbb{Z}/2\mathbb{Z}$ is the field of two elements
- The total Stiefel-Whitney class $w(E)$ is defined as the formal sum of all Stiefel-Whitney classes: $w(E) = 1 + w_1(E) + w_2(E) + \dots + w_n(E)$, where n is the rank of the vector bundle
- Stiefel-Whitney classes are natural under pullbacks, similar to Chern classes
- The first Stiefel-Whitney class $w_1(E)$ classifies the orientability of the vector bundle E : the bundle is orientable if and only if $w_1(E) = 0$

- The top Stiefel-Whitney class $w_n(E)$ of a rank n real vector bundle E is equal to the mod 2 Euler class of the bundle

Computation and Applications

- The Whitney product formula for Stiefel-Whitney classes is analogous to the one for Chern classes: $w(E \oplus F) = w(E) \smile w(F)$, where $\smile$ denotes the cup product
- The vanishing of all Stiefel-Whitney classes is a necessary and sufficient condition for a real vector bundle to be trivial, similar to the case of Chern classes for complex vector bundles
- Stiefel-Whitney classes provide obstructions to the existence of certain geometric structures on manifolds, such as spin structures and pin structures
- Example: The real projective space $\mathbb{R}P^n$ has a tautological line bundle γ_R^1 whose first Stiefel-Whitney class generates the cohomology ring of $\mathbb{R}P^n$

Chern vs Stiefel-Whitney Classes

Relationship between Chern and Stiefel-Whitney Classes

- For a complex vector bundle E , the mod 2 reduction of the total Chern class $c(E)$ is equal to the total Stiefel-Whitney class of the underlying real vector bundle: $\rho_2(c(E)) = w(E_R)$, where ρ_2 is the mod 2 reduction homomorphism and E_R is the underlying real vector bundle of E
- The odd Stiefel-Whitney classes of the underlying real vector bundle of a complex vector bundle are all zero: $w_{2i+1}(E_R) = 0$ for all i
- The even Stiefel-Whitney classes of the underlying real vector bundle of a complex vector bundle are the mod 2 reductions of the Chern classes: $w_{2i}(E_R) = \rho_2(c_i(E))$ for all i

Similarities and Differences

- Both Chern and Stiefel-Whitney classes are characteristic classes associated with vector bundles, providing a way to measure the twisting and non-triviality of the bundle
- Chern classes are defined for complex vector bundles, while Stiefel-Whitney classes are defined for real vector bundles
- Chern classes take values in integer cohomology, while Stiefel-Whitney classes take values in mod 2 cohomology
- The vanishing of all Chern or Stiefel-Whitney classes is a necessary and sufficient condition for a vector bundle to be trivial
- Both Chern and Stiefel-Whitney classes satisfy naturality under pullbacks and have similar Whitney sum formulas
- Example: The tangent bundle of the complex projective space $\mathbb{C}P^n$ has non-zero Chern classes, while the tangent bundle of the real projective space $\mathbb{R}P^n$ has non-zero Stiefel-Whitney classes

9.3 The Thom isomorphism theorem

The Thom isomorphism theorem is a key result in vector bundle theory. It connects the cohomology of a vector bundle's base space to the cohomology of its Thom space, providing a powerful tool for calculations.

This theorem has wide-ranging applications in algebraic topology. It's crucial for understanding characteristic classes, cobordism theory, and manifold topology, bridging the gap between vector bundles and cohomology computations.

Thom Isomorphism Theorem

Statement and Isomorphism

- The Thom isomorphism theorem asserts that for an oriented real vector bundle E over a compact base space B , there exists an isomorphism between the cohomology of the base space B and the cohomology of the Thom space of E , shifted by the rank of the bundle
- Precisely, if E is an oriented real vector bundle of rank k over a compact base space B , then there is an isomorphism: $H^i(B; R) \cong H^{i+k}(Th(E); R)$ for all i , where $Th(E)$ represents the Thom space of E and R is the coefficient ring
- For instance, if E is a rank 2 bundle over a sphere S^2 , then $H^i(S^2; R) \cong H^{i+2}(Th(E); R)$ for all i
- The isomorphism is given by the cup product with the Thom class of the vector bundle E , which is a cohomology class in $H^k(Th(E); R)$

Significance and Applications

- The Thom isomorphism theorem serves as a powerful tool in algebraic topology, establishing a relationship between the cohomology of the base space and the cohomology of the Thom space, which often possesses a simpler structure
- It enables the computation of cohomology groups of Thom spaces by leveraging the knowledge of the cohomology groups of the base space
- For example, if the cohomology groups of a base manifold are known, the Thom isomorphism can be employed to determine the cohomology groups of the associated Thom space
- The theorem finds applications in various areas of topology, such as characteristic classes, cobordism theory, and the study of manifolds

Thom Spaces and Classes

Thom Space Construction

- The Thom space $Th(E)$ of a vector bundle E over a base space B is obtained by collapsing the complement of the zero section in the total space of E to a single point
- Intuitively, the Thom space is formed by adding a "point at infinity" to each fiber of the vector bundle, effectively compactifying the fibers
- For a line bundle over a circle, the Thom space resembles a torus with a disk attached at each point of the circle
- The construction of the Thom space provides a way to study the topology of vector bundles by considering the cohomology of the associated Thom space

Thom Class and its Properties

- The Thom class is a cohomology class $u \in H^k(Th(E); R)$, where k is the rank of the vector bundle E and R is the coefficient ring

- It is characterized by the property that its restriction to each fiber of the vector bundle is a generator of the cohomology of the fiber (which is isomorphic to R)
- In other words, the Thom class evaluates to a generator of R on each fiber of the bundle
- For an oriented vector bundle, the Thom class is unique and generates the cohomology of the Thom space as a free module over the cohomology of the base space
- The Thom class plays a crucial role in the statement and proof of the Thom isomorphism theorem, acting as the bridge between the cohomology of the base space and the cohomology of the Thom space

Cohomology Computations with Thom Isomorphism

Computing Cohomology of Thom Spaces

- The Thom isomorphism allows for the computation of the cohomology groups of the Thom space $Th(E)$ in terms of the cohomology groups of the base space B
- Given the cohomology groups of the base space B , one can apply the Thom isomorphism to determine the cohomology groups of the Thom space by shifting the degrees by the rank of the vector bundle
- For instance, if E is an oriented rank k bundle over B , and the cohomology groups of B are known, then $H^{i+k}(Th(E); R) \cong H^i(B; R)$ for all i
- This computational technique simplifies the process of determining the cohomology of Thom spaces, as it reduces the problem to understanding the cohomology of the base space

Integration with Other Tools

- In practice, the Thom isomorphism is frequently used in combination with other tools, such as the Serre spectral sequence or the Gysin sequence, to compute the cohomology of more complex spaces
- The Serre spectral sequence is a powerful tool for computing the cohomology of fiber bundles, and it can be applied in conjunction with the Thom isomorphism to study the cohomology of Thom spaces associated with vector bundles
- For example, the Serre spectral sequence can be used to relate the cohomology of the base space, fiber, and total space of a fiber bundle, and the Thom isomorphism can then be applied to compute the cohomology of the associated Thom space
- The Gysin sequence, on the other hand, is a long exact sequence that relates the cohomology of a manifold to the cohomology of a submanifold and its normal bundle, and it can be used in tandem with the Thom isomorphism to study the cohomology of submanifolds and their neighborhoods

Thom Isomorphism vs Poincaré Duality

Poincaré Duality

- Poincaré duality is a fundamental result in algebraic topology that establishes a relationship between the homology and cohomology of a closed, oriented manifold
- For a closed, oriented manifold M of dimension n , Poincaré duality states that there is an isomorphism between the homology and cohomology groups: $H_i(M; R) \cong H^{n-i}(M; R)$ for all i , where R is the coefficient ring

- For example, for a closed, oriented surface of genus g , Poincaré duality implies that $H_1(M; \mathbb{R}) \cong H^1(M; \mathbb{R})$
- Poincaré duality provides a deep connection between the homology and cohomology of a manifold, allowing for the computation of one from the other

Relationship between Thom Isomorphism and Poincaré Duality

- The Thom isomorphism theorem can be viewed as a generalization of Poincaré duality to the setting of vector bundles
- In the case of the tangent bundle of a closed, oriented manifold M , the Thom space of the tangent bundle is homotopy equivalent to the suspension of M , and the Thom class corresponds to the fundamental class of M under this equivalence
- The suspension of a manifold M is obtained by taking the product of M with an interval and collapsing the ends to points
- By applying the Thom isomorphism to the tangent bundle of a closed, oriented manifold, one can recover the Poincaré duality isomorphism between the homology and cohomology of the manifold
- The Thom isomorphism and Poincaré duality are closely related and provide a powerful framework for studying the topology of manifolds and vector bundles
- While Poincaré duality relates the homology and cohomology of a manifold, the Thom isomorphism extends this relationship to the setting of vector bundles and their associated Thom spaces

9.4 Applications to manifolds

Characteristic classes are powerful tools in algebraic topology, helping us understand vector bundles over manifolds. They provide crucial information about a manifold's topology, including orientability and the existence of non-vanishing vector fields.

This section explores applications of characteristic classes to manifolds, focusing on Stiefel-Whitney, Chern, and Pontryagin classes. We'll see how these classes relate to important theorems like the hairy ball theorem and Poincaré-Hopf theorem.

Characteristic Classes for Manifold Topology

Cohomology Classes and Vector Bundles

- Characteristic classes are cohomology classes associated to vector bundles over a manifold that provide information about the topology of the manifold
- The Stiefel-Whitney classes are characteristic classes associated to real vector bundles
- Stiefel-Whitney classes are elements of the cohomology ring with $\mathbb{Z}/2\mathbb{Z}$ coefficients
- The i -th Stiefel-Whitney class $w_i(E)$ of a real vector bundle E is an element of $H^i(M; \mathbb{Z}/2\mathbb{Z})$, where M is the base manifold
- The Chern classes are characteristic classes associated to complex vector bundles
- Chern classes are elements of the cohomology ring with integer coefficients
- The i -th Chern class $c_i(E)$ of a complex vector bundle E is an element of $H^{2i}(M; \mathbb{Z})$, where M is the base manifold

- The Pontryagin classes are characteristic classes associated to real vector bundles that are derived from the Chern classes of the complexification of the bundle
- Pontryagin classes are elements of the cohomology ring with integer coefficients
- The i -th Pontryagin class $p_i(E)$ of a real vector bundle E is an element of $H^{4i}(M; \mathbb{Z})$, where M is the base manifold

