

Elementary Algebraic Geometry

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Unit 1 – Intro to Algebraic Geometry

1.1 Historical overview and motivation

Algebraic geometry has evolved from ancient Greek studies of curves to a powerful field connecting algebra and geometry. It's a journey through time, from Descartes and Fermat's coordinate systems to Grothendieck's revolutionary schemes and category theory.

This historical overview sets the stage for our dive into algebraic geometry. We'll explore how polynomial equations reveal geometric truths, bridging abstract algebra and visual geometry in ways that impact number theory, cryptography, and even physics.

Algebraic Geometry's Historical Evolution

Origins in Ancient Greek Mathematics

- Algebraic geometry has its roots in ancient Greek mathematics
- Apollonius and Diophantus studied curves and equations, laying the foundation for future developments in the field

Development of Analytic Geometry in the 17th Century

- In the 17th century, Descartes and Fermat developed analytic geometry
- Connected algebra and geometry by introducing coordinate systems
- Enabled the study of algebraic curves and surfaces using algebraic equations
- Analytic geometry provided a new framework for studying geometric objects and their properties

Advancements in the 19th Century

- The 19th century saw significant advancements in algebraic geometry
- Key contributors include Plücker, Cayley, and Riemann
- Introduced new concepts and techniques, such as the study of algebraic curves and surfaces in higher dimensions
- Riemann introduced the concept of Riemann surfaces and studied their topology, laying the foundation for complex algebraic geometry
- These advancements expanded the scope and depth of algebraic geometry

Progress in the Early 20th Century

- In the early 20th century, the Italian school of algebraic geometry made important progress
- Led by Enriques and Severi, they focused on the classification of algebraic surfaces
- Developed new techniques for studying the geometry and topology of surfaces
- Contributed to the understanding of the birational geometry of surfaces

Modern Algebraic Geometry

- Modern algebraic geometry, influenced by the work of Zariski, Weil, and Grothendieck, incorporates tools from commutative algebra and homological algebra
- Zariski developed the modern approach to algebraic geometry
- Emphasized the use of commutative algebra and introduced the Zariski topology
- Laid the foundation for the algebraic study of geometric objects
- Grothendieck revolutionized algebraic geometry by introducing schemes and developing the theory using tools from homological algebra and category theory
- Schemes provide a unified framework for studying algebraic varieties and their generalizations
- Grothendieck's work led to a more abstract and general approach to algebraic geometry

Key Figures in Algebraic Geometry

Pioneers of Analytic Geometry

- René Descartes (1596-1650) and Pierre de Fermat (1607-1665) developed analytic geometry
- Connected algebra and geometry by introducing coordinate systems
- Enabled the study of geometric objects using algebraic equations
- Their work laid the foundation for the development of algebraic geometry

Bernhard Riemann and Complex Algebraic Geometry

- Bernhard Riemann (1826-1866) made significant contributions to complex algebraic geometry
- Introduced the concept of Riemann surfaces
- Surfaces that locally resemble the complex plane but may have a more complex global structure
- Studied the topology of Riemann surfaces, leading to important results in complex analysis and geometry
- Riemann's work laid the foundation for the study of complex algebraic varieties

David Hilbert and the Foundations of Algebraic Geometry

- David Hilbert (1862-1943) made significant contributions to the foundations of algebraic geometry
- Introduced Hilbert's Nullstellensatz, a fundamental theorem relating algebraic sets and ideals in polynomial rings
- Establishes a correspondence between geometric objects (algebraic sets) and algebraic objects (ideals)
- Plays a crucial role in the algebraic study of geometric problems
- Developed the theory of Hilbert schemes, which parametrize algebraic varieties and their subschemes
- Provides a tool for studying families of algebraic varieties and their moduli spaces

Oscar Zariski and the Modern Approach to Algebraic Geometry

- Oscar Zariski (1899-1986) developed the modern approach to algebraic geometry
- Emphasized the use of commutative algebra in the study of algebraic varieties
- Introduced the Zariski topology, a topology on algebraic varieties defined by polynomial equations
- Developed the theory of valuations and local rings, which play a central role in the local study of algebraic varieties
- Zariski's work laid the foundation for the algebraic and geometric study of algebraic varieties

Alexandre Grothendieck and the Revolution of Algebraic Geometry

- Alexandre Grothendieck (1928-2014) revolutionized algebraic geometry
- Introduced the concept of schemes, generalizing algebraic varieties
- Schemes are locally ringed spaces that generalize both algebraic varieties and arithmetic objects
- Provide a unified framework for studying algebraic geometry in various settings (classical, arithmetic, etc.)
- Developed the theory of algebraic geometry using tools from homological algebra and category theory
- Introduced the notion of abelian categories and derived functors, which play a central role in modern algebraic geometry
- Developed the theory of étale cohomology, a powerful tool for studying the topology of algebraic varieties

Motivation and Applications of Algebraic Geometry

Understanding Geometric Properties through Algebra

- Algebraic geometry seeks to understand the geometric properties of solutions to polynomial equations
- Bridges the gap between algebra and geometry by studying geometric objects using algebraic techniques
- Allows for the study of geometric properties (dimension, singularities, etc.) using algebraic tools (rings, modules, etc.)
- Provides a way to translate geometric problems into algebraic language, enabling the use of powerful algebraic techniques

Applications in Number Theory

- The study of algebraic curves and surfaces has important applications in number theory
- Diophantine equations, which are polynomial equations with integer coefficients, can be studied using algebraic geometry
- Algebraic geometry provides tools for understanding the structure and solutions of Diophantine equations

- Elliptic curves, a special class of algebraic curves, play a central role in modern number theory and cryptography
- Algebraic geometry has led to significant advances in the study of rational points on algebraic varieties and the arithmetic of algebraic curves and surfaces

Connections to Complex Geometry

- Algebraic geometry plays a crucial role in the study of complex geometry
- Many complex manifolds can be described as algebraic varieties
- Algebraic varieties are defined by polynomial equations and can be studied using algebraic techniques
- The study of complex algebraic varieties leads to important results in complex geometry and topology
- The interplay between algebraic geometry and complex geometry has led to significant advances in both fields

Applications in Other Fields

- The tools and techniques of algebraic geometry have found applications in various fields beyond pure mathematics
- Coding theory and cryptography
- Algebraic geometry provides a framework for constructing error-correcting codes and cryptographic protocols
- Algebraic curves and varieties are used in the design of efficient and secure coding and cryptographic schemes
- Mathematical physics
- Algebraic geometry is used in the study of string theory and quantum field theory
- Algebraic varieties and their moduli spaces appear naturally in the description of physical systems and their symmetries

Appreciating the Connections and Developments

- Understanding the motivation behind algebraic geometry helps to appreciate its historical development and its connections to other areas of mathematics and science
- Algebraic geometry has evolved over centuries, with contributions from diverse fields such as algebra, geometry, topology, and number theory
- The interplay between these fields has led to significant advances and new perspectives in algebraic geometry
- The historical development of algebraic geometry reflects the broader trends and developments in mathematics and science
- Recognizing the applications and connections of algebraic geometry provides a deeper understanding of its importance and potential impact in various domains

1.2 Basic algebraic structures and geometric concepts

Algebraic geometry blends algebra and geometry, using rings, fields, and ideals to study geometric shapes. It's like using math tools to build and analyze intricate structures, giving us a new way to see shapes and spaces.

In this chapter, we'll explore how algebraic concepts connect to geometric objects. We'll see how polynomial equations define curves and surfaces, and how algebraic tools help us understand their properties and relationships.

Algebraic Structures in Geometry

Rings and Fields

- Rings are algebraic structures with addition and multiplication operations that satisfy certain axioms, such as associativity, commutativity, and distributivity
- Examples include the integers, polynomial rings, and coordinate rings of algebraic varieties
- The axioms ensure that arithmetic operations behave consistently and predictably within the structure
- Fields are rings in which every nonzero element has a multiplicative inverse
- Examples include the rational numbers, real numbers, and complex numbers
- Fields are used as the base field for defining algebraic varieties, providing a foundation for geometric constructions

Ideals and Modules

- Ideals are subsets of rings that are closed under addition and absorption (multiplication by ring elements)
- They play a crucial role in defining algebraic varieties and studying their properties
- Principal ideals are generated by a single element, while prime and maximal ideals have special properties related to factorization and quotient rings
- Modules are generalizations of vector spaces, where the scalars come from a ring instead of a field
- They are used to study the properties of algebraic varieties and their sheaves
- Important examples include free modules, which are direct sums of copies of the ring, and torsion modules, which contain elements annihilated by non-zero ring elements

Geometric Concepts and Their Algebraic Counterparts

Varieties and Morphisms

- Affine varieties are geometric objects defined by the vanishing of a set of polynomials in an affine space
- They correspond to ideals in polynomial rings, with the ideal consisting of all polynomials that vanish on the variety
- Examples include curves, surfaces, and hypersurfaces defined by polynomial equations

- Projective varieties are geometric objects defined by the vanishing of a set of homogeneous polynomials in a projective space
- They are the projective counterparts of affine varieties and are used to study the global properties of algebraic varieties
- Projective varieties are defined by homogeneous ideals and have a well-defined dimension and degree
- Morphisms between algebraic varieties are maps that preserve the algebraic structure
- They correspond to ring homomorphisms between the coordinate rings of the varieties
- Examples include regular maps, which are defined by polynomials, and rational maps, which are defined by ratios of polynomials

Sheaves and Cohomology

- Sheaves are geometric objects that assign algebraic data (e.g., rings or modules) to open sets of a topological space, such as an algebraic variety
- They are used to study the local properties of algebraic varieties and to define cohomology theories
- Important examples include the structure sheaf, which assigns the local ring to each point, and the sheaf of regular functions, which assigns the ring of regular functions to each open set
- Cohomology theories, such as sheaf cohomology and de Rham cohomology, are algebraic tools for studying the global properties of algebraic varieties
- They assign algebraic invariants (e.g., vector spaces or modules) to algebraic varieties that capture their topological and geometric features, such as their connectivity and the existence of certain types of differential forms
- Cohomology groups measure the obstructions to solving certain algebraic or geometric problems, such as finding global sections of a sheaf or constructing meromorphic functions with prescribed poles and zeros

Connecting Algebra and Geometry

Nullstellensatz and Coordinate Rings

- The Nullstellensatz is a fundamental theorem that establishes a correspondence between ideals in polynomial rings and algebraic subsets of affine space
- It states that every maximal ideal in a polynomial ring over an algebraically closed field is the ideal of a point in the corresponding affine space
- The Nullstellensatz allows us to study algebraic varieties using the tools of commutative algebra
- The coordinate ring of an affine variety is the quotient of a polynomial ring by the ideal defining the variety
- It encodes the algebraic properties of the variety, such as its dimension and singularities
- The coordinate ring is a finitely generated k -algebra, where k is the base field, and its prime ideals correspond to subvarieties of the original variety

Function Fields and Zariski Topology

- The function field of an algebraic variety is the field of rational functions on the variety
- It is the field of fractions of the coordinate ring and encodes the birational geometry of the variety
- Two varieties are birationally equivalent if their function fields are isomorphic, which means they have the same rational curves and rational maps
- The Zariski topology is a topology on algebraic varieties defined by taking the closed sets to be the algebraic subsets (i.e., subsets defined by polynomial equations)
- It is used to study the global properties of algebraic varieties, such as their irreducible components and dimension
- The Zariski topology is coarser than the classical topology, meaning it has fewer open sets, but it is better suited for studying algebraic properties

Solving Geometric Problems with Algebra

Gröbner Bases and Resolutions

- Gröbner bases are special generating sets of ideals in polynomial rings that have useful computational properties
- They are used to solve systems of polynomial equations, compute the dimension and degree of algebraic varieties, and study their singularities
- Gröbner bases can be computed using algorithms such as Buchberger's algorithm and Faugère's F5 algorithm
- Resolutions of singularities are methods for transforming singular algebraic varieties into smooth ones by a sequence of blowups
- They are used to study the local properties of singularities and to compute invariants such as the genus and the canonical class
- Examples of resolution techniques include embedded resolution, which resolves singularities of subvarieties, and the minimal model program, which aims to find the simplest birational model of a variety

Intersection Theory and Cohomology

- Intersection theory is the study of how algebraic varieties intersect each other
- It involves computing intersection numbers, which are numerical invariants that measure the multiplicity of the intersection
- Intersection theory is used to study the geometry of algebraic curves and surfaces and to compute their invariants, such as the degree and the genus
- Cohomology theories, such as sheaf cohomology and de Rham cohomology, are algebraic tools for studying the global properties of algebraic varieties
- They assign algebraic invariants (e.g., vector spaces or modules) to algebraic varieties that capture their topological and geometric features, such as their connectivity and the existence of certain types of differential forms

- Examples of cohomology theories include Čech cohomology, which is defined using open covers, and étale cohomology, which is defined using algebraic analogues of topological coverings

1.3 Polynomial rings and ideals

Polynomial rings and ideals are the building blocks of algebraic geometry. They provide a powerful framework for studying geometric objects through algebraic equations, bridging the gap between algebra and geometry.

In this context, we'll explore how polynomial rings encode geometric information and how ideals represent sets of solutions. We'll see how these concepts form the foundation for understanding algebraic varieties and their properties.