Euler Class and Orientability

- The Euler class is a characteristic class associated to oriented real vector bundles that measures the twisting of the bundle
- The Euler class $e(E)$ of an oriented real vector bundle E is an element of the top cohomology group $H^n(M; \mathbb{Z})$, where n is the dimension of the base manifold M
- The Euler class is related to the obstruction to the existence of a non-vanishing section of the vector bundle
- The top Stiefel-Whitney class of the tangent bundle of a closed manifold vanishes if and only if the manifold is orientable
- A manifold is orientable if its tangent bundle admits a consistent choice of orientation
- The vanishing of the top Stiefel-Whitney class $w_n(TM)$ is equivalent to the existence of an orientation on the manifold M
- The Euler characteristic of a closed manifold can be computed as the integral of the Euler class of its tangent bundle
- The Euler characteristic $\chi(M)$ of a closed manifold M is a topological invariant that measures the alternating sum of the dimensions of the cohomology groups
- The Gauss-Bonnet theorem states that $\chi(M) = \int_M e(TM)$, where $e(TM)$ is the Euler class of the tangent bundle of M

Vector Fields on Spheres

Hairy Ball Theorem

- The hairy ball theorem states that there is no non-vanishing continuous tangent vector field on an even-dimensional sphere
- A non-vanishing vector field is a continuous assignment of a non-zero tangent vector at each point of the manifold
- The theorem implies that it is impossible to comb the hair on a spherical object (like a tennis ball) without creating a cowlick or bald spot
- The proof of the hairy ball theorem relies on the Euler characteristic and the Poincaré-Hopf theorem
- The Poincaré-Hopf theorem relates the Euler characteristic to the sum of the indices of the zeros of a vector field
- The index of a zero of a vector field measures the local behavior of the vector field around the zero (winding number)

- The Euler characteristic of an even-dimensional sphere is 2, while the Euler characteristic of an odd-dimensional sphere is 0
- The Euler characteristic of the n -sphere S^n is given by $\chi(S^n) = 1 + (-1)^n$
- For even n , $\chi(S^n) = 2$, and for odd n , $\chi(S^n) = 0$

Poincaré-Hopf Theorem and Applications

- The Poincaré-Hopf theorem implies that any continuous tangent vector field on an even-dimensional sphere must have at least one zero
- If a non-vanishing vector field existed, the sum of the indices of its zeros would be zero, contradicting the Euler characteristic
- The existence of a zero of the vector field proves the hairy ball theorem
- The Poincaré-Hopf theorem has various applications in topology and geometry
- It can be used to prove the fundamental theorem of algebra (every non-constant polynomial has a root)
- It is related to the study of singularities of vector fields and the Morse theory of functions on manifolds

The Euler Class and its Applications

Definition and Properties

- The Euler class is a characteristic class associated to oriented real vector bundles that measures the twisting of the bundle
- The Euler class $e(E)$ of an oriented real vector bundle E is an element of the top cohomology group $H^n(M; \mathbb{Z})$, where n is the dimension of the base manifold M
- The Euler class is an obstruction to the existence of a non-vanishing section of the vector bundle
- If a vector bundle admits a non-vanishing section, then its Euler class vanishes
- The Euler class of an oriented real vector bundle is an element of the top cohomology group of the base space with integer coefficients
- The Euler class is a cohomology class with integer coefficients, unlike the Stiefel-Whitney classes which have $\mathbb{Z}/2\mathbb{Z}$ coefficients
- The Euler class captures more refined information about the twisting of the bundle compared to the Stiefel-Whitney classes

Relation to Characteristic Numbers

- The Euler class of the tangent bundle of an oriented closed manifold is equal to the Euler characteristic of the manifold times the generator of the top cohomology group
- For an oriented closed n -manifold M , the Euler class of the tangent bundle $e(TM)$ satisfies $e(TM) = \chi(M) \cdot \mu$, where μ is the generator of $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$
- This relation connects the Euler class to the Euler characteristic, a fundamental topological invariant
- The Euler class can be used to prove the Poincaré-Hopf theorem and the hairy ball theorem

- The vanishing of the Euler class of the tangent bundle of an even-dimensional sphere implies the hairy ball theorem
- The Poincaré-Hopf theorem can be derived from the properties of the Euler class and the Gauss-Bonnet theorem

Characteristic Classes vs Obstructions

Obstructions and Geometric Structures

- Characteristic classes provide information about the obstructions to the existence of certain geometric structures on a manifold
- Geometric structures include orientations, non-vanishing vector fields, almost complex structures, symplectic structures, etc.
- The vanishing of certain characteristic classes is a necessary condition for the existence of these structures
- The vanishing of certain characteristic classes is a necessary condition for the existence of certain geometric structures
- If a characteristic class (like the Euler class or the top Stiefel-Whitney class) does not vanish, it obstructs the existence of the corresponding geometric structure
- However, the vanishing of a characteristic class is not always sufficient for the existence of the structure, as there may be higher obstructions

Examples of Obstructions

- The Euler class is related to the obstruction to the existence of a non-vanishing section of an oriented real vector bundle
- If the Euler class of a vector bundle vanishes, it is a necessary condition for the existence of a non-vanishing section
- The vanishing of the Euler class is not always sufficient, as there may be higher obstructions (like the higher Stiefel-Whitney classes)
- The top Stiefel-Whitney class is related to the obstruction to the existence of an orientation on a manifold
- A manifold is orientable if and only if its top Stiefel-Whitney class vanishes
- The vanishing of the top Stiefel-Whitney class is both necessary and sufficient for orientability
- The Pontryagin classes are related to the obstructions to the existence of certain geometric structures on a manifold
- The vanishing of certain Pontryagin classes is necessary for the existence of almost complex structures or symplectic structures
- The Pontryagin classes provide information about the topology of the moduli space of geometric structures on a manifold

Unit 10 – Homological Algebra

10.1 Chain complexes and homology

Chain complexes are the backbone of homological algebra. They're sequences of abelian groups connected by boundary maps, with the key property that composing two consecutive maps gives zero. This structure allows us to define and study cycles, boundaries, and homology groups.

Homology groups measure the "holes" in chain complexes by comparing cycles and boundaries. They're powerful tools for understanding topological spaces and algebraic structures. Chain maps and induced homology homomorphisms help us relate different complexes and their homologies, forming the basis for more advanced techniques in algebraic topology.

Chain complexes and properties

Definition and structure

- A chain complex is an algebraic structure consisting of a sequence of abelian groups or modules (C_n) connected by homomorphisms called boundary maps or differentials ($\partial_n: C_n \rightarrow C_{n-1}$)
- The boundary maps satisfy the fundamental property of chain complexes, also known as the "square zero" property: the composition of any two consecutive maps is zero ($\partial_{n-1} \circ \partial_n = 0$ for all n)
- Elements in the kernel of ∂_n are called n -cycles ($\ker(\partial_n)$), while elements in the image of ∂_{n+1} are called n -boundaries ($\text{im}(\partial_{n+1})$)
- The "square zero" property implies that every n -boundary is also an n -cycle, as $\partial_n(\partial_{n+1}(x)) = 0$ for any x in C_{n+1}

Exactness and triviality

- A chain complex is exact at C_n if the n -cycles are precisely the n -boundaries ($\ker(\partial_n) = \text{im}(\partial_{n+1})$)
- In other words, every n -cycle is the boundary of some $(n+1)$ -chain
- A chain complex is called exact if it is exact at every position (C_n) in the sequence
- An acyclic or exact sequence is a chain complex whose homology groups are all trivial ($H_n(C) = 0$ for all n)
- This means that every n -cycle is an n -boundary, and there are no non-trivial "holes" or "obstructions" in the complex

Computing homology groups

Definition and process

- The n -th homology group of a chain complex C , denoted $H_n(C)$, is the quotient group of n -cycles modulo n -boundaries ($\ker(\partial_n) / \text{im}(\partial_{n+1})$)
- To compute homology groups:
 1. Determine the kernel of each boundary map ∂_n (elements x in C_n such that $\partial_n(x) = 0$)

2. Determine the image of each boundary map ∂_{n+1} (elements y in C_n such that $y = \partial_{n+1}(z)$ for some z in C_{n+1})
3. Compute the quotient group $\ker(\partial_n) / \text{im}(\partial_{n+1})$ to obtain the homology group $H_n(C)$

Interpretation and properties

- Homology groups measure the "holes" or "obstructions" in the chain complex at each dimension
- Non-trivial homology groups indicate the presence of n -cycles that are not n -boundaries
- Homology groups are invariants of the chain complex, preserved under chain homotopy equivalences
- The rank of the n -th homology group, called the n -th Betti number, counts the number of independent "holes" in the complex at the n -th dimension
- For example, the 0th Betti number counts the number of connected components, while the 1st Betti number counts the number of independent loops or tunnels

Chain maps vs homology homomorphisms

Chain maps

- A chain map between chain complexes C and D is a family of homomorphisms $f_n: C_n \rightarrow D_n$ that commute with the boundary maps ($f_{n-1} \circ \partial_n = \partial_n \circ f_n$ for all n)
- Chain maps preserve the structure of chain complexes, acting as morphisms in the category of chain complexes
- Chain maps preserve cycles and boundaries:
- If x is an n -cycle in C , then $f_n(x)$ is an n -cycle in D
- If x is an n -boundary in C , then $f_n(x)$ is an n -boundary in D

Induced homology homomorphisms

- A chain map $f: C \rightarrow D$ induces homomorphisms on homology, $H(f)_n$ or f_n^* , between the corresponding homology groups ($H(f)_n: H_n(C) \rightarrow H_n(D)$ for each n)
- Induced homomorphisms are defined by $H(f)_n([x]) = [f_n(x)]$, where $[x]$ denotes the homology class of an n -cycle x in $H_n(C)$
- Induced homomorphisms are well-defined due to chain maps preserving cycles and boundaries
- Induced homomorphisms are functorial:
- Composition: If $f: C \rightarrow D$ and $g: D \rightarrow E$ are chain maps, then $H(g \circ f)_n = H(g)_n \circ H(f)_n$ for all n
- Identity: If $\text{id}_C: C \rightarrow C$ is the identity chain map, then $H(\text{id}_C)_n: H_n(C) \rightarrow H_n(C)$ is the identity homomorphism for all n

Fundamental Theorem of Homological Algebra

Statement and implications

- The Fundamental Theorem states that a short exact sequence of chain complexes $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ induces a long exact sequence in homology: $\dots \rightarrow H_n(A) \rightarrow H_n(B) \rightarrow H_n(C) \rightarrow H_{n-1}(A) \rightarrow \dots \rightarrow H_0(C) \rightarrow 0$

- The maps in the long exact sequence are the induced homomorphisms on homology arising from the chain maps in the short exact sequence
- The connecting homomorphism $H_n(C) \rightarrow H_{n-1}(A)$, defined by the "snake lemma" construction, measures the obstruction to lifting n -cycles in C to n -cycles in B
- The Fundamental Theorem allows computing homology groups by relating them to simpler or known chain complexes via short exact sequences

Applications and importance

- The Fundamental Theorem is used to prove:
 - The long exact sequence of a pair (X, A) in singular homology
 - The excision theorem in singular homology
 - The computation of the homology of a quotient complex C/A in terms of the homology of C and A
 - The Fundamental Theorem underlies the construction of spectral sequences, advanced algebraic tools for computing homology groups
- Mastery of the Fundamental Theorem is crucial for understanding and applying advanced homological techniques in algebraic topology and other areas of mathematics, such as:
 - Derived functors and derived categories
 - Grothendieck spectral sequences
 - Cohomology theories and their properties

10.2 Exact sequences and the Snake Lemma

Exact sequences and the Snake Lemma are key tools in homological algebra. They help us understand relationships between different mathematical objects by connecting their homology groups. These concepts are crucial for analyzing complex structures and solving problems in algebraic topology.