Polynomial rings and their properties

Definition and notation

- A polynomial ring, denoted as $R[x]$, is a ring formed by polynomials with coefficients from a ring R
- The elements of a polynomial ring are polynomials, and the operations of addition and multiplication are performed on these polynomials
- For example, if R is the ring of integers, then $R[x]$ consists of polynomials with integer coefficients, such as $3x^2 + 2x - 1$

Properties inherited from the coefficient ring

- Polynomial rings inherit many properties from the coefficient ring R , such as commutativity and the existence of identity elements for addition and multiplication
- If R is commutative, then $R[x]$ is also commutative, meaning that for any polynomials $f(x)$ and $g(x)$ in $R[x]$, $f(x)g(x) = g(x)f(x)$
- The zero polynomial and the constant polynomial 1 serve as the additive and multiplicative identity elements in $R[x]$, respectively
- If R is a field, then $R[x]$ is a principal ideal domain (PID), meaning that every ideal in $R[x]$ is generated by a single polynomial
- For instance, if R is the field of real numbers, then every ideal in $R[x]$ is of the form $(f(x))$, where $f(x)$ is a polynomial in $R[x]$

Degree of a polynomial

- The degree of a polynomial is the highest power of the variable in the polynomial, and it plays a crucial role in determining the properties of the polynomial ring
- For example, the polynomial $3x^2 + 2x - 1$ has degree 2, as the highest power of x is 2
- Polynomials of the same degree can be compared and ordered based on their leading coefficients, allowing for the division algorithm and the notion of greatest common divisors (GCDs) in $R[x]$

Ideals in algebraic geometry

Definition and properties

- An ideal I in a ring R is a subset of R that is closed under addition and multiplication by elements of R
- For any $a, b \in I$ and $r \in R$, $a + b \in I$ and $ra \in I$
- Ideals generalize the concept of multiples in the ring of integers and allow for the study of congruences and quotient rings
- In the ring of integers $\mathbb{Z}$, the ideal (n) consists of all multiples of n , and the quotient ring $\mathbb{Z}/(n)$ represents the congruence classes modulo n

Correspondence with algebraic sets

- In algebraic geometry, ideals in polynomial rings are used to define algebraic sets, which are the solution sets of systems of polynomial equations
- For example, the ideal $I = (x^2 + y^2 - 1, y - x^2)$ in $\mathbb{R}[X, Y]$ defines the algebraic set $V(I)$, which is the unit circle intersected with the parabola $y = x^2$
- The correspondence between ideals and algebraic sets is a fundamental concept in algebraic geometry, known as the Zariski topology
- Closed sets in the Zariski topology are precisely the algebraic sets, and the topology is defined by taking finite unions and arbitrary intersections of algebraic sets

Prime and maximal ideals

- Prime ideals, which are ideals P such that for any $a, b \in R$, if $ab \in P$, then either $a \in P$ or $b \in P$, play a crucial role in understanding the geometry of algebraic sets
- Prime ideals correspond to irreducible algebraic sets, which cannot be written as the union of two proper subsets
- Maximal ideals, which are ideals that are maximal with respect to inclusion, correspond to points in the affine space defined by the polynomial ring
- For example, in $\mathbb{C}[X, Y]$, the maximal ideal $(x - 1, y - 2)$ corresponds to the point $(1, 2)$ in the complex affine plane

Operations on ideals

Sum, product, and intersection

- The sum of two ideals I and J , denoted as $I + J$, is the smallest ideal containing both I and J
- $I + J = \{a + b : a \in I, b \in J\}$
- The product of two ideals I and J , denoted as IJ , is the ideal generated by all products of elements from I and J
-

$$IJ = \left\{ \sum_{i=1}^n a_i b_i : a_i \in I, b_i \in J, n \in \mathbb{N} \right\}$$

- The intersection of two ideals I and J , denoted as $I \cap J$, is the largest ideal contained in both I and J
- $I \cap J = \{a : a \in I \text{ and } a \in J\}$

Quotient and radical

- The quotient of two ideals I and J , denoted as $(I : J)$, is the ideal consisting of all elements r in R such that rJ is a subset of I
- $(I : J) = \{r \in R : rJ \subseteq I\}$
- The radical of an ideal I , denoted as $\sqrt{I}$, is the ideal consisting of all elements r in R such that some power of r belongs to I
- $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}$
- Operations on ideals allow for the manipulation and simplification of systems of polynomial equations in algebraic geometry
- For instance, the radical of an ideal corresponds to the vanishing set of the ideal, which is the set of all points where all polynomials in the ideal evaluate to zero

Geometric meaning of ideals

Algebraic sets and vanishing ideals

- The algebraic set defined by an ideal I , denoted as $V(I)$, is the set of all points in the affine space that satisfy all the polynomial equations in I
- $V(I) = \{(a_1, \dots, a_n) \in k^n : f(a_1, \dots, a_n) = 0 \text{ for all } f \in I\}$, where k is the underlying field
- The ideal of an algebraic set S , denoted as $I(S)$, is the set of all polynomials that vanish on every point of S
- $I(S) = \{f \in k[x_1, \dots, x_n] : f(a_1, \dots, a_n) = 0 \text{ for all } (a_1, \dots, a_n) \in S\}$

Zariski topology and irreducible sets

- The Zariski topology on the affine space is defined by taking algebraic sets as the closed sets, establishing a correspondence between ideals and closed sets
- The closure of a set in the Zariski topology is the smallest algebraic set containing it, and it corresponds to the radical of the ideal of the set
- Irreducible algebraic sets, which cannot be written as the union of two proper subsets, correspond to prime ideals in the polynomial ring
- An algebraic set is irreducible if and only if its ideal is a prime ideal

Dimension and local properties

- The dimension of an algebraic set is related to the height of its corresponding prime ideal, providing a way to study the geometric properties of the set
- The dimension of an irreducible algebraic set is the transcendence degree of its function field over the base field

- The local ring at a point in an algebraic set is obtained by localizing the polynomial ring at the maximal ideal corresponding to that point, allowing for the study of local properties of the set
- The localization of a ring R at a prime ideal P , denoted as R_P , consists of elements of the form $\frac{a}{b}$, where $a \in R$ and $b \in R \setminus P$, and it captures the local behavior of the algebraic set near the point corresponding to P

Unit 2 – Affine Varieties

2.1 Definition and examples of affine varieties

Affine varieties are the building blocks of algebraic geometry, defined by polynomial equations over algebraically closed fields. They represent geometric objects like lines, curves, and surfaces, forming a bridge between algebra and geometry.

Understanding affine varieties is crucial for grasping more complex concepts in algebraic geometry. We'll explore their definition, examples, and key properties, including the Zariski topology and the correspondence between varieties and ideals.

Affine varieties and algebraic sets

Definition of affine varieties

- An affine variety is the set of solutions of a system of polynomial equations in n variables over an algebraically closed field k
- The algebraic set of an affine variety V is the set of all points $(x_1, \dots, x_n)$ in k^n that satisfy the polynomial equations defining V
- For example, the algebraic set of the affine variety defined by the equation $x^2 + y^2 = 1$ over the field of complex numbers is the unit circle in the complex plane
- An affine variety V is irreducible if it cannot be written as the union of two proper subvarieties
- Equivalently, the ideal $I(V)$ is a prime ideal

Zariski topology and ideals

- The ideal $I(V)$ of an affine variety V is the set of all polynomials $f(x_1, \dots, x_n)$ that vanish on V , meaning $f(a_1, \dots, a_n) = 0$ for all points $(a_1, \dots, a_n)$ in V
- For instance, the ideal of the affine variety defined by $x^2 + y^2 = 1$ is the ideal generated by the polynomial $x^2 + y^2 - 1$
- The Zariski topology on k^n is defined by taking the closed sets to be the affine varieties (algebraic sets)
- In this topology, the closure of a set is the smallest affine variety containing it
- The Zariski topology is coarser than the usual Euclidean topology on k^n

Examples of affine varieties

Affine varieties in low dimensions

- In the affine plane (dimension 2), examples of affine varieties include lines, parabolas, ellipses, and hyperbolas
- The line $y = mx + b$ is an affine variety defined by the linear equation $y - mx - b = 0$
- The parabola $y = x^2$ is an affine variety defined by the quadratic equation $y - x^2 = 0$

- In 3-dimensional affine space, examples include planes, spheres, ellipsoids, and cubic surfaces
- The plane $ax + by + cz + d = 0$ is an affine variety defined by a linear equation
- The sphere $x^2 + y^2 + z^2 = r^2$ is an affine variety defined by a quadratic equation

Trivial and compound affine varieties

- The empty set and the entire affine space k^n are both trivial examples of affine varieties in any dimension n
- The intersection of two affine varieties is always an affine variety
- For example, the intersection of the plane $z = 0$ and the sphere $x^2 + y^2 + z^2 = 1$ is the unit circle in the XY -plane
- The union of two affine varieties is an affine variety if and only if one is contained in the other

Ideals and affine varieties

Correspondence between affine varieties and radical ideals

- There is a one-to-one correspondence (bijection) between affine varieties in k^n and radical ideals in the polynomial ring $k[x_1, \dots, x_n]$
- Given an affine variety V , its ideal $I(V)$ is always a radical ideal
- Conversely, given any radical ideal I in $k[x_1, \dots, x_n]$, the set $V(I)$ of points where all polynomials in I vanish is an affine variety
- The correspondence $V \mapsto I(V)$ and $I \mapsto V(I)$ are inclusion-reversing
- If $V_1 \subseteq V_2$ then $I(V_2) \subseteq I(V_1)$
- If $I_1 \subseteq I_2$ then $V(I_2) \subseteq V(I_1)$

Hilbert's Nullstellensatz

- Hilbert's Nullstellensatz states that if k is algebraically closed, the correspondence between affine varieties and radical ideals is a bijection
- In other words, every radical ideal is the ideal of some affine variety, and every affine variety is the zero set of some radical ideal
- This fundamental theorem establishes a deep connection between algebraic geometry and commutative algebra

Dimension and degree of affine varieties

Dimension of affine varieties

- The dimension of an affine variety V is the Krull dimension of its coordinate ring $k[V] = k[x_1, \dots, x_n]/I(V)$, which is the supremum of the lengths of chains of prime ideals in $k[V]$
- Intuitively, the dimension measures the number of independent directions in which V extends
- The dimension of V is equal to the number of independent variables in a set of equations defining V , after eliminating redundant variables

- For example, the parabola $y = x^2$ has dimension 1, since it can be described by a single independent variable x

Degree of affine varieties

- If V is an affine variety of dimension d in k^n , then almost all linear subspaces of k^n of codimension d intersect V in a finite number of points, called the degree of V
- The degree measures the size or complexity of V
- For instance, a line has degree 1, a quadratic curve has degree 2, and a cubic surface has degree 3
- The degree of V is equal to the leading coefficient of the Hilbert polynomial of $k[V]$, which describes the dimension of the vector space of polynomials on V of a given degree
- The Hilbert polynomial encodes important information about the structure of V and its coordinate ring

2.2 Zariski topology on affine space

The Zariski topology on affine space is a key concept in algebraic geometry. It defines closed sets as zero sets of polynomials, creating a unique framework for studying geometric objects algebraically.

This topology bridges algebra and geometry, allowing us to analyze varieties using polynomial equations. It's less intuitive than other topologies but provides powerful tools for understanding the structure of algebraic sets.

Zariski Topology on Affine Space

Definition and Closed Sets

- The Zariski topology on affine n -space over a field k , denoted $A^n(k)$, is defined by specifying the closed sets
- A subset V of $A^n(k)$ is closed in the Zariski topology if and only if it is the set of common zeros of a collection of polynomials in $k[x_1, \dots, x_n]$
- In other words, V is closed if and only if there exists a set $S \subseteq k[x_1, \dots, x_n]$ such that $V = V(S) = \{(a_1, \dots, a_n) \in A^n(k) : f(a_1, \dots, a_n) = 0 \text{ for all } f \in S\}$
- Example: In $A^2(\mathbb{R})$, the set $V = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is closed as it is the zero set of the polynomial $f(x, y) = x^2 + y^2 - 1$

Properties of Closed Sets

- The empty set and the whole space $A^n(k)$ are both closed in the Zariski topology, as they are the zero sets of the polynomials 1 and 0, respectively
- The Zariski topology is a topology, meaning it satisfies the axioms of a topological space
- The empty set and the whole space are closed
- Arbitrary intersections of closed sets are closed
- Finite unions of closed sets are closed
- Example: In $A^2(\mathbb{C})$, the union of the zero sets of X and Y , $V(X) \cup V(Y) = \{(x, y) \in \mathbb{C}^2 : xy = 0\}$, is closed

Open and Closed Sets in Zariski Topology

Open Sets and Basic Open Sets

- A subset U of $A^n(k)$ is open in the Zariski topology if and only if its complement $A^n(k) \setminus U$ is closed
- Basic open sets in the Zariski topology are sets of the form $D(f) = \{(a_1, \dots, a_n) \in A^n(k) : f(a_1, \dots, a_n) \neq 0\}$, where f is a polynomial in $k[x_1, \dots, x_n]$
- The basic open set $D(f)$ is the complement of the closed set $V(f) = \{(a_1, \dots, a_n) \in A^n(k) : f(a_1, \dots, a_n) = 0\}$
- Example: In $A^1(\mathbb{R})$, the basic open set $D(x) = \{x \in \mathbb{R} : x \neq 0\}$ is the complement of the closed set $V(x) = \{0\}$
- Every open set in the Zariski topology is a union of basic open sets

Relationship between Open and Closed Sets

- Closed sets in the Zariski topology are finite intersections of zero sets of polynomials
- Open sets in the Zariski topology are arbitrary unions of basic open sets
- Example: In $A^2(\mathbb{Q})$, the set $U = D(x) \cup D(y) = \{(x, y) \in \mathbb{Q}^2 : xy \neq 0\}$ is open, while its complement $V = V(xy) = \{(x, y) \in \mathbb{Q}^2 : xy = 0\}$ is closed