The Snake Lemma, in particular, allows us to create long exact sequences from short ones. This technique is super useful for computing homology groups and making connections between seemingly unrelated mathematical objects. It's like a secret weapon for tackling tricky algebraic problems!

Exact Sequences

Short and Long Exact Sequences

- A short exact sequence consists of three objects and two morphisms in an abelian category
- The image of the first morphism equals the kernel of the second morphism
- The first morphism is injective while the second morphism is surjective
- A long exact sequence is an infinite exact sequence of objects and morphisms in an abelian category
- Often arises from short exact sequences of chain complexes
- Connects homology groups of different degrees
- The connecting homomorphism in a long exact sequence measures the failure of exactness in the corresponding short exact sequence of chain complexes

- Defined using a diagram chase
- Plays a crucial role in relating homology groups of different chain complexes

Splitting Lemmas and Applications

- Splitting lemmas provide conditions under which a short exact sequence splits
- A split exact sequence is isomorphic to a direct sum of the first and third objects
- Splitting occurs when there exists a section (right inverse) or a retraction (left inverse) of one of the morphisms
- The long exact sequence in homology relates the homology groups of a chain complex to those of a subcomplex and the corresponding quotient complex
- Constructed using the Snake Lemma or the Zigzag Lemma
- Allows for computation of homology groups in terms of simpler chain complexes (Mayer-Vietoris sequence)

Snake Lemma for Homology

Statement and Diagram Chase

- The Snake Lemma studies the relationships between homology groups in a commutative diagram of abelian groups or modules with exact rows
- Constructs a long exact sequence involving the kernels and cokernels of the vertical maps
- The connecting homomorphism is defined using a diagram chase
- The Snake Lemma is named after the zigzag pattern formed by the kernels, images, and cokernels in the diagram
- The "snake" connects the objects in the top row to those in the bottom row
- Provides a way to relate the homology groups of different chain complexes

Applications to Long Exact Sequences

- The Snake Lemma can be used to derive the long exact sequence in homology associated with a short exact sequence of chain complexes
- The short exact sequence of chain complexes induces short exact sequences of homology groups
- The Snake Lemma connects these short exact sequences into a long exact sequence
- The long exact sequence in homology is a powerful tool for computing homology groups
- Relates the homology of a chain complex to the homology of simpler chain complexes (subcomplex and quotient complex)
- Allows for inductive arguments and the comparison of homology groups

Five Lemma for Exactness

Statement and Commutative Diagrams

- The Five Lemma is a result about commutative diagrams in an abelian category
- If certain maps are isomorphisms and the rows are exact, then the remaining map is also an isomorphism
- Provides a way to prove that a certain sequence is exact by comparing it to a known exact sequence
- The Five Lemma is often used in conjunction with other diagram lemmas (Snake Lemma, Short Five Lemma)
- Helps to establish the exactness of sequences in homological algebra
- Can be used to prove the functoriality of long exact sequences

Variants and Generalizations

- Variants of the Five Lemma handle similar situations with different numbers of objects and morphisms
- The Four Lemma deals with four objects and three morphisms
- The Six Lemma deals with six objects and five morphisms
- The Five Lemma can be generalized to longer exact sequences and more complex commutative diagrams
- The Nine Lemma and the 3×3 Lemma are examples of such generalizations
- These generalizations are useful in the study of spectral sequences and derived categories

Exact Sequences and Homological Algebra

Central Role of Exact Sequences

- Homological algebra studies sequences of abelian groups or modules connected by homomorphisms
- Focuses on exact sequences and derived functors
- Provides a framework for studying the structure of mathematical objects
- Exact sequences, particularly long exact sequences, encode important structural information about the objects involved
- Relate the homology or cohomology groups of different objects
- Allow for the computation of invariants using simpler objects

Tools and Techniques

- The Snake Lemma and the Five Lemma are key tools in homological algebra
- Used to work with exact sequences and prove exactness
- Enable the construction and study of long exact sequences
- Derived functors, such as Ext and Tor , measure the failure of a functor to be exact

- Can be studied using long exact sequences
- Provide additional invariants and structural information
- Spectral sequences are collections of exact sequences related by differentials
- Provide a powerful computational tool in homological algebra
- Enable the calculation of homology and cohomology groups in complex situations (Serre spectral sequence, Adams spectral sequence)

10.3 Ext and Tor functors

Ext and Tor functors are key tools in homological algebra, measuring how certain operations fail to be exact. They help us understand module structures and connect various concepts in algebraic topology.

These functors are crucial for computing homology and cohomology groups. They appear in the universal coefficient theorem, linking homology with coefficients to regular homology, and play a role in spectral sequences.

Ext and Tor Functors

Definition and Properties

- Ext and Tor functors measure the failure of exactness of certain functors in homological algebra
- Ext derives from the Hom functor and measures the extent to which the Hom functor fails to be exact
- Tor derives from the tensor product functor and measures the extent to which the tensor product functor fails to be exact
- Projective resolutions define Ext, while injective resolutions define Tor
- Ext and Tor operate as bifunctors, meaning they are functors in two variables (the first and second arguments)
- Example: For modules A and B , $\text{Ext}(A, B)$ and $\text{Tor}(A, B)$ are bifunctors
- The Ext functor is contravariant in the first argument and covariant in the second argument
- Example: Given a morphism $f: A \rightarrow A'$, $\text{Ext}(f, B): \text{Ext}(A', B) \rightarrow \text{Ext}(A, B)$ is a contravariant functor
- The Tor functor is covariant in both arguments
- Example: Given morphisms $f: A \rightarrow A'$ and $g: B \rightarrow B'$, $\text{Tor}(f, g): \text{Tor}(A, B) \rightarrow \text{Tor}(A', B')$ is a covariant functor

Importance in Homological Algebra

- Ext and Tor functors play a crucial role in understanding the structure of modules and chain complexes
- They provide a way to measure the deviation from exactness in certain settings
- Example: If the Hom functor is not exact, Ext groups will be non-zero
- Ext and Tor functors connect various concepts in homological algebra, such as projective and injective resolutions, derived categories, and spectral sequences

- Understanding Ext and Tor is essential for studying more advanced topics in algebraic topology and algebraic geometry

Computing Ext and Tor Groups

Computation using Resolutions

- Finding projective or injective resolutions of the modules involved is the first step in computing Ext and Tor groups
- For Ext, start with a projective resolution of the first module, apply the contravariant Hom functor with the second module, and take the homology of the resulting complex
- Example: Given modules A and B, find a projective resolution $P_\bullet$ of A, compute $\text{Hom}(P_\bullet, B)$, and take the homology to obtain $\text{Ext}(A, B)$
- For Tor, start with a projective resolution of one of the modules, tensor it with the other module, and take the homology of the resulting complex
- Example: Given modules A and B, find a projective resolution $P_\bullet$ of A, compute $P_\bullet \otimes B$, and take the homology to obtain $\text{Tor}(A, B)$

Special Cases

- In the case of abelian groups (i.e., $\mathbb{Z}$ -modules), projective resolutions can be constructed using free abelian groups
- Example: Every abelian group has a free resolution, which can be used to compute Ext and Tor
- For finitely generated modules over a principal ideal domain (PID), projective and injective resolutions can be constructed using the structure theorem
- Example: Over a PID, every finitely generated module is the direct sum of cyclic modules, which admits a simple projective resolution
- In some cases, Ext and Tor groups can be computed directly using their definitions and properties, without explicitly finding resolutions
- Example: For a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the long exact sequence of Ext groups can be used to compute $\text{Ext}(A, -)$ and $\text{Ext}(C, -)$ in terms of $\text{Ext}(B, -)$

Ext, Tor, and Homology

Relationship to Homology and Cohomology

- Ext and Tor functors are closely related to homology and cohomology in algebraic topology
- For a chain complex C and an abelian group G, the homology groups of C with coefficients in G can be computed as $H_n(C; G) = \text{Tor}_n(\mathbb{Z}, G; C)$, where $\mathbb{Z}$ is the trivial $\mathbb{Z}$ -module
- Example: The singular homology groups of a topological space X with coefficients in G can be computed as $H_n(X; G) = \text{Tor}_n(\mathbb{Z}, G; C_\bullet(X))$, where $C_\bullet(X)$ is the singular chain complex of X
- Similarly, the cohomology groups of C with coefficients in G can be computed as $H^n(C; G) = \text{Ext}^n(\mathbb{Z}, G; C)$

- Example: The singular cohomology groups of a topological space X with coefficients in G can be computed as $H^n(X; G) = \text{Ext}^n(\mathbb{Z}, G; C_\bullet(X))$

Applications in Algebraic Topology

- Ext and Tor functors provide a powerful tool for computing homology and cohomology groups in various settings
- In the context of topological spaces, the singular homology and cohomology groups can be computed using Tor and Ext, respectively, applied to the singular chain complex
- Example: For a CW complex X , the cellular homology groups can be computed using Tor applied to the cellular chain complex
- Ext and Tor also appear in the construction of spectral sequences, which are important computational tools in algebraic topology
- Example: The Serre spectral sequence, which relates the homology of a fiber bundle to the homology of its base and fiber, is constructed using Ext and Tor

Universal Coefficient Theorem for Homology and Cohomology

Statement of the Theorem

- The universal coefficient theorem relates homology and cohomology groups of a chain complex C with coefficients in an abelian group G
- For homology, the universal coefficient theorem states that there is a short exact sequence:

$$0 \rightarrow H_n(C) \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}(H_{n-1}(C), G) \rightarrow 0$$
- For cohomology, the universal coefficient theorem states that there is a split short exact sequence:

$$0 \rightarrow \text{Ext}(H_{n-1}(C), G) \rightarrow H^n(C; G) \rightarrow \text{Hom}(H_n(C), G) \rightarrow 0$$

Applications and Consequences

- The universal coefficient theorems allow for the computation of homology and cohomology groups with coefficients in G , given the homology groups of C and the Tor or Ext groups
- Example: If the homology groups $H_n(C)$ are known, and G is a simple abelian group (e.g., $\mathbb{Z}$ or a field), the universal coefficient theorem can be used to quickly compute $H_n(C; G)$ and $H^n(C; G)$
- In the case where G is a field (e.g., $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for a prime p), the Tor and Ext terms vanish, simplifying the universal coefficient theorem
- Example: If G is a field, then $H_n(C; G) \cong H_n(C) \otimes G$ and $H^n(C; G) \cong \text{Hom}(H_n(C), G)$
- The universal coefficient theorem is particularly useful when working with CW complexes, as the homology groups of a CW complex are often easier to compute than its cohomology groups
- Example: For a CW complex X , the universal coefficient theorem can be used to compute the cohomology groups $H^n(X; G)$ from the homology groups $H_n(X)$
- The universal coefficient theorem also plays a role in the construction of the Künneth formula, which relates the homology and cohomology of a product space to the homology and cohomology of its factors

10.4 Derived categories

Derived categories are a powerful tool in homological algebra, extending abelian categories by inverting quasi-isomorphisms. They allow us to study complexes and morphisms in a more flexible setting, preserving important homological information while simplifying certain computations.