Properties of Zariski Topology

Separation Properties

- The Zariski topology is a T0 space, meaning that for any two distinct points, there exists an open set containing one point but not the other
- The Zariski topology is not Hausdorff (T2), as two distinct points may not have disjoint open neighborhoods
- For example, in $A^1(k)$, the zero sets of x and $x - 1$ are disjoint, but their complements (the basic open sets $D(x)$ and $D(x - 1)$) intersect
- This property distinguishes the Zariski topology from other common topologies, such as the Euclidean topology

Noetherian Property and Irreducible Varieties

- The Zariski topology is Noetherian, meaning that every descending chain of closed subsets stabilizes after finitely many steps
- This property is a consequence of the Hilbert Basis Theorem, which states that $k[x_1, \dots, x_n]$ is a Noetherian ring
- The Noetherian property has important implications for the structure of affine varieties and their subvarieties
- Irreducible closed sets in the Zariski topology are called irreducible varieties
- A closed set is irreducible if it cannot be written as the union of two proper closed subsets

- Example: In $A^2(\mathbb{C})$, the zero set of $y - x^2$, $V(y - x^2) = \{(x, y) \in \mathbb{C}^2 : y = x^2\}$, is an irreducible variety, while the zero set of xy , $V(xy) = \{(x, y) \in \mathbb{C}^2 : xy = 0\}$, is not irreducible as it is the union of the zero sets of x and y

Zariski Topology for Affine Varieties

Definition and Properties of Affine Varieties

- An affine variety is a closed subset of $A^n(k)$ in the Zariski topology, defined as the zero set of a collection of polynomials in $k[x_1, \dots, x_n]$
- The Zariski topology provides a natural framework for studying the geometric properties of affine varieties
- Example: The set $V = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3 - x\}$ is an affine variety in $A^2(\mathbb{C})$, as it is the zero set of the polynomial $f(x, y) = y^2 - x^3 + x$

Dimension and Morphisms

- The dimension of an affine variety can be defined using the Zariski topology and the concept of irreducible varieties
- The dimension of an irreducible variety V is the transcendence degree of its function field $k(V)$ over the base field k
- Intuitively, the dimension corresponds to the number of independent parameters needed to describe points on the variety
- Morphisms between affine varieties can be studied using the Zariski topology and the notion of regular functions
- A regular function on an affine variety V is a function that can be expressed as a polynomial in the coordinate functions
- A morphism between affine varieties is a map that preserves the structure of regular functions
- Example: The projection map $\pi : V \rightarrow \mathbb{C}$, $(x, y) \mapsto x$, where $V = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3 - x\}$, is a morphism of affine varieties

Algebraic Geometry and the Zariski Topology

- The Zariski topology allows for the development of a cohesive theory of algebraic geometry, connecting the algebraic properties of polynomial rings with the geometric properties of affine varieties
- Many important results in algebraic geometry, such as the Nullstellensatz and the correspondence between ideals and varieties, rely on the properties of the Zariski topology
- The study of the Zariski topology on affine space serves as a foundation for more advanced topics in algebraic geometry, such as the study of projective varieties, schemes, and sheaves

2.3 Irreducibility and decomposition

Irreducibility and decomposition are key concepts in understanding affine varieties. They help us break down complex geometric objects into simpler parts, making them easier to study and analyze.

By examining whether a variety can be split into smaller pieces, we gain insights into its structure and properties. This knowledge is crucial for solving equations, classifying varieties, and exploring their

geometric features.

Irreducibility of Affine Varieties

Definition and Characterization

- An affine variety V is irreducible if it cannot be expressed as the union of two proper closed subsets
- Equivalently, an affine variety V is irreducible if its coordinate ring $A(V)$ has no zero divisors, meaning $A(V)$ is an integral domain
- The empty set and a single point (origin in $\mathbb{A}^n$) are considered irreducible affine varieties
- The Zariski closure of any irreducible subset of an affine space is an irreducible affine variety (irreducible algebraic set in $\mathbb{A}^n$)
- The irreducibility of an affine variety is equivalent to the primality of its ideal in the corresponding polynomial ring ($I(V)$ is prime in $k[x_1, \dots, x_n]$)

Properties and Examples

- An affine variety V is irreducible if and only if its coordinate ring $A(V)$ is an integral domain
- Example: The affine line $\mathbb{A}^1$ is irreducible because its coordinate ring $k[x]$ is an integral domain
- The graph of a non-constant polynomial function $f(x_1, \dots, x_n)$ is an irreducible affine variety in $\mathbb{A}^{n+1}$
- Example: The parabola $V(y - x^2)$ in $\mathbb{A}^2$ is irreducible
- The union of two distinct irreducible affine varieties is reducible (not irreducible)
- Example: The union of the coordinate axes $V(xy)$ in $\mathbb{A}^2$ is reducible

Determining Irreducibility

Algebraic Techniques

- To determine the irreducibility of an affine variety V , examine its defining ideal $I(V)$ in the polynomial ring
- If $I(V)$ is a prime ideal, then V is irreducible. Conversely, if V is irreducible, then $I(V)$ is a prime ideal
- Use the Nullstellensatz to relate the irreducibility of V to the primality of $I(V)$
- Apply algebraic techniques to check the primality of $I(V)$
- Gröbner bases can be used to compute a reduced Gröbner basis of $I(V)$ and check if it generates a prime ideal
- Primary decomposition can be used to decompose $I(V)$ into primary components and check if there is only one minimal prime component

Geometric Intuition

- Geometric intuition can sometimes be used to argue for the irreducibility or reducibility of an affine variety

- Example: The union of two distinct parallel lines in $\mathbb{A}^2$ is reducible because it can be decomposed into two irreducible components
- Example: The cone $V(x^2 + y^2 - z^2)$ in $\mathbb{A}^3$ is irreducible because it cannot be decomposed into proper closed subsets
- Visualizing the affine variety and its potential decompositions can provide insights into its irreducibility
- Example: The circle $V(x^2 + y^2 - 1)$ in $\mathbb{A}^2$ is irreducible because it forms a single connected component

Decomposition into Irreducible Components

Existence and Uniqueness

- Every affine variety V can be uniquely decomposed into a finite union of irreducible closed subsets, called its irreducible components
- The irreducible components of V are the maximal irreducible closed subsets of V with respect to inclusion
- The decomposition of V into irreducible components is unique up to reordering
- The dimension of V is equal to the maximum of the dimensions of its irreducible components
- Example: The dimension of the union of a plane and a line in $\mathbb{A}^3$ is 2, the dimension of the plane component

Correspondence with Prime Ideals

- The irreducible components of V correspond to the minimal prime ideals containing $I(V)$ in the polynomial ring
- Each minimal prime ideal $P_i \supset I(V)$ defines an irreducible component $V(P_i)$ of V
- The decomposition of V into irreducible components corresponds to the primary decomposition of $I(V)$ into minimal prime components
- The prime ideals associated with the irreducible components can be used to study their properties and intersections
- Example: The ideal $I(V) = \langle xy, xz \rangle$ in $k[x, y, z]$ has two minimal prime components $\langle x \rangle$ and $\langle y, z \rangle$, corresponding to the irreducible components $V(x)$ and $V(y, z)$

Finding Irreducible Components

Algebraic Methods

- To find the irreducible components of an affine variety V , decompose its defining ideal $I(V)$ into primary components
- Use primary decomposition algorithms to find the minimal prime ideals containing $I(V)$
- Each minimal prime ideal corresponds to an irreducible component of V
- The irreducible components can be described by their defining prime ideals or by their generating sets of polynomials

- Example: The irreducible components of $V(xy, xz)$ in $\mathbb{A}^3$ are $V(x)$ and $V(y, z)$, defined by the prime ideals $\langle x \rangle$ and $\langle y, z \rangle$

Geometric Visualization

- Geometrically, sketch the irreducible components to visualize the decomposition of V
- Example: The affine variety $V(xy, xz)$ in $\mathbb{A}^3$ can be visualized as the union of the YZ -plane and the X -axis
- Apply the concept of dimension to determine the highest-dimensional irreducible components of V
- Example: In the decomposition of $V(xy, x(z - 1))$ in $\mathbb{A}^3$, the component $V(x)$ (the YZ -plane) has dimension 2, while the component $V(y, z - 1)$ (a line parallel to the X -axis) has dimension 1

2.4 Coordinate rings of affine varieties

Coordinate rings are the mathematical backbone of affine varieties. They're like a secret code that unlocks the mysteries of these geometric objects. By studying these rings, we can uncover important properties and relationships within affine varieties.

Think of coordinate rings as a bridge between algebra and geometry. They allow us to translate geometric problems into algebraic ones, making it easier to analyze and solve complex issues in affine varieties. It's a powerful tool that connects different areas of math.

Coordinate Rings of Affine Varieties

Definition and Properties

- The coordinate ring of an affine variety V , denoted by $A(V)$, is the ring of regular functions on V
- Regular functions on an affine variety V are functions that can be expressed as polynomials in the coordinate variables $(x_1, \dots, x_n)$
- The coordinate ring $A(V)$ is the quotient ring $k[x_1, \dots, x_n]/I(V)$, where $I(V)$ is the ideal of polynomials vanishing on V
- The elements of $A(V)$ are equivalence classes of polynomials, where two polynomials are equivalent if their difference lies in $I(V)$
- The coordinate ring $A(V)$ is an integral domain if and only if V is an irreducible variety
- An irreducible variety cannot be expressed as the union of two proper subvarieties
- The dimension of the coordinate ring $A(V)$ is equal to the dimension of the affine variety V
- The dimension of an affine variety is the maximum length of a chain of irreducible subvarieties

Examples

- Consider the affine variety $V = V(y - x^2) \subseteq \mathbb{A}^2$. The coordinate ring $A(V)$ is isomorphic to $k[x]$, the polynomial ring in one variable
- For the affine space $\mathbb{A}^n$, the coordinate ring $A(\mathbb{A}^n)$ is isomorphic to the polynomial ring $k[x_1, \dots, x_n]$

Coordinate Rings vs Polynomial Rings

Relationship

- The coordinate ring $A(V)$ is a quotient ring of the polynomial ring $k[x_1, \dots, x_n]$
- The quotient map $k[x_1, \dots, x_n] \rightarrow A(V)$ sends a polynomial to its equivalence class in the coordinate ring
- The ideal $I(V)$ used to define the coordinate ring consists of all polynomials that vanish on the affine variety V
- A polynomial f vanishes on V if $f(p) = 0$ for all points $p \in V$
- The coordinate ring $A(V)$ inherits the grading from the polynomial ring $k[x_1, \dots, x_n]$, making it a graded ring
- The grading is defined by the degree of polynomials: $A(V) = \bigoplus_{n=0}^{\infty} A(V)_n$, where $A(V)_n$ consists of equivalence classes of polynomials of degree n

Nullstellensatz

- The Nullstellensatz establishes a correspondence between radical ideals in $k[x_1, \dots, x_n]$ and affine varieties in k^n
- Every radical ideal $I \subseteq k[x_1, \dots, x_n]$ is the ideal of an affine variety $V(I)$
- Every affine variety $V \subseteq k^n$ is the zero set of a radical ideal $I(V) \subseteq k[x_1, \dots, x_n]$

Computing Coordinate Rings

Determining the Ideal

- To compute the coordinate ring $A(V)$, first determine the ideal $I(V)$ of polynomials vanishing on V
- Express the affine variety V as the zero set of a collection of polynomials $f_1, \dots, f_m$ in $k[x_1, \dots, x_n]$
- The ideal $I(V)$ is the radical of the ideal generated by the polynomials $f_1, \dots, f_m$
- The radical of an ideal I , denoted by $\sqrt{I}$, is the set of all polynomials f such that $f^n \in I$ for some integer $n \geq 1$

Computing the Quotient Ring

- Compute the quotient ring $k[x_1, \dots, x_n]/I(V)$ to obtain the coordinate ring $A(V)$
- The elements of the quotient ring are equivalence classes of polynomials, where two polynomials are equivalent if their difference lies in $I(V)$
- Use Gröbner basis techniques to simplify the computation of the quotient ring and its elements
- A Gröbner basis is a special generating set of an ideal that allows for efficient computation in the quotient ring
- Gröbner bases can be computed using algorithms like Buchberger's algorithm or the F4/F5 algorithms

Examples

- For the affine variety $V = V(y^2 - x^3 - x) \subseteq \mathbb{A}^2$, the ideal $I(V)$ is generated by the polynomial $y^2 - x^3 - x$

- The coordinate ring $A(V)$ is isomorphic to $k[x, y]/(y^2 - x^3 - x)$
- Consider the affine variety $V = V(x^2 + y^2 - 1) \subseteq A^2$. The ideal $I(V)$ is generated by the polynomial $x^2 + y^2 - 1$
- The coordinate ring $A(V)$ is isomorphic to $k[x, y]/(x^2 + y^2 - 1)$

Coordinate Rings for Studying Affine Varieties

Correspondence with Points and Subvarieties

- The Hilbert Nullstellensatz relates the maximal ideals of $A(V)$ to the points of the affine variety V
- Every maximal ideal of $A(V)$ corresponds to a point of V , and every point of V corresponds to a maximal ideal of $A(V)$
- The prime ideals of $A(V)$ correspond to the irreducible subvarieties of V
- An ideal $I \subseteq A(V)$ is prime if and only if $V(I)$ is an irreducible subvariety of V
- The height of a prime ideal in $A(V)$ equals the codimension of the corresponding irreducible subvariety

Regular Functions and Local Properties

- The units in the coordinate ring $A(V)$ are precisely the non-vanishing regular functions on V
- A regular function $f \in A(V)$ is a unit if and only if $f(p) \neq 0$ for all points $p \in V$
- The coordinate ring $A(V)$ can be used to study the local properties of V , such as the tangent space and the local ring at a point
- The tangent space at a point $p \in V$ is the dual of the maximal ideal corresponding to p in $A(V)$
- The local ring at a point $p \in V$ is the localization of $A(V)$ at the maximal ideal corresponding to p

Morphisms and Coordinate Rings

- Morphisms between affine varieties can be studied using homomorphisms between their coordinate rings
- A morphism $\phi: V \rightarrow W$ between affine varieties induces a homomorphism $\phi^*: A(W) \rightarrow A(V)$ between their coordinate rings
- The properties of the morphism ϕ (injectivity, surjectivity, isomorphism) are reflected in the properties of the induced homomorphism ϕ^*

Examples

- For the affine variety $V = V(y - x^2) \subseteq A^2$, the point $(a, a^2) \in V$ corresponds to the maximal ideal $(x - a, y - a^2) \subseteq A(V)$
- Consider the affine varieties $V = V(y^2 - x^3 - x)$ and $W = V(y - x^2)$. The morphism $\phi: V \rightarrow W$ given by $(x, y) \mapsto (x, y^2)$ induces a homomorphism $\phi^*: A(W) \rightarrow A(V)$ that sends x to x and y to y^2

Unit 3 – Projective Varieties

3.1 Projective space and homogeneous coordinates

Projective space and homogeneous coordinates are key concepts in algebraic geometry. They extend Euclidean space by adding points at infinity, allowing parallel lines to intersect. This unified framework simplifies geometric problems and provides a powerful tool for studying algebraic varieties.