Derived functors, constructed using derived categories, help us understand how functors behave homologically. They're crucial for computing Ext and Tor groups, studying sheaf cohomology, and solving problems in algebraic geometry and representation theory.

Derived Categories and Morphisms

Definition and Construction

- A derived category $D(A)$ is constructed from an abelian category A by formally inverting quasi-isomorphisms in the category of complexes $K(A)$
- The localization process ensures that quasi-isomorphic complexes become isomorphic in the derived category
- Objects in a derived category are complexes of objects in the original abelian category A
- Morphisms in a derived category are obtained by localizing the morphisms in $K(A)$ with respect to quasi-isomorphisms

Properties and Structure

- Quasi-isomorphisms are morphisms of complexes that induce isomorphisms on all cohomology groups
- Derived categories are triangulated categories, which means they possess a shift functor and a class of distinguished triangles satisfying certain axioms
- The shift functor in a derived category shifts the degree of a complex by one (e.g., $C^\bullet[1]^n = C^{\bullet+1}$)
- Distinguished triangles in a derived category are analogous to short exact sequences in an abelian category and satisfy similar properties (e.g., the long exact sequence of cohomology)

Construction of Derived Functors

Definition and Motivation

- Derived functors are obtained by applying a functor to a complex and then taking the cohomology of the resulting complex
- The process of deriving a functor is a way to extend the functor from the original abelian category to the derived category while preserving certain homological properties
- Deriving a functor allows for the study of its homological behavior, such as exactness and preservation of quasi-isomorphisms

Examples and Computation

- The most common examples of derived functors are the derived tensor product ($\otimes^L$), derived Hom ($RHom$), and derived global sections ($R\Gamma$)

- To compute derived functors, one typically replaces the original complex with a quasi-isomorphic complex that has better properties (e.g., a projective or injective resolution) and then applies the functor to this new complex
- For example, to compute the derived tensor product $M \otimes^L N$, one replaces M or N with a flat resolution and then takes the tensor product of the resolutions
- Similarly, to compute the derived Hom $RHom(M, N)$, one replaces M with a projective resolution or N with an injective resolution and then takes the Hom of the resolutions

Applications of Derived Functors

Homological Algebra

- Derived functors can be used to compute Ext and Tor groups, which measure the failure of exactness of certain functors
- $Ext^n(M, N)$ is the n -th cohomology of $RHom(M, N)$ and measures the failure of the Hom functor to be exact
- $Tor_n(M, N)$ is the n -th homology of $M \otimes^L N$ and measures the failure of the tensor product to be exact
- The derived Hom functor can be used to study the homological algebra of Hom spaces and to define the Hochschild cohomology of algebras

Sheaf Cohomology

- The derived global sections functor $R\Gamma$ can be used to study the cohomology of sheaves on a space
- For a sheaf $\mathcal{F}$ on a topological space X , the cohomology groups $H^n(X, \mathcal{F})$ are defined as the cohomology of the complex $R\Gamma(X, \mathcal{F})$
- Derived functors can also be used to define the hypercohomology of a complex of sheaves, which generalizes sheaf cohomology to the derived setting

Derived Categories in Modern Mathematics

Algebraic Geometry

- Derived categories provide a natural framework for studying the homological algebra of sheaves on algebraic varieties, leading to the development of derived algebraic geometry
- Derived categories have been used to prove important results in algebraic geometry, such as the reconstruction theorem for smooth projective varieties and the derived McKay correspondence
- The derived category of coherent sheaves $D^b(X)$ on a smooth projective variety X encodes important geometric information about X , such as its Hodge structure and Chow groups

Representation Theory

- In representation theory, derived categories are used to study the homological properties of representations of algebras and groups
- Tilting theory, which studies equivalences between derived categories, has applications in representation theory, such as the classification of certain classes of algebras (e.g., quasihereditary)

algebras)

- Derived categories have been used to study the structure of module categories over algebras, such as the Auslander-Reiten quiver and the Happel functor

Unit 11 – Sheaves and Sheaf Cohomology

11.1 Sheaves and sheaf cohomology

Sheaves and sheaf cohomology are powerful tools in algebraic topology. They provide a way to study local-to-global properties of spaces, connecting local data to global information. This framework is crucial for understanding complex geometric and topological structures.

Sheaf cohomology generalizes other cohomology theories, offering a unified approach to various mathematical concepts. It's particularly useful in algebraic geometry, allowing us to extract meaningful information from geometric objects and their relationships.

Sheaves on Topological Spaces

Definition and Properties of Sheaves

- A sheaf F on a topological space X consists of:
 - A set $F(U)$ for each open set U in X
 - Restriction maps $\text{res}_{U,V} : F(U) \rightarrow F(V)$ for each inclusion $V \subseteq U$ of open sets
 - These sets and maps must satisfy the following conditions:
 - (Identity) For every open set U , the restriction map $\text{res}_{U,U}$ is the identity map on $F(U)$
 - (Composition) If $W \subseteq V \subseteq U$ are open sets, then $\text{res}_{V,W} \circ \text{res}_{U,V} = \text{res}_{U,W}$
 - (Gluing) If $\{U_i\}$ is an open cover of an open set U and $s_i \in F(U_i)$ are elements such that $\text{res}_{U_i, U_i \cap U_j}(s_i) = \text{res}_{U_j, U_i \cap U_j}(s_j)$ for all i, j , then there exists a unique $s \in F(U)$ such that $\text{res}_{U, U_i}(s) = s_i$ for all i

Morphisms, Abelian Category, and Stalks

- A morphism $\phi : F \rightarrow G$ between sheaves on X is a collection of maps $\phi_U : F(U) \rightarrow G(U)$ for each open set U , compatible with the restriction maps
- Sheaves form an abelian category, with kernels, cokernels, and exact sequences defined pointwise
- This allows for the use of homological algebra techniques in the study of sheaves
- A sheaf F is flasque (flabby) if for every inclusion $V \subseteq U$ of open sets, the restriction map $\text{res}_{U,V}$ is surjective
- Flasque sheaves are used in the construction of injective resolutions
- The stalk of a sheaf F at a point $x \in X$, denoted F_x , is the direct limit of the sets $F(U)$ over all open neighborhoods U of x , with the induced restriction maps
- Stalks provide local information about the sheaf at each point

Sheaf Cohomology Groups

Injective Sheaves and Resolutions

- An injective sheaf is a sheaf I such that for any monomorphism $\phi : F \rightarrow G$ and any morphism $\psi : F \rightarrow I$, there exists a morphism $\chi : G \rightarrow I$ such that $\chi \circ \phi = \psi$
- Injective sheaves are analogous to injective modules in homological algebra
- An injective resolution of a sheaf F is an exact sequence $0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$, where each I^i is an injective sheaf
- Injective resolutions are used to define sheaf cohomology groups

Definition and Computation of Sheaf Cohomology

- Given a sheaf F on a topological space X , the sheaf cohomology groups $H^i(X, F)$ are defined as the right derived functors of the global sections functor $\Gamma(X, -)$
- To compute the sheaf cohomology groups: 1. Choose an injective resolution $0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ of F 2. Apply the global sections functor to obtain a cochain complex $0 \rightarrow \Gamma(X, I^0) \rightarrow \Gamma(X, I^1) \rightarrow \dots$ 3. The cohomology groups of this cochain complex are the sheaf cohomology groups $H^i(X, F)$
- The sheaf cohomology groups are independent of the choice of injective resolution

Computing Sheaf Cohomology

Simple Cases and Interpretations

- For a constant sheaf A on a topological space X , the sheaf cohomology groups $H^i(X, A)$ are isomorphic to the singular cohomology groups $H^i(X, A)$ with coefficients in the abelian group A
- This allows for the computation of sheaf cohomology using singular cohomology in certain cases
- For a locally constant sheaf F on a locally connected space X , the sheaf cohomology group $H^0(X, F)$ is isomorphic to the set of global sections $\Gamma(X, F)$
- Global sections are the elements that are compatible with all restriction maps
- On a contractible space X , the higher sheaf cohomology groups $H^i(X, F)$ vanish for any sheaf F and $i > 0$
- Contractible spaces (such as the real line $\mathbb{R}$) have trivial higher cohomology
- The first sheaf cohomology group $H^1(X, F)$ classifies the isomorphism classes of F -torsors on X , which are sheaves of sets locally isomorphic to F
- F -torsors can be thought of as twisted versions of the sheaf F

Čech-to-Derived Functor Spectral Sequence

- The Čech-to-derived functor spectral sequence relates Čech cohomology and sheaf cohomology
- It provides a way to compute sheaf cohomology using Čech cohomology of a cover and the higher direct images of the sheaf

- The higher direct images measure the failure of the sheaf to be acyclic on the intersections of the cover
- The spectral sequence starts with the Čech cohomology groups and converges to the sheaf cohomology groups
- This allows for the computation of sheaf cohomology in terms of simpler cohomological data

Sheaf Cohomology vs Other Theories

Generalizations and Connections

- Sheaf cohomology generalizes singular cohomology and Čech cohomology
- For a constant sheaf, sheaf cohomology recovers singular cohomology
- For a good cover, sheaf cohomology is isomorphic to Čech cohomology
- De Rham's theorem states that for a smooth manifold X , the de Rham cohomology groups (defined using differential forms) are isomorphic to the sheaf cohomology groups of the constant sheaf $\mathbb{R}$ on X
- This establishes a connection between differential geometry and sheaf theory

Applications to Vector Bundles

- Sheaf cohomology can be used to compute the cohomology of vector bundles
- For a vector bundle E on a topological space X , the sheaf cohomology groups $H^i(X, E)$ of the sheaf of sections of E are isomorphic to the cohomology groups of E
- This provides a sheaf-theoretic approach to studying vector bundles
- The cohomology groups of vector bundles have important applications in geometry and physics
- They classify topological invariants (characteristic classes) and measure obstructions to the existence of global sections

11.2 The Čech-to-derived functor spectral sequence

The Čech-to-derived functor spectral sequence bridges Čech cohomology and derived functor cohomology for sheaves. It uses a double complex to compute sheaf cohomology, connecting local and global perspectives on cohomological information.

This powerful tool allows us to understand sheaf cohomology through different lenses. By relating Čech and derived functor approaches, it provides insights into the structure of sheaves and their cohomology on topological spaces.