Homogeneous coordinates represent points in projective space using tuples defined up to scalar multiples. This system elegantly handles points at infinity and enables the study of projective transformations, which preserve incidence relations between points and lines in projective space.

Projective space

Extension of Euclidean space

- Projective space extends Euclidean space by including points at infinity
- Allows for the representation of parallel lines intersecting at infinity
- In projective space, two parallel lines always intersect at a unique point called a point at infinity or an ideal point (line at infinity)
- Projective space has the property that any two distinct points determine a unique line, and any two distinct lines intersect at a unique point, possibly at infinity (duality)

Projective plane and transformations

- The projective plane, denoted as RP^2 , is a two-dimensional projective space
- Can be visualized as a Euclidean plane with additional points at infinity
- Projective transformations, such as projections and perspectivities, preserve the incidence relations between points and lines in projective space
- Examples of projective transformations include homographies and collineations
- Projective geometry studies the properties of figures that are invariant under projective transformations

Homogeneous coordinates

Representing points in projective space

- Homogeneous coordinates are a coordinate system used to represent points in projective space
- A point is represented by a tuple of numbers $(x_0, x_1, \dots, x_n)$ defined up to a non-zero scalar multiple
- In the projective plane (RP^2), a point is represented by a triple (x_0, x_1, x_2) , where at least one coordinate is non-zero
- Two sets of homogeneous coordinates $(x_0, x_1, \dots, x_n)$ and $(\lambda x_0, \lambda x_1, \dots, \lambda x_n)$ represent the same point in projective space for any non-zero scalar λ

Lines and points at infinity

- Points at infinity in projective space are represented by homogeneous coordinates where the last coordinate (x_n) is equal to zero
- Lines in the projective plane are represented by linear equations in homogeneous coordinates, such as $ax_0 + bx_1 + cx_2 = 0$
- The line at infinity in the projective plane is given by the equation $x_2 = 0$
- The intersection of a line with the line at infinity determines its point at infinity, representing its direction

Homogeneous vs affine coordinates

Conversion between coordinate systems

- Affine coordinates are the standard Cartesian coordinates used in Euclidean space, while homogeneous coordinates are used in projective space
- To convert from homogeneous coordinates $(x_0, x_1, \dots, x_n)$ to affine coordinates $(y_1, y_2, \dots, y_n)$, divide each coordinate by the last non-zero coordinate (usually x_n): $y_i = x_i/x_n$ for $i = 1, 2, \dots, n-1$
- To convert from affine coordinates $(y_1, y_2, \dots, y_n)$ to homogeneous coordinates $(x_0, x_1, \dots, x_n)$, append a 1 as the last coordinate: $(x_0, x_1, \dots, x_n) = (y_1, y_2, \dots, y_n, 1)$

Limitations of affine coordinates

- Points at infinity in projective space cannot be represented using affine coordinates, as they correspond to homogeneous coordinates with $x_n = 0$
- Affine coordinates are not well-suited for representing the intersection of parallel lines or the direction of lines
- Homogeneous coordinates provide a more unified framework for dealing with points at infinity and projective transformations

Points at infinity

Definition and properties

- Points at infinity, also called ideal points, are additional points added to Euclidean space to create projective space
- Allow parallel lines to intersect, providing a more unified treatment of geometric concepts
- In the projective plane, points at infinity lie on a special line called the line at infinity, which is the set of all points with homogeneous coordinates $(x_0, x_1, 0)$
- Each set of parallel lines in Euclidean space corresponds to a unique point at infinity in projective space, representing their common direction

Significance in projective geometry

- The inclusion of points at infinity in projective space allows for a more unified treatment of geometric concepts, such as the intersection of lines and the transformation of figures

- Points at infinity play a crucial role in the study of projective geometry and its applications, such as computer vision and computer graphics
- Understanding points at infinity is essential for working with projective transformations and analyzing the behavior of geometric objects in projective space
- Examples of applications include perspective projection, camera calibration, and 3D reconstruction from 2D images

3.2 Projective varieties and their properties

Projective varieties are the geometric objects at the heart of algebraic geometry. They're defined as zero sets of homogeneous polynomials in projective space, giving us a powerful way to study geometric shapes using algebraic tools.

These varieties have fascinating properties like dimension, degree, and singularities. We can break them down into irreducible pieces, study their intersections, and even define a special ring structure called the Chow ring to capture their geometry.

Projective varieties and ideals

Definition and ideal-theoretic description

- A projective variety is a subset of projective space that is the zero locus of a collection of homogeneous polynomials
- The ideal generated by the homogeneous polynomials that vanish on a projective variety is called the defining ideal of the variety
- The projective coordinate ring of a projective variety is the quotient ring of the polynomial ring by the defining ideal of the variety

Topology and irreducibility

- The Zariski topology on projective space is defined by taking projective varieties as the closed sets
- A projective variety is irreducible if it cannot be written as the union of two proper subvarieties (i.e., it is not the union of two smaller projective varieties)

Properties of projective varieties

Dimension, degree, and singularities

- The dimension of a projective variety is the maximum length of a chain of irreducible subvarieties, minus one
- The degree of a projective variety is the number of points in its intersection with a general linear subspace of complementary dimension (i.e., a linear subspace whose dimension plus the dimension of the variety equals the dimension of the ambient projective space)
- A point on a projective variety is singular if the Jacobian matrix of the defining polynomials has rank less than the codimension of the variety at that point
- The codimension is the difference between the dimension of the ambient projective space and the dimension of the variety

- The singular locus of a projective variety is the set of all singular points, which forms a subvariety of lower dimension
- The tangent space at a smooth point of a projective variety is the kernel of the Jacobian matrix at that point

Decomposition into irreducible components

- Every projective variety can be uniquely decomposed as a finite union of irreducible components
- The irreducible components of a projective variety correspond to the minimal prime ideals of its defining ideal
- The dimension of a reducible projective variety is the maximum of the dimensions of its irreducible components

Irreducibility vs Reducibility

Definitions and examples

- A projective variety is irreducible if it cannot be written as the union of two proper subvarieties
- Example: the projective curve defined by the equation $y^2z = x^3 - xz^2$ (an elliptic curve) is irreducible
- A projective variety is reducible if it can be written as the union of two proper subvarieties
- Example: the projective curve defined by the equation $XY = 0$ is reducible, as it is the union of the lines $x = 0$ and $y = 0$

Relationship to prime ideals

- The irreducible components of a projective variety correspond to the minimal prime ideals of its defining ideal
- A prime ideal is an ideal P such that if $ab \in P$, then either $a \in P$ or $b \in P$
- A minimal prime ideal is a prime ideal that does not contain any smaller prime ideals

Intersection of projective varieties

Bézout's theorem and intersection multiplicity

- Bézout's theorem states that the intersection of two projective varieties of complementary dimension consists of a finite number of points, counted with multiplicity
- The multiplicity of a point in the intersection of two projective varieties can be computed using the Hilbert polynomial of the local ring at that point
- The Hilbert polynomial measures the growth of the dimensions of the homogeneous components of the local ring
- The intersection multiplicity satisfies certain axioms, such as being additive on unions and multiplicative on products

Intersection product and Chow ring

- The intersection product of two subvarieties of a projective variety is a cycle that represents their set-theoretic intersection with multiplicities
- A cycle is a formal linear combination of subvarieties with integer coefficients
- The Chow ring of a projective variety is the ring generated by the classes of subvarieties modulo rational equivalence, with the intersection product as multiplication
- Two cycles are rationally equivalent if they can be connected by a family of cycles parametrized by a rational curve

3.3 Relationship between affine and projective varieties

Affine and projective varieties are closely linked in algebraic geometry. Homogenization transforms affine varieties into projective ones, adding points at infinity and preserving geometric properties. This process allows us to study affine varieties in a more complete setting.

Projective varieties can be seen as completions of affine ones. By using affine cones, patches, and dehomogenization, we can switch between these two perspectives. This relationship helps us understand both local and global properties of algebraic varieties.

Homogenization of Polynomials

Converting Affine Varieties to Projective Varieties

- Homogenization converts an affine variety into a projective variety by introducing an additional variable to each term of the defining polynomial(s)
- The additional variable makes all terms have the same degree
- To homogenize a polynomial $f(x_1, \dots, x_n)$ of degree d , multiply each term by the homogenizing variable x_0 raised to the power of d minus the degree of the term
- For example, homogenizing the polynomial $f(x, y) = x^2 + xy + y$ of degree 2 results in $F(x_0, x, y) = x^2 + x_0xy + x_0^2y$

Properties of Homogenized Varieties

- The resulting homogeneous polynomial $F(x_0, x_1, \dots, x_n)$ defines a projective variety in $\mathbb{P}^n$
- The original affine variety is embedded as the subset where $x_0 \neq 0$
- Homogenization preserves the geometric properties of the affine variety
- Homogenization allows for the study of the affine variety's behavior "at infinity" in the projective space

Affine vs Projective Varieties

Embedding Affine Varieties into Projective Spaces

- Every affine variety can be embedded into a projective space by homogenizing its defining polynomials
- The resulting projective variety contains the original affine variety as a dense open subset

- The affine variety can be recovered from the projective variety by dehomogenizing the defining polynomials and considering the subset where the homogenizing variable is non-zero

Projective Varieties as Completions of Affine Varieties

- Projective varieties can be thought of as completions of affine varieties
- Projective varieties add points "at infinity" to capture the limiting behavior of the affine variety
- The dimension of a projective variety is one less than the dimension of its associated affine cone
- The affine cone is formed by taking the union of all scalar multiples of points in the projective variety

Affine Cone and Patch

Affine Cone

- The affine cone of a projective variety $V \subset \mathbb{P}^n$ is the preimage of V under the projection map from $\mathbb{A}^{n+1} \setminus \{0\}$ to $\mathbb{P}^n$
- The affine cone is the set of all points $(x_0, x_1, \dots, x_n)$ in $\mathbb{A}^{n+1}$ such that $[x_0 : x_1 : \dots : x_n] \in V$
- The affine cone is a union of lines through the origin in $\mathbb{A}^{n+1}$
- The defining equations of the affine cone are the homogeneous polynomials that define the projective variety

Affine Patch

- An affine patch of a projective variety V is the intersection of V with the subset of $\mathbb{P}^n$ where one of the homogeneous coordinates is non-zero
- For example, $U_i = \{[x_0 : x_1 : \dots : x_n] \in V \mid x_i \neq 0\}$
- Each affine patch is isomorphic to an affine variety
- The affine variety is obtained by dehomogenizing the defining polynomials of V with respect to the non-zero coordinate
- The affine patches form an open cover of the projective variety
- Affine patches allow for the study of local properties using affine coordinates

Dehomogenization and Homogenization

Dehomogenization

- Dehomogenization converts a projective variety into an affine variety
- To dehomogenize, set one of the homogeneous coordinates to 1 and consider the resulting affine equations in the remaining variables
- To dehomogenize a homogeneous polynomial $F(x_0, x_1, \dots, x_n)$ with respect to x_i , set $x_i = 1$ and simplify the equation
- The simplified equation is a non-homogeneous polynomial $f(x_0, \dots, \hat{x}_i, \dots, x_n)$, where $\hat{x}_i$ denotes the omission of x_i

- The dehomogenized polynomial f defines an affine variety, which is isomorphic to the affine patch of the projective variety where $x_i \neq 0$

Relationship between Homogenization and Dehomogenization

- Homogenization and dehomogenization are inverse processes
- These processes allow for the conversion between affine and projective varieties while preserving the underlying geometric structure
- The choice of the coordinate for dehomogenization (or the homogenizing variable) affects the specific affine patch obtained
- For example, dehomogenizing with respect to x_0 yields the affine patch U_0 , while dehomogenizing with respect to x_1 yields U_1
- The resulting affine varieties from different dehomogenizations are isomorphic and provide equivalent local descriptions of the projective variety

3.4 Projective closure and homogenization

Projective closure and homogenization are key concepts in bridging affine and projective geometry. They allow us to extend affine varieties into projective space, adding points at infinity to study their global structure and behavior.