Čech-to-Derived Functor Spectral Sequence

Construction of the Spectral Sequence

- The Čech-to-derived functor spectral sequence computes the sheaf cohomology of a sheaf F on a topological space X
- Involves a double complex $C^{p,q} = C^p(U, I^q F)$
- U is an open cover of X

- $I^q F$ is the q -th injective resolution of F
- $C^p(U, -)$ denotes the Čech complex
- Spectral sequence arises from two filtrations on the double complex
- "Horizontal" filtration by p
- "Vertical" filtration by q
- E_1 page of the spectral sequence: $E_1^{p,q} = H^q(C^p(U, I^* F))$
- Cohomology groups of the Čech complex with coefficients in the injective resolution
- E_2 page: $E_2^{p,q} = H^p(X, H^q(I^* F))$
- $H^q(I^* F)$ is the q -th derived functor of F
- Spectral sequence converges to the sheaf cohomology $H^*(X, F)$

Convergence and Edge Homomorphisms

- The Čech-to-derived functor spectral sequence converges to the sheaf cohomology $H^*(X, F)$
- Convergence is often denoted by $E_\infty^{p,q}$, representing the associated graded object of $H^*(X, F)$ with respect to a suitable filtration
- Edge homomorphisms of the spectral sequence relate the sheaf cohomology to the Čech cohomology and derived functor cohomology
- Provide a way to extract information about $H^*(X, F)$ from the E_2 page
- In favorable situations, the spectral sequence may degenerate at the E_2 page
- All differentials on later pages are zero
- Sheaf cohomology can be read off directly from the E_2 page

Computing Sheaf Cohomology

Using the Spectral Sequence

- The Čech-to-derived functor spectral sequence computes the sheaf cohomology groups $H^n(X, F)$ for a sheaf F on a topological space X
- E_2 page provides a way to compute sheaf cohomology in terms of Čech cohomology and derived functor cohomology
- $E_2^{p,q} = H^p(X, H^q(I^* F))$ relates Čech cohomology (p -index) to derived functor cohomology (q -index)
- Differentials on the E_r pages for $r \geq 2$ give relations between the cohomology groups
- Used to determine the sheaf cohomology
- Example: If the spectral sequence degenerates at the E_2 page, sheaf cohomology can be read off directly
- Degeneration means all differentials on later pages are zero

Favorable Situations and Degeneration

- In some cases, the Čech-to-derived functor spectral sequence may degenerate at the E_2 page
- Happens when all differentials on later pages (E_r for $r \geq 3$) are zero
- Implies that the E_2 page directly gives the sheaf cohomology
- Degeneration at the E_2 page is a favorable situation for computing sheaf cohomology
- Allows for direct computation without the need to consider higher differentials
- Examples of situations where degeneration may occur:
 - When the space X has suitable acyclicity properties (Stein spaces)
 - When the sheaf F is flasque or soft

Čech vs Derived Functor Cohomology

Relationship between Čech and Derived Functor Cohomology

- The Čech-to-derived functor spectral sequence connects Čech cohomology and derived functor cohomology
- Čech cohomology $H^p(U, F)$:
 - Computed using open covers
 - Measures the global behavior of the sheaf F
- Derived functor cohomology $H^q(X, F)$:
 - Defined using injective resolutions
 - Captures the sheaf-theoretic cohomology
- E_2 page of the spectral sequence: $E_2^{p,q} = H^p(X, H^q(I^* F))$
- Relates Čech cohomology (p -index) to derived functor cohomology (q -index)

Coincidence and Comparison

- When the spectral sequence degenerates at the E_2 page, Čech cohomology and derived functor cohomology coincide
- Degeneration implies that the differentials on later pages are zero
- E_2 page directly gives the sheaf cohomology
- In general, the spectral sequence provides a way to compute derived functor cohomology using Čech cohomology and additional data from the differentials
- Differentials give relations between the cohomology groups
- Can be used to extract information about the derived functor cohomology
- Comparison between Čech and derived functor cohomology:
 - Čech cohomology is more computable and intuitive

- Derived functor cohomology is more abstract and sheaf-theoretic
- Spectral sequence bridges the gap between the two approaches

Spectral Sequence Terms and Differentials

Meaning of the Terms

- The terms in the Čech-to-derived functor spectral sequence have specific meanings related to the cohomology of the sheaf F
- E_1 page terms $E_1^{p,q}$:
 - Represent the cohomology of the Čech complex with coefficients in the injective resolution of F
 - Computed as $E_1^{p,q} = H^q(C^p(U, I^* F))$
- E_2 page terms $E_2^{p,q}$:
 - Represent the Čech cohomology of the derived functors of F
 - Given by $E_2^{p,q} = H^p(X, H^q(I^* F))$
- Higher page terms $E_r^{p,q}$ for $r \geq 3$:
 - Obtained by taking cohomology with respect to the differentials
 - Represent successive approximations to the sheaf cohomology

Role of the Differentials

- The differentials in the Čech-to-derived functor spectral sequence provide additional relations between the cohomology groups
- Differentials on the E_r pages: $d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$
- Maps between the terms on the E_r page
- Satisfy the condition $d_r^{p+r, q-r+1} \circ d_r^{p,q} = 0$ (differentials square to zero)
- Differentials on the E_2 page: $d_2^{p,q} : H^p(X, H^q(I^* F)) \rightarrow H^{p+2}(X, H^{q-1}(I^* F))$
- Relate the Čech cohomology and derived functor cohomology
- Provide a way to compute the derived functor cohomology using Čech cohomology
- Kernel and image of the differentials:
 - Determine the successive pages of the spectral sequence
 - Used to compute the higher page terms $E_r^{p,q}$ for $r \geq 3$
 - Ultimately determine the sheaf cohomology $H^*(X, F)$
- Convergence of the spectral sequence:
 - Represented by $E_\infty^{p,q}$
 - Obtained by taking the limit of the higher page terms

- Represents the associated graded object of the sheaf cohomology $H^*(X, \mathcal{F})$ with respect to a suitable filtration

11.3 Applications to algebraic geometry

Sheaf cohomology is a powerful tool in algebraic geometry, connecting local and global properties of varieties. It allows us to study coherent sheaves, vector bundles, and important invariants like Chern classes and Hodge numbers.

This section explores applications of sheaf cohomology in algebraic geometry. We'll see how it's used to classify vector bundles, compute invariants of varieties, and relate to other cohomology theories like Čech, singular, étale, and de Rham cohomology.

Sheaf cohomology for coherent sheaves

Properties and applications of coherent sheaves

- Coherent sheaves are sheaves of modules over the structure sheaf of an algebraic variety that are locally finitely generated and locally finitely presented
- This means they can be described locally by a finite number of generators and relations
- Examples of coherent sheaves include the structure sheaf itself and sheaves of sections of vector bundles
- Sheaf cohomology provides a powerful tool to study global sections and higher cohomology groups of coherent sheaves on algebraic varieties
- The zeroth cohomology group $H^0(X, \mathcal{F})$ represents the global sections of the sheaf $\mathcal{F}$
- Higher cohomology groups $H^i(X, \mathcal{F})$ measure the obstruction to extending local sections globally
- The dimension of the cohomology groups of a coherent sheaf can be used to determine properties such as the rank, degree, and Euler characteristic of the sheaf
- The rank of a coherent sheaf is the dimension of the stalk at a generic point (the fiber of the sheaf over that point)
- The degree of a coherent sheaf on a projective variety is the degree of the corresponding cycle in the Chow ring
- The Euler characteristic $\chi(X, \mathcal{F})$ is the alternating sum of the dimensions of the cohomology groups

Vanishing theorems and duality

- Vanishing theorems, such as the Kodaira vanishing theorem and the Serre vanishing theorem, provide conditions under which certain cohomology groups of coherent sheaves vanish
- The Kodaira vanishing theorem states that for an ample line bundle L on a smooth projective variety X over $\mathbb{C}$, $H^i(X, K_X \otimes L) = 0$ for $i > 0$, where K_X is the canonical bundle
- The Serre vanishing theorem states that for a coherent sheaf $\mathcal{F}$ on a projective variety X and a sufficiently ample line bundle L , $H^i(X, \mathcal{F} \otimes L) = 0$ for $i > 0$
- Serre duality relates the cohomology groups of a coherent sheaf to the cohomology groups of its dual sheaf, providing a powerful tool for computing cohomology

- For a coherent sheaf $\mathcal{F}$ on a smooth projective variety X of dimension n over a field k , there are isomorphisms $H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee$, where $\mathcal{F}^\vee$ is the dual sheaf and ω_X is the canonical sheaf
- This allows the computation of cohomology groups by reducing to the dual sheaf and using vanishing theorems

Sheaf cohomology in vector bundle classification

Vector bundles and the Picard group

- Vector bundles are locally free sheaves, which means they are locally isomorphic to a direct sum of copies of the structure sheaf
- A rank r vector bundle on a variety X is a sheaf $\mathcal{E}$ that is locally isomorphic to $\mathcal{O}_X^{\oplus r}$
- Line bundles are vector bundles of rank 1 and play a crucial role in the classification of varieties
- The set of isomorphism classes of vector bundles on an algebraic variety forms an abelian group called the Picard group, which can be studied using sheaf cohomology
- The Picard group $\text{Pic}(X)$ is the group of isomorphism classes of line bundles on X with the tensor product operation
- There is an isomorphism $\text{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$, where $\mathcal{O}_X^\times$ is the sheaf of invertible functions on X

Chern classes and the Riemann-Roch theorem

- The first Chern class of a vector bundle is an element of the second cohomology group of the variety with coefficients in the sheaf of invertible functions, and it provides an important invariant for classifying vector bundles
- The first Chern class $c_1(\mathcal{E})$ of a vector bundle $\mathcal{E}$ is an element of $H^2(X, \mathcal{O}_X^\times)$ that measures the obstruction to the existence of a global trivialization of $\mathcal{E}$
- The first Chern class of a line bundle L is the image of the isomorphism class of L under the isomorphism $\text{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$
- Extensions of vector bundles are classified by the first cohomology group of the sheaf of homomorphisms between the bundles, which can be computed using sheaf cohomology
- An extension of vector bundles $0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$ is classified by an element of $H^1(X, \text{Hom}(\mathcal{E}'', \mathcal{E}'))$
- The sheaf of homomorphisms $\text{Hom}(\mathcal{E}'', \mathcal{E}')$ can be studied using sheaf cohomology to determine the possible extensions
- The Riemann-Roch theorem relates the Euler characteristic of a vector bundle to its Chern classes, providing a powerful tool for studying the geometry of vector bundles
- For a vector bundle $\mathcal{E}$ on a smooth projective variety X of dimension n , the Riemann-Roch theorem states that $\chi(X, \mathcal{E}) = \int_X \text{ch}(\mathcal{E}) \cdot \text{td}(X)$, where $\text{ch}(\mathcal{E})$ is the Chern character of $\mathcal{E}$ and $\text{td}(X)$ is the Todd class of X
- The Chern character and Todd class are expressed in terms of Chern classes and can be computed using sheaf cohomology