By homogenizing equations, we can transform affine varieties into projective ones. This process reveals hidden properties, like asymptotes and singularities, and enables us to apply powerful tools from projective geometry to analyze affine varieties more comprehensively.

Projective closure of an affine variety

Definition and properties

- The projective closure of an affine variety V is the smallest projective variety $\bar{V}$ containing V as a dense open subset
- Obtained by embedding the affine variety into projective space and taking the closure in the Zariski topology
- Adds points at infinity to the original affine variety, completing it in a sense (compactification)
- The projective closure is unique up to isomorphism and is independent of the choice of embedding
- Allows for the study of the global structure and behavior of the affine variety

Relationship between affine and projective varieties

- Every affine variety can be embedded into a projective space, and its projective closure can be obtained
- The original affine variety is a dense open subset of its projective closure
- The complement of the affine variety in its projective closure consists of the points at infinity
- The projective closure provides a natural way to compactify an affine variety and study its behavior at infinity

Homogenization of defining equations

Process of homogenization

- To compute the projective closure, first homogenize the defining equations of the affine variety by introducing a new variable X_0
- Homogenization is done by multiplying each monomial in the equations by an appropriate power of X_0 to make all terms have the same total degree
- The resulting homogeneous equations define a projective variety, which is the projective closure of the original affine variety
- The original affine variety can be recovered by setting $X_0 = 1$ in the homogeneous equations (dehomogenization)

Examples of homogenization

- For the affine variety defined by $x^2 + y^2 = 1$, the homogenized equation is $x^2 + y^2 = z^2$
- The affine variety defined by $xy = 1$ has the homogenized equation $xy = z^2$
- The affine variety defined by $y = x^2$ has the homogenized equation $yz = x^2$

Geometric meaning of points at infinity

Interpretation of points at infinity

- The points at infinity added in the projective closure correspond to the directions in which the affine variety "goes off to infinity"
- These points at infinity can be thought of as the limit points of the affine variety when viewed from increasingly distant vantage points
- The behavior of the variety near the points at infinity can provide information about its global structure and asymptotic properties
- The points at infinity can be studied by looking at the intersection of the projective closure with the hyperplane at infinity defined by $X_0 = 0$

Examples of points at infinity

- For the affine variety defined by $y = x^2$, there is a single point at infinity $[0 : 1 : 0]$, corresponding to the vertical direction
- The affine variety defined by $xy = 1$ has two points at infinity: $[1 : 0 : 0]$ and $[0 : 1 : 0]$, corresponding to the horizontal and vertical directions
- The affine variety defined by $x^2 + y^2 = 1$ has no real points at infinity, but has two complex points at infinity $[i : 1 : 0]$ and $[-i : 1 : 0]$

Homogenization for studying varieties at infinity

Applications of homogenization

- Homogenization allows for the study of the behavior of affine varieties near their points at infinity

- By analyzing the homogeneous defining equations, one can determine the multiplicity and structure of the points at infinity
- The intersection multiplicity of the projective closure with the hyperplane at infinity can provide information about the singularities and tangent directions of the variety at infinity
- Homogenization techniques can be used to study the degree and genus of projective curves, as well as the intersection theory of projective varieties

Importance of studying varieties at infinity

- The study of varieties at infinity using homogenization is particularly useful in understanding the global topology and geometry of the varieties
- The behavior at infinity can reveal important properties such as asymptotes, branches, and singularities
- Studying the points at infinity can provide insight into the overall structure and shape of the variety
- Homogenization allows for the application of powerful tools from projective geometry to study affine varieties in a more complete and unified setting

Unit 4 – Coordinate Rings

4.1 Coordinate rings of affine and projective varieties

Coordinate rings are the algebraic backbone of varieties, connecting geometry to algebra. They encode crucial information about a variety's structure, dimension, and properties, allowing us to study geometric objects using powerful algebraic techniques.

For affine varieties, the coordinate ring consists of polynomial functions. For projective varieties, we use homogeneous polynomials in the graded coordinate ring. These rings provide a bridge between the concrete geometry of varieties and abstract algebra.

Coordinate rings of varieties

Definition and structure

- The coordinate ring of an affine variety V , denoted $A(V)$, is the ring of polynomial functions on V , the set of all polynomial functions from V to the base field k
- $A(V)$ is a finitely generated k -algebra, generated by the coordinate functions on V (e.g., x, y , and z for a variety in $\mathbb{A}^3$)
- The coordinate ring $A(V)$ is isomorphic to the quotient ring $k[x_1, \dots, x_n]/I(V)$, where $I(V)$ is the ideal of polynomials vanishing on V
- This isomorphism allows for the study of the algebraic properties of $A(V)$ using the tools of commutative algebra
- Example: For the variety $V = V(xy - 1) \subset \mathbb{A}^2$, $A(V) \cong k[x, y]/(xy - 1)$

Properties and correspondences

- The Nullstellensatz establishes a correspondence between affine varieties and radical ideals in $k[x_1, \dots, x_n]$
- Every radical ideal I corresponds to a unique affine variety $V(I)$
- Every affine variety V corresponds to a unique radical ideal $I(V)$
- The dimension of an affine variety V is equal to the Krull dimension of its coordinate ring $A(V)$
- The Krull dimension is the supremum of the lengths of chains of prime ideals in $A(V)$
- Example: The variety $V = V(x^2 + y^2 - 1) \subset \mathbb{A}^2$ has dimension 1, as $A(V) \cong k[x, y]/(x^2 + y^2 - 1)$ has Krull dimension 1
- The coordinate ring $A(V)$ is an integral domain if and only if V is an irreducible variety
- An irreducible variety cannot be written as the union of two proper subvarieties
- Example: The variety $V = V(xy) \subset \mathbb{A}^2$ is reducible, as $V = V(x) \cup V(y)$, and $A(V) \cong k[x, y]/(xy)$ is not an integral domain
- The function field of an irreducible affine variety V is the field of fractions of $A(V)$
- The function field consists of rational functions on V , i.e., quotients of polynomials in $A(V)$

- Example: For the irreducible variety $V = V(y - x^2) \subset \mathbb{A}^2$, the function field is $k(V) = \text{Frac}(k[x, y]/(y - x^2))$

Homogeneous coordinate rings

Definition and grading

- The homogeneous coordinate ring of a projective variety V , denoted $S(V)$, is the graded ring of homogeneous polynomial functions on V
- $S(V)$ is a graded k -algebra, with the grading given by the degree of the homogeneous polynomials
- A polynomial $f \in k[x_0, \dots, x_n]$ is homogeneous of degree d if $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$ for all $\lambda \in k$
- Example: The polynomial $x_0^2 + x_1 x_2$ is homogeneous of degree 2
- The homogeneous coordinate ring $S(V)$ is isomorphic to the quotient ring $k[x_0, \dots, x_n]/I(V)$, where $I(V)$ is the homogeneous ideal of polynomials vanishing on V
- A homogeneous ideal is an ideal generated by homogeneous polynomials
- Example: For the projective variety $V = V(x_0 x_2 - x_1^2) \subset \mathbb{P}^2$, $S(V) \cong k[x_0, x_1, x_2]/(x_0 x_2 - x_1^2)$

Hilbert function and polynomial

- The Hilbert function of $S(V)$ encodes information about the dimensions of the graded components of $S(V)$
- The Hilbert function $h_V : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ is defined by $h_V(d) = \dim_k S(V)_d$, where $S(V)_d$ is the d -th graded component of $S(V)$
- Example: For the projective variety $V = V(x_0 x_2 - x_1^2) \subset \mathbb{P}^2$, the Hilbert function is $h_V(d) = d + 1$ for all $d \geq 0$
- The Hilbert polynomial of $S(V)$ is a polynomial that agrees with the Hilbert function for sufficiently large degrees and provides information about the dimension and degree of V
- The Hilbert polynomial $P_V(t)$ is defined as the unique polynomial such that $P_V(d) = h_V(d)$ for all $d \gg 0$
- The degree of $P_V(t)$ is equal to the dimension of V , and the leading coefficient of $P_V(t)$ is related to the degree of V
- Example: For the projective variety $V = V(x_0 x_2 - x_1^2) \subset \mathbb{P}^2$, the Hilbert polynomial is $P_V(t) = t + 1$, indicating that V has dimension 1 and degree 1

Affine vs projective rings

Embedding affine varieties into projective space

- Any affine variety V can be embedded into a projective space as a quasi-projective variety, denoted $\tilde{V}$
- The embedding is given by the map $\varphi : V \rightarrow \mathbb{P}^n, (a_1, \dots, a_n) \mapsto (1 : a_1 : \dots : a_n)$
- The image of φ is an open subset of the projective closure of V , which is the smallest projective variety containing $\varphi(V)$

- Example: The affine variety $V = V(y - x^2) \subset \mathbb{A}^2$ can be embedded into $\mathbb{P}^2$ as $\tilde{V} = V(x_1x_2 - x_0x_1) \subset \mathbb{P}^2$

Homogenization and dehomogenization

- The homogeneous coordinate ring $S(\tilde{V})$ of the projective closure $\tilde{V}$ is related to the affine coordinate ring $A(V)$ by the process of homogenization and dehomogenization of polynomials
- The affine coordinate ring $A(V)$ can be recovered from $S(\tilde{V})$ by dehomogenizing with respect to a non-vanishing homogeneous coordinate
- Dehomogenization of a homogeneous polynomial $f(x_0, \dots, x_n)$ with respect to x_0 is the polynomial $f(1, x_1, \dots, x_n)$
- Example: Dehomogenizing the polynomial $x_0x_2 - x_1^2$ with respect to x_0 yields the polynomial $x_2 - x_1^2$
- The projective coordinate ring $S(\tilde{V})$ can be obtained from $A(V)$ by homogenizing the polynomials in $A(V)$ with respect to a new variable
- Homogenization of a polynomial $f(x_1, \dots, x_n)$ of degree d with respect to x_0 is the polynomial $x_0^d f(x_1/x_0, \dots, x_n/x_0)$
- Example: Homogenizing the polynomial $y - x^2$ with respect to x_0 yields the polynomial $x_0y - x^2$

Applications of coordinate rings

Determining geometric properties

- Use the coordinate ring to determine the dimension, irreducibility, and singularities of a variety
- The dimension of a variety is equal to the Krull dimension of its coordinate ring
- A variety is irreducible if and only if its coordinate ring is an integral domain
- Singularities of a variety correspond to prime ideals in its coordinate ring that are not maximal
- Example: The variety $V = V(y^2 - x^3) \subset \mathbb{A}^2$ has a singularity at the origin, as the maximal ideal (x, y) in $A(V) \cong k[x, y]/(y^2 - x^3)$ is not a regular local ring

Computing Hilbert functions and polynomials

- Compute the Hilbert function and Hilbert polynomial of a projective variety to study its geometric properties
- The Hilbert function provides information about the dimensions of the graded components of the homogeneous coordinate ring
- The Hilbert polynomial encodes the dimension and degree of the projective variety
- Example: For the projective variety $V = V(x_0^2x_2 - x_1^3) \subset \mathbb{P}^2$, the Hilbert polynomial is $P_V(t) = 3t + 1$, indicating that V has dimension 1 and degree 3

Correspondence between ideals and varieties

- Utilize the correspondence between ideals and varieties to solve problems related to the structure of coordinate rings

- The Nullstellensatz establishes a bijective correspondence between radical ideals and affine varieties
- The projective Nullstellensatz establishes a bijective correspondence between homogeneous radical ideals and projective varieties
- Example: To find the ideal of polynomials vanishing on the affine variety $V = V(x^2 + y^2 - 1) \subset \mathbb{A}^2$, compute the radical of the ideal $(x^2 + y^2 - 1)$

Applying the Nullstellensatz

- Apply the Nullstellensatz to establish the relationship between the ideal-theoretic and geometric properties of varieties
- The Nullstellensatz states that for any ideal $I \subset k[x_1, \dots, x_n]$, $I(V(I)) = \sqrt{I}$
- The projective Nullstellensatz states that for any homogeneous ideal $I \subset k[x_0, \dots, x_n]$, $I(V(I)) = \sqrt{I}$
- Example: To show that the affine varieties $V(x^2 - y)$ and $V(x - y^2)$ in $\mathbb{A}^2$ are not isomorphic, prove that their coordinate rings are not isomorphic using the Nullstellensatz

Graded rings and projective varieties

- Use the properties of graded rings to study the homogeneous coordinate ring of a projective variety
- The homogeneous coordinate ring of a projective variety is a graded ring, with the grading given by the degree of the homogeneous polynomials
- Graded modules over the homogeneous coordinate ring correspond to coherent sheaves on the projective variety
- Example: The twisted cubic curve $V = V(x_1x_3 - x_2^2, x_0x_3 - x_1x_2, x_0x_2 - x_1^2) \subset \mathbb{P}^3$ has homogeneous coordinate ring $S(V) \cong k[x_0, x_1, x_2, x_3]/(x_1x_3 - x_2^2, x_0x_3 - x_1x_2, x_0x_2 - x_1^2)$, which is a graded ring

Switching between affine and projective settings

- Employ the techniques of homogenization and dehomogenization to switch between affine and projective settings
- Homogenization allows for the study of affine varieties using tools from projective geometry
- Dehomogenization allows for the study of projective varieties using tools from affine geometry
- Example: To find the singular points of the affine variety $V = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2$, homogenize the defining equation to obtain the projective variety $\tilde{V} = V(x_0x_2^2 - x_1^3 - x_0x_1^2) \subset \mathbb{P}^2$, find the singular points of $\tilde{V}$, and then dehomogenize to obtain the singular points of V

Function fields of irreducible varieties

- Solve problems involving the function field of an irreducible variety using the properties of the coordinate ring
- The function field of an irreducible affine variety is the field of fractions of its coordinate ring
- The function field of an irreducible projective variety is the field of fractions of the degree 0 part of its homogeneous coordinate ring

- Example: To find the genus of the smooth projective curve $V = V(x_0x_2 - x_1^2) \subset \mathbb{P}^2$, compute the dimension of the space of global sections of the canonical sheaf on V using the properties of the function field $k(V) = \text{Frac}(k[x_0, x_1, x_2]/(x_0x_2 - x_1^2))$

4.2 Prime and maximal ideals in coordinate rings

Prime and maximal ideals in coordinate rings are crucial for understanding the structure of affine varieties. They bridge the gap between algebra and geometry, allowing us to translate geometric properties into algebraic language and vice versa.

These ideals help us identify important subsets of varieties, like irreducible components and individual points. By studying them, we gain insights into the dimension, irreducibility, and local properties of varieties, which are essential for deeper algebraic geometry concepts.

Prime and maximal ideals in coordinate rings

Defining prime and maximal ideals

- A proper ideal P in a ring R is prime if for any $a, b \in R$ such that $ab \in P$, either $a \in P$ or $b \in P$
- Example: In the ring $\mathbb{Z}$, the ideal (2) is prime since if $ab \in (2)$, then either a or b must be even
- A proper ideal M in a ring R is maximal if there is no proper ideal I such that $M \subsetneq I \subsetneq R$
- Example: In the ring $\mathbb{Z}$, the ideal (5) is maximal since there is no proper ideal strictly between (5) and $\mathbb{Z}$

Correspondence with subvarieties and points

- In the coordinate ring $k[V]$ of an affine variety V , prime ideals correspond to irreducible subvarieties of V
- Example: In the coordinate ring $\mathbb{C}[x, y]/(y^2 - x^3 - x)$ of the cubic curve $V(y^2 - x^3 - x)$, the prime ideal (x, y) corresponds to the irreducible subvariety consisting of the origin
- Maximal ideals in $k[V]$ correspond to points of the affine variety V
- Example: In the coordinate ring $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ of the unit circle, the maximal ideal $(x - 1, y)$ corresponds to the point $(1, 0)$ on the circle
- The maximal ideals in the coordinate ring $k[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$
- Example: In $\mathbb{C}[x, y]$, the maximal ideal $(x - 2, y + 3)$ corresponds to the point $(2, -3)$ in $\mathbb{C}^2$

Spectrum of a coordinate ring

Definition and notation

- The spectrum $\text{Spec}(R)$ of a ring R is the set of all prime ideals of R
- The spectrum of a coordinate ring $k[V]$ is denoted by $\text{Spec}(k[V])$ and consists of all prime ideals in $k[V]$
- Example: For the coordinate ring $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ of the unit circle, $\text{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 - 1))$ contains prime ideals like (0) and $(x - 1, y)$

Zariski topology

- The spectrum $\text{Spec}(k[V])$ can be given the Zariski topology, where closed sets are defined by ideals in $k[V]$
- The Zariski topology on $\text{Spec}(k[V])$ is defined by taking the closed sets to be $V(I) = \{P \in \text{Spec}(k[V]) : I \subseteq P\}$ for ideals I in $k[V]$
- The Zariski topology makes $\text{Spec}(k[V])$ into a topological space
- There is a one-to-one correspondence between irreducible closed subsets of $\text{Spec}(k[V])$ and prime ideals in $k[V]$
- Example: In $\text{Spec}(\mathbb{C}[x, y])$, the irreducible closed subset $V(x - 1, y)$ corresponds to the prime ideal $(x - 1, y)$
- The maximal ideals in $\text{Spec}(k[V])$ are closed points in the Zariski topology
- Example: In $\text{Spec}(\mathbb{R}[x])$, the maximal ideal $(x - 2)$ is a closed point

Ideals and subvarieties

Correspondence between ideals and subvarieties

- Every ideal I in the coordinate ring $k[V]$ defines a subvariety $V(I) = \{x \in V : f(x) = 0 \text{ for all } f \in I\}$ of the affine variety V
- Example: In $\mathbb{C}[x, y]$, the ideal $I = (x^2 + y^2 - 1)$ defines the unit circle $V(I) = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 1\}$
- Conversely, every subvariety W of V defines an ideal $I(W) = \{f \in k[V] : f(x) = 0 \text{ for all } x \in W\}$ in $k[V]$
- Example: The parabola $W = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$ defines the ideal $I(W) = (y - x^2)$ in $\mathbb{R}[x, y]$
- The correspondence between ideals and subvarieties reverses inclusions: if $I \subseteq J$ are ideals in $k[V]$, then $V(J) \subseteq V(I)$
- Example: In $\mathbb{C}[x, y]$, $(x) \subseteq (x, y)$, so $V(x, y) = \{(0, 0)\} \subseteq V(x) = \{(0, y) : y \in \mathbb{C}\}$

Irreducibility and dimension

- The subvariety $V(I)$ is irreducible if and only if the ideal I is prime
- Example: The ideal $(x^2 - y^2) = (x - y)(x + y)$ in $\mathbb{C}[x, y]$ is not prime, so $V(x^2 - y^2)$ is reducible
- The dimension of a subvariety $V(P)$ defined by a prime ideal P is equal to the Krull dimension of the coordinate ring $k[V]/P$
- Example: The dimension of the parabola $V(y - x^2)$ in $\mathbb{C}^2$ is equal to the Krull dimension of $\mathbb{C}[x, y]/(y - x^2)$, which is 1

Properties of prime and maximal ideals

Nullstellensatz and irreducible components

- The Nullstellensatz establishes a correspondence between maximal ideals in $k[V]$ and points of the affine variety V
- The Weak Nullstellensatz states that if k is algebraically closed, then every maximal ideal in $k[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$
- The Strong Nullstellensatz states that if k is algebraically closed and I is an ideal in $k[x_1, \dots, x_n]$, then $I(V(I)) = \sqrt{I}$
- The prime ideals in $k[V]$ determine the irreducible components of the affine variety V
- Example: The irreducible components of $V(xy)$ in $\mathbb{C}^2$ are $V(x)$ and $V(y)$, corresponding to the prime ideals (x) and (y) in $\mathbb{C}[x, y]$

Height, dimension, and localization

- The height of a prime ideal P in $k[V]$ is equal to the codimension of the subvariety $V(P)$ in V
- Example: In $\mathbb{C}[x, y, z]$, the prime ideal (x, y) has height 2 since $V(x, y)$ has codimension 2 in $\mathbb{C}^3$
- The Krull dimension of the coordinate ring $k[V]$ is equal to the dimension of the affine variety V
- Example: The Krull dimension of $\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ is 2, equal to the dimension of the unit sphere in $\mathbb{R}^3$
- The localization of $k[V]$ at a prime ideal P corresponds to the local ring of the subvariety $V(P)$ at the generic point
- Example: The localization of $\mathbb{C}[x, y]$ at the prime ideal $(x - 1, y)$ is isomorphic to the local ring of the variety $V(x - 1, y)$ at the point $(1, 0)$

4.3 Local rings and localization

Local rings are crucial in algebraic geometry, focusing on properties around specific points on varieties. They have a unique maximal ideal containing all non-units, allowing us to study local behavior. Localization creates new rings by inverting elements, often used to analyze coordinate rings at prime ideals corresponding to points on varieties.

Localizing coordinate rings at prime ideals produces local rings that capture a variety's behavior near specific points. This process helps us understand geometric properties like dimension, regularity, and singularities. The correspondence between points and maximal ideals allows us to study variety geometry through coordinate ring algebra.

Local rings and their properties

Definition and basic properties of local rings

- A local ring is a commutative ring with a unique maximal ideal
- The unique maximal ideal consists of all non-units in the local ring
- Units are elements that have multiplicative inverses within the ring

- Non-units are elements that do not have multiplicative inverses
- The quotient of a local ring by its maximal ideal is a field, called the residue field
- This residue field captures the behavior of the ring "locally" around the maximal ideal
- Example: For the local ring $\mathbb{Q}[x]_{(x)}$ (localization of $\mathbb{Q}[x]$ at the prime ideal (x)), the residue field is $\mathbb{Q}$

Additional properties and examples of local rings

- Local rings are Noetherian if they satisfy the ascending chain condition on ideals
- Ascending chain condition: Every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \dots$ eventually stabilizes, i.e., there exists an N such that $I_n = I_N$ for all $n \geq N$
- Noetherian rings have favorable properties, such as the Hilbert basis theorem and the existence of primary decompositions
- Localization of a ring at a prime ideal produces a local ring
- The localization of a ring R at a prime ideal $\mathfrak{p}$ is denoted as $R_{\mathfrak{p}}$
- Example: The localization of $\mathbb{Z}$ at the prime ideal (p) is the local ring $\mathbb{Z}_{(p)}$, consisting of fractions with denominators not divisible by p

Localization of coordinate rings

The process of localization

- Localization is a process that creates a new ring from an existing ring by inverting a subset of elements
- Given a ring R and a multiplicatively closed subset S (i.e., $1 \in S$ and $a, b \in S \Rightarrow ab \in S$), the localization of R at S is the ring $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$
- The elements of S become units in the localized ring $S^{-1}R$
- For a prime ideal $\mathfrak{p}$ in a ring R , the localization of R at $\mathfrak{p}$ is the ring $R_{\mathfrak{p}}$ obtained by inverting all elements of R not in $\mathfrak{p}$
- The multiplicatively closed subset in this case is $S = R \setminus \mathfrak{p}$
- Example: For the ring $\mathbb{Z}$ and the prime ideal (2) , the localization $\mathbb{Z}_{(2)}$ consists of fractions with odd denominators

Localization of coordinate rings at prime ideals

- The coordinate ring of an affine variety can be localized at a prime ideal corresponding to a point on the variety
- For an affine variety $V \subseteq \mathbb{A}^n$, its coordinate ring is $A(V) = k[x_1, \dots, x_n]/I(V)$, where $I(V)$ is the ideal of polynomials vanishing on V
- Each point $P \in V$ corresponds to a maximal ideal $\mathfrak{m}_P$ in $A(V)$
- Localizing the coordinate ring at a prime ideal produces the local ring at that point
- The local ring at a point $P \in V$ is $\mathcal{O}_{V,P} = A(V)_{\mathfrak{m}_P}$

- This local ring captures the local behavior of the variety near the point P
- The maximal ideal of the localized ring consists of functions vanishing at the corresponding point
- For a point $P = (a_1, \dots, a_n) \in V$, the maximal ideal $\mathfrak{m}_P$ of $\mathcal{O}_{V,P}$ is generated by $\{x_1 - a_1, \dots, x_n - a_n\}$
- Functions in $\mathfrak{m}_P$ have a zero at P , while functions outside $\mathfrak{m}_P$ are non-vanishing at P

Local rings and geometric points

Correspondence between points and maximal ideals

- There is a one-to-one correspondence between points on an affine variety and maximal ideals in its coordinate ring
- Each point $P \in V$ corresponds to a maximal ideal $\mathfrak{m}_P$ in $A(V)$
- Conversely, each maximal ideal $\mathfrak{m}$ in $A(V)$ corresponds to a point $P_{\mathfrak{m}} \in V$
- This correspondence allows us to study the geometry of the variety through the algebraic properties of its coordinate ring

Properties of local rings and geometric implications

- The dimension of the local ring at a point is equal to the dimension of the variety at that point
- The dimension of a local ring $\mathcal{O}_{V,P}$ is the Krull dimension, i.e., the supremum of the lengths of chains of prime ideals in $\mathcal{O}_{V,P}$
- The dimension of a variety V at a point P is the dimension of the tangent space $T_P V$, which is isomorphic to the Zariski cotangent space $\mathfrak{m}_P / \mathfrak{m}_P^2$
- The residue field of the local ring at a point P is isomorphic to the field of rational functions on V defined at P
- The residue field $k(P) = \mathcal{O}_{V,P} / \mathfrak{m}_P$ is the field of fractions of the quotient ring $A(V) / \mathfrak{m}_P$
- Elements of $k(P)$ can be viewed as rational functions on V that are well-defined at P
- Properties of the local ring, such as regularity and normality, reflect geometric properties of the variety at the corresponding point
- A point $P \in V$ is regular (or smooth) if the local ring $\mathcal{O}_{V,P}$ is a regular local ring, i.e., its maximal ideal is generated by a regular sequence
- A variety V is normal if all of its local rings are integrally closed domains

Applications of local rings in algebraic geometry

Studying local properties of varieties

- Determine whether a given ring is local by checking for a unique maximal ideal
- A ring R is local if and only if it has a unique maximal ideal
- To check if a ring is local, find all maximal ideals and verify that there is only one

- Compute the localization of a given ring at a specified prime ideal
- To localize a ring R at a prime ideal $\mathfrak{p}$, form the multiplicatively closed subset $S = R \setminus \mathfrak{p}$ and construct the localized ring $R_{\mathfrak{p}} = S^{-1}R$
- Example: To localize $\mathbb{Z}[X]$ at the prime ideal (X) , invert all polynomials with a non-zero constant term to obtain $\mathbb{Z}[X]_{(X)}$
- Determine the maximal ideal and residue field of a local ring
- For a local ring R with maximal ideal $\mathfrak{m}$, the residue field is the quotient $R/\mathfrak{m}$
- Example: For the local ring $\mathbb{Q}[X, Y]_{(X, Y)}$, the maximal ideal is (X, Y) , and the residue field is $\mathbb{Q}$

Applying local rings to solve geometric problems

- Use local rings to study the local properties of a variety at a specific point
- Compute the dimension of the variety at a point by finding the Krull dimension of the local ring
- Determine the regularity or singularity of a point by examining the local ring
- Apply localization to solve problems related to the local structure of varieties, such as determining the dimension or singularity type at a point
- Example: To find the dimension of a variety V at a point P , localize the coordinate ring $A(V)$ at the maximal ideal $\mathfrak{m}_P$ and compute the Krull dimension of $\mathcal{O}_{V, P}$
- Use local rings to compute the tangent space and tangent cone of a variety at a point
- The tangent space $T_P V$ is isomorphic to the Zariski cotangent space $\mathfrak{m}_P / \mathfrak{m}_P^2$
- The tangent cone $C_P V$ is the spectrum of the associated graded ring

$$\mathrm{gr}_{\mathfrak{m}_P}(\mathcal{O}_{V, P}) = \bigoplus_{i=0}^{\infty} \mathfrak{m}_P^i / \mathfrak{m}_P^{i+1}$$

- Solve problems involving the relationship between local rings and the geometry of varieties, such as determining the multiplicity of a point or the intersection multiplicity of curves
- The multiplicity of a point P on a variety V is the length of the local ring $\mathcal{O}_{V, P}$ as a module over itself
- The intersection multiplicity of two curves at a point can be computed using the Hilbert-Samuel multiplicity of the local ring at the intersection point

4.4 Function fields and rational functions

Function fields and rational functions are the backbone of algebraic geometry. They're like the DNA of algebraic varieties, containing all the essential info about their structure and properties. Understanding these concepts is key to unlocking the secrets of geometric objects.