Sheaf cohomology for algebraic variety invariants

Cohomology of the structure sheaf and Hodge numbers

- The cohomology groups of the structure sheaf of an algebraic variety provide important invariants, such as the dimension, genus, and arithmetic genus of the variety
- The dimension of a variety X is the largest integer n such that $H^n(X, \mathcal{O}_X) \neq 0$
- The genus of a smooth projective curve C is $g = \dim H^1(C, \mathcal{O}_C)$
- The arithmetic genus of a variety X is $p_a(X) = (-1)^{\dim X} (\chi(X, \mathcal{O}_X) - 1)$
- The Hodge numbers of a smooth projective variety can be computed using the Hodge decomposition of the cohomology groups of the sheaf of differential forms
- The Hodge numbers $h^{p,q}(X)$ of a smooth projective variety X are the dimensions of the cohomology groups $H^q(X, \Omega_X^p)$, where Ω_X^p is the sheaf of holomorphic p -forms on X
- The Hodge decomposition states that $H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X, \Omega_X^p)$, allowing the computation of Hodge numbers using sheaf cohomology

Canonical bundle and Kodaira dimension

- The canonical bundle of a variety, which is the determinant of the cotangent bundle, plays a crucial role in the classification of varieties and can be studied using sheaf cohomology
- The canonical bundle ω_X of a smooth variety X is the determinant of the cotangent bundle Ω_X^1 , i.e., $\omega_X = \det \Omega_X^1$
- The pluricanonical bundles $\omega_X^{\otimes k}$ are tensor powers of the canonical bundle and their cohomology groups provide important invariants
- The Kodaira dimension of a variety, which measures the growth of pluricanonical forms, can be computed using the dimensions of the cohomology groups of the pluricanonical bundles
- The Kodaira dimension $\kappa(X)$ of a variety X is the maximum dimension of the image of X under the rational maps defined by the pluricanonical linear systems $|m\omega_X|$ for sufficiently divisible $m > 0$, or $-\infty$ if all pluricanonical linear systems are empty
- The Kodaira dimension can be computed using the asymptotic behavior of the dimensions of the cohomology groups $H^0(X, \omega_X^{\otimes m})$ as $m \rightarrow \infty$

Sheaf cohomology vs other cohomology theories

Čech and singular cohomology

- Sheaf cohomology is related to Čech cohomology, which is defined using open covers of a variety and provides a more explicit way to compute cohomology groups
- Čech cohomology $H^i(X, \mathcal{F})$ of a sheaf $\mathcal{F}$ on a variety X is defined using an open cover $\mathcal{U}$ of X and the Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$
- For a sufficiently fine open cover, sheaf cohomology is isomorphic to Čech cohomology, i.e., $H^i(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$

- For smooth varieties, sheaf cohomology is isomorphic to singular cohomology, which is defined using singular chains and provides a topological perspective on cohomology
- Singular cohomology $H^i(X, \mathbb{Z})$ of a topological space X is defined using the dual of the singular chain complex, which is constructed using continuous maps from simplices to X
- For a smooth variety X over $\mathbb{C}$, there are isomorphisms $H^i(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^q(X, \Omega_X^p)$, relating singular cohomology to sheaf cohomology and Hodge theory

Étale and crystalline cohomology

- Étale cohomology is a cohomology theory for algebraic varieties that takes into account the arithmetic properties of the variety and is related to sheaf cohomology through the étale topology
- Étale cohomology $H_{\text{ét}}^i(X, F)$ of a sheaf F on a variety X is defined using the étale site of X , which is a Grothendieck topology that captures the arithmetic properties of X
- For a smooth proper variety X over a field k , there are comparison theorems relating étale cohomology to sheaf cohomology, such as the isomorphism $H_{\text{ét}}^i(X, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell \cong H^i(X, \mathbb{Q}_\ell)$ for $\ell \neq \text{char}(k)$
- Crystalline cohomology is a p-adic cohomology theory that is related to sheaf cohomology through the theory of crystals and provides a way to study varieties over fields of positive characteristic
- Crystalline cohomology $H_{\text{cris}}^i(X/W)$ of a smooth proper variety X over a perfect field k of characteristic $p > 0$ is defined using the crystalline site of X over the ring of Witt vectors $W = W(k)$
- There are comparison theorems relating crystalline cohomology to other cohomology theories, such as the de Rham-Witt complex and the Hodge-Witt cohomology, which are analogues of de Rham cohomology and Hodge theory in positive characteristic

De Rham cohomology

- De Rham cohomology, which is defined using differential forms, is isomorphic to sheaf cohomology for smooth varieties over fields of characteristic zero
- De Rham cohomology $H_{\text{dR}}^i(X)$ of a smooth variety X over a field k of characteristic zero is defined as the hypercohomology of the de Rham complex $\Omega_X^\bullet$
- The algebraic de Rham theorem states that there is an isomorphism $H_{\text{dR}}^i(X) \cong H^i(X, \mathbb{C})$ for a smooth variety X over $\mathbb{C}$, relating de Rham cohomology to sheaf cohomology
- The Hodge filtration on de Rham cohomology induces the Hodge decomposition on the cohomology groups of the sheaf of differential forms, providing a connection to Hodge theory

11.4 Sheaf operations and derived functors

Sheaf operations and derived functors are crucial tools in algebraic topology. They allow us to manipulate and analyze sheaves, which track local data on topological spaces. These operations help us understand how information flows between different parts of a space.

Derived functors measure how well sheaf operations preserve exactness. They give us a way to extend sheaf cohomology beyond just global sections. This deeper understanding of sheaves connects local and global properties, bridging different areas of mathematics.

Basic sheaf operations

Sheaves and their properties

- A sheaf is a tool for systematically tracking locally defined data attached to the open sets of a topological space
- Sheaves satisfy gluing conditions that allow local data to be uniquely determined by their restrictions to smaller open sets
- The stalk of a sheaf F at a point x , denoted F_x , captures the local behavior of the sheaf near x
- Sheaf morphisms are maps between sheaves that preserve the sheaf structure and can be defined locally
- Sheaves form an abelian category, which allows for the use of homological algebra techniques

Operations on sheaves

- For a continuous map $f: X \rightarrow Y$ and a sheaf F on X , the direct image sheaf f_*F on Y is defined by $(f_*F)(V) = F(f^{-1}(V))$ for each open set V in Y
- The inverse image sheaf f^*F is defined as the sheafification of the presheaf $U \mapsto \lim_{V \supset U} F(V)$ where the limit is taken over all open sets V containing U
- The restriction of a sheaf F on X to an open subset U , denoted $F|_U$, is defined by $(F|_U)(V) = F(V)$ for each open set V in U
- The extension by zero of a sheaf F on an open set U to X is denoted by $j_!F$, where j is the inclusion map. It is the sheaf on X that agrees with F on U and is zero outside of U
- Sheaf operations such as direct sums, tensor products, and sheaf Hom can be defined using the corresponding operations on the stalks or local sections

Derived functors of sheaf operations

Derived functors and exactness

- Derived functors are a way to measure the failure of a functor to be exact. They provide additional information about the sheaves involved
- The right derived functors of a left exact functor F are denoted by R^iF and can be computed using injective resolutions
- The left derived functors of a right exact functor G are denoted by L_iG and can be computed using projective resolutions
- Derived functors preserve exactness in the appropriate sense: R^iF preserves exactness of injective resolutions, while L_iG preserves exactness of projective resolutions
- The long exact sequence of derived functors is a powerful tool for computing and comparing the cohomology of sheaves

Important derived functors

- The derived functors of the global section functor are the sheaf cohomology functors $H^i(X, F)$

- The derived functors of the direct image functor f_* are the higher direct image functors $R^i f_*$, which can be used to compute sheaf cohomology on the target space Y
- The derived functors of the inverse image functor f^* are the higher inverse image functors $L^i f^*$, which play a role in the base change theorem
- The derived functors of the tensor product and sheaf Hom functors are the Tor and Ext functors, respectively
- The local-to-global spectral sequence relates the cohomology of a sheaf to the cohomology of its stalks

Computing derived functors

Simple cases and examples

- For a constant sheaf A on a contractible space X , the sheaf cohomology $H^i(X, A)$ is isomorphic to A for $i=0$ and is zero for $i>0$
- For a locally constant sheaf F on a circle S^1 , the sheaf cohomology $H^0(S^1, F)$ is isomorphic to the global sections of F , $H^1(S^1, F)$ is isomorphic to the coinvariants of the monodromy action on a stalk, and $H^i(S^1, F) = 0$ for $i>1$
- For a sheaf F on a CW complex X , the sheaf cohomology $H^i(X, F)$ can be computed using a cellular resolution of F , which is a complex of sheaves constructed from the cellular structure of X
- The Čech cohomology of a sheaf F with respect to an open cover U of X is isomorphic to the sheaf cohomology $H^i(X, F)$ when the cover is sufficiently fine (Leray theorem)
- The cohomology of a sheaf on a projective variety can often be computed using the Čech complex associated to a suitable affine open cover

Interpreting sheaf cohomology

- The interpretation of sheaf cohomology often depends on the specific context, such as the geometry or topology of the underlying space, or the algebraic properties of the sheaf
- In some cases, sheaf cohomology has a geometric interpretation, such as the classification of certain vector bundles or the obstruction to the existence of global sections
- Sheaf cohomology can also be related to other cohomology theories, such as singular cohomology or de Rham cohomology, via comparison theorems
- The vanishing or non-vanishing of certain sheaf cohomology groups can provide information about the structure of the underlying space or the properties of the sheaf
- In the context of algebraic geometry, sheaf cohomology is a fundamental tool for studying the geometry of varieties and their subvarieties

Spectral sequences for sheaf cohomology

Spectral sequences and their structure

- A spectral sequence is a tool for computing homology or cohomology by successive approximations. It consists of a sequence of pages, each containing a bigraded complex, with differentials between the pages

- The E_r page of a spectral sequence is a bigraded complex with differentials d_r of bidegree $(r, 1 - r)$. The cohomology of d_r is isomorphic to the E_{r+1} page
- A spectral sequence is said to converge to a graded object H^* if there exists an r_0 such that $E_r^{p,q} \cong E_{r_0}^{p,q}$ for all $r \geq r_0$ and $H^n \cong \bigoplus_{p+q=n} E_{r_0}^{p,q}$
- The differentials and convergence of a spectral sequence often provide additional information about the relationship between the objects involved, such as the topology of the spaces or the properties of the sheaves
- Spectral sequences can be constructed from various sources, such as double complexes, filtrations, or exact couples

Important spectral sequences

- The Leray spectral sequence is associated with a continuous map $f: X \rightarrow Y$ and a sheaf F on X . Its E_2 page is given by $E_2^{p,q} = H^p(Y, R^q f_* F)$, and it converges to the sheaf cohomology $H^{p+q}(X, F)$
- The Grothendieck spectral sequence is associated with a composition of functors $F \circ G$. Its E_2 page is given by $E_2^{p,q} = (R^p F)((R^q G)(A))$, and it converges to $R^{p+q}(F \circ G)(A)$
- The Serre spectral sequence is associated with a fibration $F \rightarrow E \rightarrow B$ and a sheaf $\mathcal{F}$ on E . Its E_2 page is given by $E_2^{p,q} = H^p(B, R^q f_* \mathcal{F})$, where $f: E \rightarrow B$ is the projection, and it converges to the sheaf cohomology $H^{p+q}(E, \mathcal{F})$
- The Hodge-de Rham spectral sequence relates the Hodge cohomology and de Rham cohomology of a complex manifold or algebraic variety
- The Eilenberg-Moore spectral sequence is associated with a pullback square of spaces and relates the cohomology of the total space to the cohomology of the base spaces and the fiber product

Unit 12 – Morse Theory

12.1 Morse functions and critical points

Morse functions are smooth real-valued functions on manifolds with non-degenerate critical points. They're key tools in studying manifold topology, revealing how the structure changes as we move through different function values.

Critical points of Morse functions are where the fun happens. These points, classified by their index, show where the topology shifts. As we pass through them, we see handles attaching and the manifold's shape evolving.

Morse functions and their properties

Definition and properties of Morse functions

- A Morse function is a smooth real-valued function on a smooth manifold whose critical points are non-degenerate
- Non-degenerate critical points are points where the Hessian matrix of second partial derivatives is non-singular
- The Hessian matrix is the matrix of second partial derivatives of the function
- Non-singularity means that the determinant of the Hessian matrix is non-zero
- Morse functions are generic in the space of smooth functions
- A small perturbation of any smooth function yields a Morse function
- This implies that Morse functions are abundant and can be easily constructed
- The Morse Lemma states that near a non-degenerate critical point, a Morse function has a canonical form up to a change of coordinates
- The canonical form is a quadratic function with a specific number of positive and negative terms
- The change of coordinates is a local diffeomorphism that simplifies the function near the critical point

Sublevel sets and handle attachments

- The sublevel sets of a Morse function undergo topological changes when passing through a critical point
- A sublevel set is the set of points where the function value is less than or equal to a given value
- Passing through a critical point of index k corresponds to attaching a k -dimensional handle to the sublevel set
- A handle is a product of a k -dimensional disc and an $(n - k)$ -dimensional disc, where n is the dimension of the manifold
- The change in topology near a critical point can be described by a handle attachment or a surgery operation

- Surgery is the process of removing a neighborhood of a submanifold and replacing it with another submanifold with the same boundary
- Handle attachments and surgeries provide a way to construct and modify manifolds using Morse functions

Critical points and their index

Definition and classification of critical points

- Critical points of a Morse function are points where the gradient vanishes
- The gradient is the vector of partial derivatives of the function
- At a critical point, all partial derivatives are zero
- The Hessian matrix at a critical point determines the type of the critical point
- The index of a critical point is the number of negative eigenvalues of the Hessian matrix
- Eigenvalues are the roots of the characteristic polynomial of the Hessian matrix
- Critical points can be classified as minima (index 0), saddles (index between 1 and $n - 1$), or maxima (index n)
- Minima are critical points with no negative eigenvalues (local minimum)
- Saddles have both positive and negative eigenvalues (neither local minimum nor maximum)
- Maxima have all negative eigenvalues (local maximum)

Morse index theorem and its implications

- The Morse index theorem relates the index of a critical point to the topology of the sublevel sets of the Morse function
- It states that the index of a critical point is equal to the number of negative eigenvalues of the Hessian matrix
- The theorem provides a link between the local behavior of the function near a critical point and the global topology of the manifold
- The index of a critical point determines the homology of the sublevel sets
- The homology groups measure the number of holes or cycles in different dimensions
- The index of a critical point is related to the change in homology when passing through that critical point
- Critical points of index k create or destroy k -dimensional holes in the sublevel sets

Topology of manifolds and critical points

Morse inequalities and Betti numbers

- The critical points of a Morse function provide information about the topology of the manifold
- The number and type of critical points are related to the homology and Betti numbers of the manifold

- The Morse inequalities relate the number of critical points of each index to the Betti numbers of the manifold
- Betti numbers are topological invariants that count the number of independent holes or cycles in each dimension
- The weak Morse inequalities state that the number of critical points of index k is greater than or equal to the k -th Betti number
- The strong Morse inequalities involve alternating sums of critical points and Betti numbers
- The Morse inequalities provide lower bounds for the number of critical points based on the topology of the manifold
- They can be used to estimate the complexity of a Morse function and the topology of the manifold
- The inequalities are sharp for certain classes of manifolds, such as spheres and projective spaces

Morse-Smale complex and cellular decomposition

- The Morse-Smale complex is a cellular decomposition of the manifold based on the flow lines connecting critical points
- A cellular decomposition is a partition of the manifold into cells of different dimensions
- Flow lines are integral curves of the gradient vector field of the Morse function
- The Morse-Smale complex is constructed by considering the stable and unstable manifolds of critical points
- The cells of the Morse-Smale complex are determined by the critical points and their indices
- Each critical point of index k corresponds to a k -dimensional cell in the complex
- The boundary of a cell consists of lower-dimensional cells corresponding to critical points of lower index
- The Morse-Smale complex provides a way to study the global structure of the manifold using the local information at critical points
- It encodes the connectivity and adjacency relations between critical points
- The homology of the manifold can be computed from the Morse-Smale complex using cellular homology

Morse function behavior near critical points

Morse Lemma and local canonical form

- The Morse Lemma provides a local canonical form for a Morse function near a critical point
- It states that in a neighborhood of a non-degenerate critical point, the Morse function can be expressed as a quadratic form
- The quadratic form has a specific number of positive and negative terms determined by the index of the critical point
- Near a critical point of index k , the Morse function has a quadratic form with k negative terms and $(n - k)$ positive terms

- The quadratic form is obtained by a change of coordinates that diagonalizes the Hessian matrix
- The canonical form simplifies the local behavior of the function and allows for a standard description of the critical point

Topological changes and handle attachments

- The sublevel sets of a Morse function undergo topological changes when passing through a critical point
- The type of topological change depends on the index of the critical point
- Passing through a critical point of index k corresponds to attaching a k -dimensional handle to the sublevel set
- Handle attachments provide a way to describe the change in topology near a critical point
- A handle of index k is a product of a k -dimensional disc and an $(n - k)$ -dimensional disc
- Attaching a handle of index k to a sublevel set means gluing the boundary of the k -dimensional disc to the boundary of the sublevel set
- The effect of a handle attachment on the homology of the sublevel set can be determined by the Morse index theorem
- Attaching a handle of index k changes the homology in dimension k and $k - 1$
- The change in homology is related to the creation or destruction of k -dimensional holes in the sublevel set

12.2 The Morse-Smale complex

Morse-Smale complexes are powerful tools in algebraic topology. They break down smooth manifolds into cells based on critical points and flow lines of Morse functions. This decomposition reveals crucial info about the manifold's structure and connectivity.

By studying the cells and their connections in a Morse-Smale complex, we can figure out important topological properties. These include Betti numbers and Euler characteristics, which tell us about holes, tunnels, and overall shape of the manifold.

Morse-Smale Complex Definition

Construction from a Morse Function

- The Morse-Smale complex is a cellular decomposition of a smooth manifold based on the critical points and flow lines of a Morse function defined on the manifold
- Involves partitioning the manifold into cells, each associated with a critical point of the Morse function
- The dimension of each cell is determined by the index of its associated critical point (minima: 0-dimensional, saddles: 1-dimensional, maxima: 2-dimensional in a 2-manifold)
- Cells are connected based on the gradient flow lines of the Morse function, which originate and terminate at critical points
- Provides a combinatorial description of the topology of the manifold
- Captures essential information about the manifold's structure and connectivity

- Allows for the computation of topological invariants (Betti numbers, Euler characteristic)

Topological Insights from the Morse-Smale Complex

- The Morse-Smale complex serves as a powerful tool for analyzing the topological properties of a manifold
- Helps determine the presence of topological features such as handles, tunnels, and cavities
- Provides a bridge between local information captured by critical points and global topological properties
- Studying the number, type, and connectivity of cells in the complex reveals important topological information
- Betti numbers represent the ranks of homology groups and provide information about connected components, holes, and higher-dimensional voids
- Euler characteristic can be computed from the alternating sum of the number of cells in each dimension

Gradient Flows in Morse-Smale Complex

Definition and Properties of Gradient Flows

- Gradient flows are integral curves of the gradient vector field of a Morse function
- Represent the direction of steepest ascent or descent at each point on the manifold
- Connect the critical points of the Morse function, forming a network that defines the structure of the Morse-Smale complex
- The stable manifold of a critical point consists of all points whose gradient flow lines converge to that critical point as time approaches positive infinity
- The unstable manifold of a critical point consists of all points whose gradient flow lines originate from that critical point as time approaches negative infinity

Role in Constructing the Morse-Smale Complex

- Gradient flow lines are integral to the construction of the Morse-Smale complex
- They connect the critical points and form the edges of the complex
- The intersection of stable and unstable manifolds of different critical points creates the cells of the complex
- Gradient flows capture the dynamics of the Morse function on the manifold
- They provide information about the flow of the gradient vector field between critical points
- Help determine the connectivity and structure of the Morse-Smale complex

Structure of the Morse-Smale Complex

Critical Points as Vertices

- Critical points of the Morse function serve as the vertices of the Morse-Smale complex

- Represent the key topological features of the manifold
- The index of a critical point determines the dimension of the cell it belongs to (minima: 0D, saddles: 1D, maxima: 2D in a 2-manifold)
- Different types of critical points play distinct roles in the complex
- Minima correspond to sinks or attractors in the gradient flow
- Saddles represent transition points between different regions of the manifold
- Maxima correspond to sources or repellers in the gradient flow

Connectivity and Cell Structure

- Gradient flow lines form the edges of the Morse-Smale complex
- They represent the flow of the gradient vector field between critical points
- Connect critical points based on the stable and unstable manifolds
- Cells of the Morse-Smale complex are glued together along their boundaries
- Boundaries are determined by the stable and unstable manifolds of the critical points
- The gluing of cells captures the topological structure and connectivity of the manifold
- The Morse-Smale complex provides a decomposition of the manifold into regions associated with critical points
- Each cell represents a subset of the manifold with similar gradient flow behavior
- The complex captures the global structure, including connectivity, holes, and critical regions

Topology of Manifolds with Morse-Smale Complex

Computing Topological Invariants

- The Morse-Smale complex allows for the computation of important topological invariants
- Betti numbers represent the ranks of homology groups and provide information about connected components, holes, and higher-dimensional voids
- Betti_0: number of connected components
- Betti_1: number of 1D holes or loops
- Betti_2: number of 2D voids or cavities
- Euler characteristic can be computed from the alternating sum of the number of cells in each dimension
- Formula:

$$\chi = \sum_{i=0}^n (-1)^i c_i$$

, where c_i is the number of i -dimensional cells

- Computing these invariants helps characterize the topology of the manifold

- Provides quantitative measures of topological features
- Allows for comparison and classification of different manifolds

Analyzing Global Topological Structure

- The Morse-Smale complex captures the global topological structure of the manifold
- Reveals the presence of topological features such as handles, tunnels, and cavities
- Helps understand the connectivity and relationships between different regions of the manifold
- By studying the structure of the Morse-Smale complex, one can gain insights into the overall topology of the manifold
- Identify the number and types of critical points and their associated cells
- Analyze the connectivity and flow patterns between critical points
- Determine the existence and location of important topological features
- The Morse-Smale complex provides a comprehensive representation of the manifold's topology
- Bridges the gap between local information at critical points and global topological properties
- Allows for a deeper understanding of the shape and structure of the manifold

12.3 The Morse inequalities

Morse inequalities connect critical points of Morse functions to Betti numbers of manifolds. They provide lower bounds for critical points based on topology, revealing insights into homology groups. This powerful tool links Morse theory to algebraic topology.