In this part of the chapter, we dive into the nitty-gritty of function fields and rational functions. We'll explore their definitions, properties, and how they relate to birational equivalence. Plus, we'll see how these ideas apply to real-world problems in algebraic geometry.

Function fields of varieties

Definition and elements of function fields

- The function field of an algebraic variety X , denoted $K(X)$, is the field of rational functions on X
- Elements of the function field $K(X)$ are equivalence classes of pairs (U, f) , where:
 - U is a non-empty open subset of X
 - f is a regular function on U
- Two pairs (U, f) and (V, g) are equivalent if f and g agree on the intersection of U and V
- The function field $K(X)$ is a field extension of the base field k , and its elements are called rational functions on X
- The function field $K(X)$ is the smallest field containing all regular functions on X

Properties and examples of function fields

- The function field $K(X)$ contains the field of regular functions on X as a subfield
- For an affine variety X defined by the ideal $I(X)$ in the polynomial ring $k[x_1, \dots, x_n]$, the function field $K(X)$ is isomorphic to the field of fractions of the coordinate ring $k[x_1, \dots, x_n]/I(X)$
- Example: For the affine plane curve X defined by the equation $y^2 = x^3 - x$, the function field $K(X)$ is isomorphic to $k(t)$, the field of rational functions in one variable t , via the isomorphism sending x to t^2 and y to $t(t^2 - 1)$
- The function field of a projective variety can be defined similarly using homogeneous coordinates
- Example: For the projective plane curve X defined by the homogeneous equation $x^3 + y^3 = z^3$, the function field $K(X)$ is isomorphic to $k(s, t)$, the field of rational functions in two variables s and t , via the isomorphism sending x/z to s and y/z to t

Rational functions and their properties

Definition and representation of rational functions

- A rational function on an algebraic variety X is an element of the function field $K(X)$
- Rational functions can be represented as the ratio of two polynomial functions, i.e., f/g , where:
 - f and g are regular functions on X
 - g is not identically zero
- The set of rational functions on X forms a field under the usual operations of addition and multiplication of functions

Domains, poles, and examples of rational functions

- Rational functions are not necessarily defined everywhere on X , as they may have poles (points where the denominator vanishes)
- The domain of a rational function is the largest open subset of X where the function is defined

- Example: On the affine plane $\mathbb{A}^2$, the rational function $f(x, y) = (x^2 + y^2)/(x - 1)$ has a pole at the point $(1, y)$ for any y , and its domain is $\mathbb{A}^2 \setminus \{(1, y) : y \in k\}$
- Rational functions can be used to define maps between algebraic varieties
- Example: The rational function $f(x, y) = (x, y/x)$ defines a rational map from the affine plane $\mathbb{A}^2$ to itself, which is not defined at the points $(0, y)$ for any y

Function fields and birational equivalence

Birational equivalence and isomorphisms of function fields

- Two algebraic varieties X and Y are birationally equivalent if there exist rational maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that:
- $g \circ f$ and $f \circ g$ are the identity maps on dense open subsets of X and Y , respectively
- If X and Y are birationally equivalent, then their function fields $K(X)$ and $K(Y)$ are isomorphic as fields
- The converse is also true: if the function fields $K(X)$ and $K(Y)$ are isomorphic, then X and Y are birationally equivalent

Properties and examples of birational equivalence

- Birational equivalence is an equivalence relation on the set of algebraic varieties, and it preserves many geometric properties of the varieties
- Example: The affine plane curve X defined by $y^2 = x^3 - x$ and the affine line $\mathbb{A}^1$ are birationally equivalent via the rational maps $f: X \rightarrow \mathbb{A}^1$, $(x, y) \mapsto x$, and $g: \mathbb{A}^1 \rightarrow X$, $t \mapsto (t^2, t(t^2 - 1))$
- The study of birational equivalence leads to the concept of birational geometry, which focuses on properties that are invariant under birational maps
- Example: The birational geometry of algebraic surfaces is closely related to the minimal model program, which aims to classify surfaces up to birational equivalence by finding their minimal models (smooth surfaces with nef canonical divisor)

Applications of function fields in algebraic geometry

Algebraic curves and the Riemann-Roch theorem

- Function fields and rational functions are essential tools in the study of algebraic curves and surfaces
- The genus of an algebraic curve can be determined using the Riemann-Roch theorem, which relates the dimension of the space of rational functions with prescribed poles to the genus of the curve
- Example: For a smooth projective curve X of genus g , the Riemann-Roch theorem states that $\dim L(D) - \dim L(K - D) = \deg D - g + 1$, where $L(D)$ is the space of rational functions with poles bounded by the divisor D , and K is the canonical divisor of X

Canonical divisors, singularities, and birational transformations

- The canonical divisor of an algebraic curve or surface can be defined using rational functions, and it plays a crucial role in the classification of algebraic varieties

- Rational functions can be used to study the singularities of algebraic varieties and to resolve them through birational transformations (e.g., blow-ups)
- Example: The blow-up of the affine plane $\mathbb{A}^2$ at the origin is a birational transformation that resolves the singularity of the curve $y^2 = x^3$ at $(0, 0)$, resulting in a smooth curve in the blown-up surface

Algebraic cycles, intersection theory, and cohomology

- The theory of function fields and rational functions is applied in the study of algebraic cycles, intersection theory, and the computation of cohomology groups of algebraic varieties
- Example: The Chow ring of a smooth projective variety X is a graded ring whose elements are algebraic cycles modulo rational equivalence, and the intersection product of cycles can be defined using rational functions
- The cohomology groups of an algebraic variety can be computed using the Čech cohomology of a cover by affine open subsets, which involves the function fields of the open subsets
- Example: The de Rham cohomology of a smooth affine variety X can be computed using the complex of regular differential forms on X , which are rational functions with certain properties

Unit 5 – Morphisms and Rational Maps

5.1 Regular functions and morphisms

Regular functions and morphisms are the building blocks of algebraic geometry. They allow us to study varieties through their function rings and maps between them. Understanding these concepts is crucial for grasping how algebraic structures relate to geometric objects.

Morphisms between varieties are the natural maps that preserve algebraic structure. They're defined using regular functions and provide a way to compare and relate different varieties. Mastering morphisms is key to understanding the deeper connections between algebraic and geometric properties of varieties.

Regular functions on varieties

Defining regular functions on affine varieties

- A regular function on an affine variety is a function that can be expressed as a polynomial in the coordinate ring of the variety
- Regular functions on affine varieties form a ring called the coordinate ring of the variety
- The coordinate ring consists of all polynomial functions on the affine variety
- Example: For the affine variety $V(y - x^2)$, the coordinate ring is $k[x, y]/(y - x^2)$
- The set of regular functions on an affine variety is denoted by $\mathcal{O}(X)$, where X is the affine variety

Defining regular functions on projective varieties

- A regular function on a projective variety is a function that can be expressed as a homogeneous polynomial of the same degree in each affine chart
- Homogeneous polynomials have the property that all terms have the same total degree
- Example: $x^2 + xy + y^2$ is a homogeneous polynomial of degree 2
- Regular functions on projective varieties are well-defined globally, meaning they are independent of the choice of affine charts
- This ensures that the regular functions are consistent across different coordinate representations of the projective variety
- The set of regular functions on a projective variety forms a graded ring
- The grading corresponds to the degree of the homogeneous polynomials
- Example: For the projective variety $V(xz - y^2)$, the graded ring is $k[x, y, z]/(xz - y^2)$

Morphisms between varieties

Defining morphisms between algebraic varieties

- A morphism between two algebraic varieties is a map that preserves the structure of the varieties
- Morphisms respect the algebraic properties and relations of the varieties

- Morphisms between affine varieties are defined by polynomial functions on the coordinate rings of the varieties
- The polynomial functions induce a ring homomorphism between the coordinate rings
- Example: A morphism from $V(y - x^2)$ to $V(w - z^3)$ can be defined by $x \mapsto z$ and $y \mapsto z^2$
- Morphisms between projective varieties are defined by homogeneous polynomial functions of the same degree on the homogeneous coordinate rings of the varieties
- The homogeneous polynomial functions induce a graded ring homomorphism between the homogeneous coordinate rings
- Example: A morphism from $V(xz - y^2)$ to $V(w^2 - uv)$ can be defined by $x \mapsto u$, $y \mapsto v$, and $z \mapsto w$
- Morphisms are required to be continuous with respect to the Zariski topology on the varieties
- The preimage of a closed set under a morphism must be a closed set

Composition and identity morphisms

- The composition of two morphisms is also a morphism
- If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are morphisms, then $g \circ f: X \rightarrow Z$ is also a morphism
- The composition of morphisms is associative: $(h \circ g) \circ f = h \circ (g \circ f)$
- The identity map is always a morphism
- For any variety X , the identity map $\text{id}_X: X \rightarrow X$ defined by $\text{id}_X(x) = x$ for all $x \in X$ is a morphism
- The identity morphism satisfies $f \circ \text{id}_X = f$ and $\text{id}_Y \circ f = f$ for any morphism $f: X \rightarrow Y$

Properties of morphisms

Injectivity of morphisms

- A morphism is injective (one-to-one) if distinct points in the domain map to distinct points in the codomain
- Formally, a morphism $f: X \rightarrow Y$ is injective if for any $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$
- Injectivity of a morphism can be determined by examining the kernel of the corresponding ring homomorphism between the coordinate rings of the varieties
- The kernel of a ring homomorphism $\varphi: A \rightarrow B$ is the set $\ker(\varphi) = \{a \in A \mid \varphi(a) = 0\}$
- A morphism is injective if and only if the kernel of the corresponding ring homomorphism is trivial (consists only of the zero element)

Surjectivity of morphisms

- A morphism is surjective (onto) if every point in the codomain is the image of some point in the domain
- Formally, a morphism $f: X \rightarrow Y$ is surjective if for any $y \in Y$, there exists an $x \in X$ such that $f(x) = y$
- Surjectivity of a morphism can be determined by examining the image of the corresponding ring homomorphism between the coordinate rings of the varieties

- The image of a ring homomorphism $\varphi : A \rightarrow B$ is the set $\text{im}(\varphi) = \{\varphi(a) \mid a \in A\}$
- A morphism is surjective if and only if the image of the corresponding ring homomorphism is equal to the codomain ring

Isomorphisms between varieties

- A morphism that is both injective and surjective is called an isomorphism, and the varieties are said to be isomorphic
- Isomorphic varieties have the same algebraic structure and properties
- Example: The affine variety $V(y - x^2)$ is isomorphic to the affine line $\mathbb{A}^1$ via the morphism defined by $x \mapsto x$
- The inverse of an isomorphism is also an isomorphism
- If $f : X \rightarrow Y$ is an isomorphism, then there exists a unique morphism $f^{-1} : Y \rightarrow X$ such that $f \circ f^{-1} = \text{id}_Y$ and $f^{-1} \circ f = \text{id}_X$

Constructing morphisms with functions

Constructing morphisms between affine varieties

- To construct a morphism from an affine variety to another, define polynomial functions on the coordinate ring of the domain variety that map to the coordinate ring of the codomain variety
- The polynomial functions induce a ring homomorphism between the coordinate rings
- Example: To construct a morphism from $V(y - x^2)$ to $V(w - z^3)$, define $x \mapsto z$ and $y \mapsto z^2$
- The polynomial functions defining the morphism must satisfy the relations of the ideal defining the codomain variety
- This ensures that the morphism is well-defined and maps points of the domain variety to points of the codomain variety
- Example: In the previous example, the polynomial functions satisfy the relation $y - x^2 \mapsto z^2 - z^2 = 0$, which is in the ideal defining $V(w - z^3)$

Constructing morphisms between projective varieties

- To construct a morphism from a projective variety to another, define homogeneous polynomial functions of the same degree on the homogeneous coordinate ring of the domain variety that map to the homogeneous coordinate ring of the codomain variety
- The homogeneous polynomial functions induce a graded ring homomorphism between the homogeneous coordinate rings
- Example: To construct a morphism from $V(xz - y^2)$ to $V(w^2 - uv)$, define $x \mapsto u$, $y \mapsto v$, and $z \mapsto w$
- The homogeneous polynomial functions defining the morphism must satisfy the relations of the homogeneous ideal defining the codomain variety
- This ensures that the morphism is well-defined and maps points of the domain variety to points of the codomain variety

- Example: In the previous example, the homogeneous polynomial functions satisfy the relation $xz - y^2 \mapsto uv - v^2$, which is in the homogeneous ideal defining $V(w^2 - uv)$

Specifying morphisms by their action on generators

- Morphisms can be constructed by specifying their action on the generators of the coordinate ring or homogeneous coordinate ring of the domain variety
- The action on the generators determines the action on all elements of the ring
- Example: To construct a morphism from $V(y - x^2)$ to $V(w - z^3)$, it suffices to specify $x \mapsto z$ and $y \mapsto z^2$, as x and y generate the coordinate ring of $V(y - x^2)$
- The specified action on the generators must be compatible with the relations of the codomain variety
- The relations of the codomain variety must be satisfied when the generators are mapped according to the specified action
- Example: In the previous example, the relation $y - x^2 = 0$ is satisfied when $x \mapsto z$ and $y \mapsto z^2$, as $z^2 - z^2 = 0$ in the coordinate ring of $V(w - z^3)$

5.2 Isomorphisms and embeddings

Isomorphisms and embeddings are crucial concepts in algebraic geometry, helping us understand the relationships between varieties. Isomorphisms reveal when varieties are essentially the same, while embeddings allow us to view one variety as part of another.