By analyzing critical points, we can estimate homological complexity. Morse inequalities help determine possible Betti number ranges and, in special cases, exact values. This approach bridges function behavior and manifold topology, offering a unique perspective on topological structures.

Morse Inequalities and Critical Points

Relating Critical Points to Betti Numbers

- The Morse inequalities relate the number of critical points of a Morse function to the Betti numbers of the manifold on which the function is defined
- For a compact manifold M and a Morse function f on M , the Morse inequalities state that for each k , the number of critical points of index k is greater than or equal to the k -th Betti number of M
- The alternating sum of the number of critical points of a Morse function is equal to the Euler characteristic of the manifold, which is also equal to the alternating sum of the Betti numbers
- For example, on a torus, a Morse function must have at least 4 critical points: 1 minimum, 2 saddle points, and 1 maximum

Providing Lower Bounds for Critical Points

- The Morse inequalities provide a lower bound for the number of critical points of each index that a Morse function must have based on the topology of the manifold

- For instance, if a manifold has non-zero k -th Betti number, any Morse function on it must have at least one critical point of index k
- By studying the critical points of a Morse function, one can obtain information about the Betti numbers and the homology groups of the manifold
- The presence of critical points of certain indices gives insight into the non-trivial homology groups of the manifold

Significance of Morse Inequalities

Connecting Morse Theory and Homology

- The Morse inequalities establish a connection between the critical points of a Morse function and the homology of the manifold, providing insight into the topology of the manifold
- The Morse inequalities are a powerful tool in algebraic topology, as they allow for the computation of homological invariants of a manifold using the critical points of a Morse function
- This connection enables the study of the topology of a manifold through the lens of Morse theory and critical points

Estimating Complexity of Homology

- The Morse inequalities provide a way to estimate the complexity of the homology of a manifold based on the behavior of a Morse function defined on it
- A manifold with a large number of critical points of a Morse function is likely to have a more intricate homology compared to a manifold with fewer critical points
- By analyzing the critical points of different indices, one can gain insights into the dimensions of the homology groups and the overall topological complexity of the manifold
- For example, a manifold with many critical points of high index suggests the presence of high-dimensional non-trivial homology groups

Applying Morse Inequalities for Betti Numbers

Obtaining Lower Bounds for Betti Numbers

- Given a Morse function on a manifold, one can use the Morse inequalities to obtain lower bounds for the Betti numbers of the manifold
- The number of critical points of each index provides a lower bound for the corresponding Betti number
- By counting the number of critical points of each index of a Morse function, the Morse inequalities provide a way to estimate the dimensions of the homology groups of the manifold
- For instance, if a Morse function has 3 critical points of index 1, the first Betti number of the manifold is at most 3

Determining Possible Range of Betti Numbers

- The Morse inequalities can be used to determine the possible range of values for the Betti numbers based on the critical points of a Morse function

- The inequalities constrain the Betti numbers to lie within certain bounds determined by the number of critical points
- In some cases, the Morse inequalities can provide exact values for the Betti numbers when the inequalities are satisfied with equality
- If the number of critical points of each index exactly matches the corresponding Betti number, the Morse function captures the complete homological information of the manifold

Equality Case of Morse Inequalities

Perfect Morse Functions

- When the Morse inequalities are satisfied with equality for all k , it implies that the Morse function is perfect, meaning that the number of critical points of each index exactly matches the corresponding Betti number
- The equality case of the Morse inequalities suggests that the Morse function captures all the essential topological information of the manifold
- In this case, the critical points of the Morse function fully determine the homology of the manifold

Simplest Possible Topology

- If a Morse function satisfies the Morse inequalities with equality, it indicates that the manifold has the simplest possible topology compatible with the number of critical points of the function
- The manifold cannot have any "extra" topological features beyond what is captured by the critical points
- The equality case implies a tight relationship between the Morse function and the topology of the manifold, with no room for additional topological complexity
- For example, if a Morse function on a surface has exactly 1 minimum, 1 maximum, and $2g$ saddle points, where g is the genus, then the surface must be a sphere ($g=0$) or a torus ($g=1$)

Rarity and Specialness

- The equality case of the Morse inequalities is a rare and special situation, and it provides a strong relationship between the Morse function and the homology of the manifold
- Most Morse functions do not satisfy the equality case, as there may be additional critical points that do not contribute to the homology
- When the equality case holds, it signifies a deep connection between the critical points and the topology of the manifold, making the Morse function particularly well-suited for studying the homological properties of the manifold
- In such cases, the Morse function provides a complete and efficient description of the homology of the manifold

12.4 Morse homology and the Morse-Witten complex

Morse homology connects the critical points of a Morse function to a manifold's topology. By analyzing gradient flows between these points, we can construct the Morse-Witten complex, whose homology mirrors the manifold's singular homology.

This powerful tool simplifies topological computations and yields insights like Morse inequalities. It bridges differential geometry and algebraic topology, offering a concrete way to understand a manifold's shape through its critical points and flows.

Morse-Witten Complex Definition

Construction from a Morse Function

- The Morse-Witten complex is a chain complex constructed from a Morse function on a smooth manifold
- A Morse function is a smooth real-valued function on a manifold with non-degenerate critical points (local minima, maxima, and saddle points)
- The chain groups of the Morse-Witten complex are generated by the critical points of the Morse function, graded by their Morse index (number of negative eigenvalues of the Hessian matrix at the critical point)
- The differential in the Morse-Witten complex is defined by counting the gradient flow lines between critical points of consecutive indices
- The gradient flow of a Morse function is a one-parameter group of diffeomorphisms obtained by integrating the negative gradient vector field
- The differential counts the number of gradient flow lines (modulo 2) between critical points, satisfying certain compactness and transversality conditions
- Example: For a height function on a torus, the differential counts the number of gradient flow lines between a saddle point and a minimum or maximum

Dependence on Additional Structures

- The construction of the Morse-Witten complex depends on the choice of the Morse function and an additional metric on the manifold
- Different Morse functions on the same manifold may yield different Morse-Witten complexes, but their homology groups are isomorphic
- The metric on the manifold is used to define the gradient vector field and the inner product structure on the tangent spaces
- The compactness and transversality conditions for the gradient flow lines may require perturbing the Morse function or the metric
- Example: On a sphere, the height function with respect to different axes yields different Morse-Witten complexes, but they all compute the homology of the sphere

Gradient Flows in Morse Homology

Connecting Critical Points

- Gradient flows connect critical points of a Morse function and determine the differential in the Morse-Witten complex
- The gradient vector field of a Morse function points in the direction of steepest ascent and is orthogonal to the level sets of the function

- The gradient flow lines are integral curves of the negative gradient vector field, flowing from higher to lower critical points
- Example: On a surface, gradient flow lines connect saddle points to minima and maxima, creating a cellular decomposition of the surface

Compactness and Transversality

- The compactness of the moduli spaces of gradient flow lines is crucial for the well-definedness of the differential in the Morse-Witten complex
- Compactness is achieved by imposing suitable boundedness and convergence conditions on the gradient flow lines
- Example: For a compact manifold, the space of gradient flow lines connecting two critical points is compact, ensuring a finite count
- The transversality of the stable and unstable manifolds of critical points ensures that the gradient flow lines are isolated and can be counted
- The stable manifold of a critical point consists of points that flow to the critical point under the gradient flow
- The unstable manifold of a critical point consists of points that flow from the critical point under the gradient flow
- Transversality means that the stable and unstable manifolds intersect in a generic way, with the expected dimension
- The Morse-Smale condition, which requires the stable and unstable manifolds to intersect transversely, is a sufficient condition for defining the Morse-Witten complex

Homology of the Morse-Witten Complex

Morse Homology Theorem

- The homology of the Morse-Witten complex is defined as the quotient of the kernel of the differential by its image
- The Morse Homology Theorem states that the homology of the Morse-Witten complex is isomorphic to the singular homology of the manifold
- This isomorphism is induced by a chain map from the Morse-Witten complex to the singular chain complex, which sends each critical point to a sum of nearby singular simplices
- Example: For the height function on a torus, the Morse homology groups are isomorphic to the singular homology groups of the torus

Morse Inequalities

- The Morse inequalities relate the Betti numbers of the manifold to the number of critical points of each index in a Morse function
- The weak Morse inequalities state that the Betti numbers are bounded above by the number of critical points of the corresponding index

- The strong Morse inequalities involve an alternating sum of the number of critical points and provide a more refined estimate of the Betti numbers
- Example: For a surface of genus g , a Morse function has at least $2g+2$ critical points, with at least one minimum, one maximum, and $2g$ saddle points
- The Morse-Witten complex provides a way to compute the homology of a manifold using the critical points of a Morse function, which can be more efficient than using singular homology

Morse Homology for Topology Analysis

Topological Information from Critical Points

- Morse homology can be used to derive topological information about a manifold from the critical points of a Morse function
- The Euler characteristic of a manifold can be expressed as the alternating sum of the number of critical points of a Morse function
- The Poincaré polynomial of a manifold, which encodes its Betti numbers, can be bounded using the Morse inequalities
- Example: The Euler characteristic of a torus is 0, which can be seen from a Morse function with one minimum, one maximum, and two saddle points

Topological Invariance and Applications

- Morse homology is independent of the choice of the Morse function and the metric on the manifold, making it a topological invariant
- Different Morse functions on the same manifold yield isomorphic Morse homology groups
- Homotopy equivalent manifolds have isomorphic Morse homology groups
- The Morse Homology Theorem implies that Morse homology is a stronger invariant than singular homology, as it captures additional information about the gradient flow
- Morse homology can be used to prove the h-cobordism theorem, which characterizes the topology of cobordisms between manifolds in terms of their homology
- A cobordism is a manifold with boundary that interpolates between two given manifolds
- The h-cobordism theorem states that if the cobordism has trivial homology relative to its boundary, then it is diffeomorphic to a product cobordism
- The functoriality of Morse homology with respect to smooth maps between manifolds allows for the study of induced homomorphisms and the construction of long exact sequences
- A smooth map between manifolds induces a chain map between their Morse-Witten complexes, which in turn induces a homomorphism between their Morse homology groups
- Long exact sequences in Morse homology can be used to study the relationship between the topology of a manifold and its submanifolds or quotient spaces