These ideas are key to the broader study of morphisms and rational maps. They provide tools for classifying varieties, simplifying their analysis, and uncovering hidden connections between seemingly different geometric objects.

Isomorphisms vs Embeddings of Varieties

Defining Isomorphisms and Embeddings

- An isomorphism between two varieties is a bijective morphism with a morphism inverse
- Meaning the two varieties are essentially the same and have identical structure
- Isomorphic varieties have the same geometric properties (dimension, degree, singularities)
- They also share algebraic properties (coordinate ring structure, Picard group, cohomology groups)
- An embedding of a variety X into a variety Y is an injective morphism from X to Y
- Allows viewing X as a subvariety of Y , inheriting some properties from the ambient space Y
- Embeddings are generally not invertible, unlike isomorphisms

Comparing Properties and Uses

- Isomorphisms preserve the intrinsic structure of varieties, while embeddings allow studying varieties as subvarieties of other spaces
- Isomorphic varieties are essentially identical, having the same geometric and algebraic properties
- Embedded varieties inherit some properties from the ambient space but may have additional structure

- Isomorphisms are used for classifying varieties up to equivalence (varieties with the same intrinsic properties)
- Help identify when seemingly different varieties are actually the same
- Simplify the study of algebraic geometry by grouping equivalent varieties
- Embeddings are used to study varieties as subvarieties of other spaces, often simplifying their analysis
- Allow studying varieties using tools and techniques available in the ambient space (linear algebra in projective spaces)

Proving Isomorphisms with Morphisms

Constructing Isomorphisms

- Two varieties X and Y are isomorphic if there exist morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$
- To prove isomorphism, construct explicit morphisms between the varieties
- Show that the compositions of the morphisms are the identity morphisms on respective varieties
- Isomorphisms can be proved using the properties of the varieties
- Defining equations of the varieties can be used to construct morphisms
- Coordinate rings of the varieties can be used to study isomorphisms algebraically

Isomorphisms and Coordinate Rings

- If two affine varieties have isomorphic coordinate rings, then the varieties are isomorphic
- Coordinate rings encode the algebraic structure of affine varieties
- Isomorphisms between coordinate rings induce isomorphisms between the corresponding affine varieties
- Isomorphisms between projective varieties can be studied using the homogeneous coordinate rings
- Homogeneous coordinate rings capture the structure of projective varieties
- Isomorphisms between homogeneous coordinate rings induce isomorphisms between the corresponding projective varieties

Examples of Embeddings: Veronese and Segre

Veronese Embedding

- The Veronese embedding is a map from the projective space $\mathbb{P}^n$ to a higher-dimensional projective space $\mathbb{P}^N$, defined by monomials of a fixed degree d
- The d -uple Veronese embedding of $\mathbb{P}^n$ is given by $[X_0 : \dots : X_n] \mapsto [\dots : X_0^{a_0} \dots X_n^{a_n} : \dots]$, where $a_0 + \dots + a_n = d$
- Allows studying higher degree hypersurfaces as linear subspaces of a higher-dimensional space

- Example: The Veronese surface is the image of $\mathbb{P}^2$ under the 2-uple Veronese embedding, given by $[x : y : z] \mapsto [x^2 : xy : y^2 : yz : z^2 : xz]$

Segre Embedding

- The Segre embedding is a map from the product of two projective spaces $\mathbb{P}^n \times \mathbb{P}^m$ to a higher-dimensional projective space $\mathbb{P}^{(n+1)(m+1)-1}$
- The Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ is given by $([X_0 : \dots : X_n], [Y_0 : \dots : Y_m]) \mapsto [\dots : X_i Y_j : \dots]$, where $0 \leq i \leq n$ and $0 \leq j \leq m$
- Allows studying the product of projective spaces as a subvariety of a single projective space
- Example: The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ is given by $([X_0 : X_1], [Y_0 : Y_1]) \mapsto [X_0 Y_0 : X_0 Y_1 : X_1 Y_0 : X_1 Y_1]$

Other Embeddings

- Rational normal curve: Embedding of $\mathbb{P}^1$ into $\mathbb{P}^n$ given by $[x : y] \mapsto [x^n : x^{n-1}y : \dots : xy^{n-1} : y^n]$
- Plücker embedding of the Grassmannian: Embedding of the Grassmannian $G(k, n)$ into a projective space using Plücker coordinates
- Canonical embedding of a variety using its canonical divisor: Embedding of a variety into a projective space using sections of the canonical line bundle

Significance of Isomorphisms in Classification

Classifying Varieties up to Isomorphism

- Isomorphisms provide a way to classify varieties up to equivalence
- Isomorphic varieties share the same intrinsic properties and are essentially the same
- Classification often involves finding invariants that distinguish non-isomorphic varieties
- Varieties with the same invariants are then shown to be isomorphic
- Isomorphisms preserve geometric properties that can be used to distinguish varieties
- Dimension, degree, and singularities are preserved under isomorphisms
- These properties can be used as invariants to classify varieties

Algebraic Invariants and Isomorphisms

- Isomorphisms preserve algebraic properties of varieties
- Structure of the coordinate ring, Picard group, and cohomology groups are preserved
- These algebraic invariants can be used to distinguish non-isomorphic varieties
- Varieties with isomorphic algebraic invariants are likely to be isomorphic
- Understanding isomorphisms helps simplify the study of algebraic geometry
- Identifies when seemingly different varieties are actually the same
- Allows focusing on representative varieties from each isomorphism class

- Reduces the complexity of the subject by grouping equivalent varieties together

5.3 Rational maps and birational equivalence

Rational maps and birational equivalence are key concepts in algebraic geometry. They allow us to study relationships between varieties that aren't necessarily isomorphic. These ideas help us understand the structure of varieties by focusing on their function fields rather than their specific geometric properties.

Birational equivalence is like a looser version of isomorphism. It lets us relate varieties that are "almost the same" in some sense. This concept is crucial for classification problems and for understanding the intrinsic properties of varieties that don't depend on specific embeddings or coordinate systems.

Rational Maps Between Varieties

Definition and Local Representation

- A rational map between two varieties X and Y is a morphism defined on an open dense subset U of X , taking values in Y
- Rational maps are locally represented by quotients of regular functions
- For affine varieties, a rational map is given by a tuple of rational functions $(f_1/g_1, \dots, f_n/g_n)$ where f_i, g_i are regular functions on X and g_i are not simultaneously zero on U
- For projective varieties, a rational map is given by a tuple of homogeneous polynomials $(F_0 : \dots : F_n)$ of the same degree, not all simultaneously zero on U

Domain of Definition and Indeterminacy Locus

- The domain of definition of a rational map is the largest open subset where the map is defined
- It is the complement of the set of points where all the defining regular functions (or homogeneous polynomials) vanish simultaneously
- The indeterminacy locus of a rational map is the complement of its domain of definition, consisting of points where the map is not defined
- For a rational map given by $(f_1/g_1, \dots, f_n/g_n)$, the indeterminacy locus is the set of points where all g_i vanish simultaneously
- The indeterminacy locus is a closed subset of X of codimension at least 2

Birational Equivalence and Maps

Definition and Equivalence Relation

- Two varieties X and Y are birationally equivalent if there exist rational maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $g \circ f$ and $f \circ g$ are the identity maps on dense open subsets of X and Y , respectively
- In other words, f and g are inverse to each other on dense open subsets
- A birational map is a rational map that admits an inverse rational map, establishing a birational equivalence between the varieties
- Birational equivalence is an equivalence relation on the class of varieties
- Reflexivity: Every variety is birationally equivalent to itself via the identity map

- Symmetry: If X is birationally equivalent to Y , then Y is birationally equivalent to X by swapping the roles of the rational maps
- Transitivity: If X is birationally equivalent to Y and Y is birationally equivalent to Z , then X is birationally equivalent to Z by composing the corresponding rational maps

Function Field Isomorphism

- Birational maps preserve the function field of varieties, inducing an isomorphism between the function fields
- If $f: X \rightarrow Y$ is a birational map, then it induces an isomorphism $f^*: K(Y) \rightarrow K(X)$ between the function fields
- The induced isomorphism preserves the field operations and the transcendence degree over the base field
- Two varieties are birationally equivalent if and only if their function fields are isomorphic
- This provides an algebraic characterization of birational equivalence

Proving Birational Equivalence

Constructing Explicit Rational Maps

- To prove that two varieties X and Y are birationally equivalent, one needs to construct explicit rational maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ and show that their compositions are the identity maps on dense open subsets
- This involves finding suitable expressions for the rational maps in terms of the coordinates or generators of the function fields
- The maps should be defined on dense open subsets and satisfy the composition property $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$ on these subsets

Dimension and Function Field Isomorphism

- The dimension of birationally equivalent varieties must be the same
- Birational maps preserve the transcendence degree of the function field over the base field, which equals the dimension of the variety
- Birational equivalence can be established by finding an isomorphism between the function fields of the varieties
- If an isomorphism $\varphi: K(X) \rightarrow K(Y)$ is found, it induces birational maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ that realize the birational equivalence
- The maps f and g can be obtained by expressing the generators of one function field in terms of the generators of the other

Properties Not Preserved

- Certain geometric properties, such as smoothness or projectivity, may not be preserved under birational equivalence

- Example: The affine line $\mathbb{A}^1$ and the punctured affine plane $\mathbb{A}^2 \setminus \{(0, 0)\}$ are birationally equivalent, but the former is smooth while the latter is not
- Example: The projective line $\mathbb{P}^1$ and the affine line $\mathbb{A}^1$ are birationally equivalent, but the former is projective while the latter is not

Properties of Rational Maps

Degree and Preimage Count

- The degree of a rational map is the degree of the corresponding function field extension
- If $f: X \rightarrow Y$ is a rational map, the degree of f is the degree of the field extension $[K(X) : f^* K(Y)]$
- For a rational map $f: X \rightarrow Y$ between projective varieties, the degree can be computed as the number of preimages of a general point in Y , counted with multiplicity
- A general point means a point outside a certain closed subset of Y of codimension at least 2
- The preimages are counted with multiplicity according to the order of vanishing of the defining homogeneous polynomials at each point

Base Points and Resolution

- Base points of a rational map are points in the domain where all the defining polynomials vanish simultaneously
- For a rational map given by $(F_0 : \dots : F_n)$, the base points are the common zeros of all F_i
- Blowing up the base points of a rational map can resolve the indeterminacy and yield a morphism
- The blow-up replaces each base point with a projective space of dimension one less than the variety
- The blow-up map is a birational morphism that resolves the indeterminacy of the rational map
- The resulting morphism is called a resolution of the rational map

Composition and Degree Multiplicativity

- The behavior of rational maps under composition, such as the multiplicativity of degrees, can be studied
- If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are rational maps, then the composition $g \circ f: X \rightarrow Z$ is also a rational map
- The degree of the composition is the product of the degrees of the individual maps:
 $\deg(g \circ f) = \deg(g) \cdot \deg(f)$
- This property follows from the multiplicativity of degrees in field extensions

5.4 Blowing up and resolution of singularities

Blowing up is a key technique in algebraic geometry for transforming varieties. It replaces points or subvarieties with higher-dimensional objects, introducing exceptional divisors. This process can improve or resolve singularities, making it crucial for studying geometric properties.

Resolution of singularities uses blowups to find smooth varieties birational to singular ones. It's always possible for curves and surfaces, while Hironaka's theorem proves it for higher dimensions in characteristic zero. This powerful tool connects to topology, classification, and other math areas.

Blowing up varieties

Definition and notation

- Blowing up is a fundamental operation in algebraic geometry that transforms a variety by replacing a point or subvariety with a higher-dimensional object, such as a projective space or an exceptional divisor
- The blowup of a variety X at a point P is denoted by $Bl_P(X)$ and is defined as the closure of the graph of the rational map from X to the projective space of lines through P
- The blowup of a variety X along a subvariety Y is denoted by $Bl_Y(X)$ and is obtained by blowing up each point of Y in X and taking the closure of the resulting space

Properties of blowups

- The blowup is an isomorphism away from the center of the blowup (the point or subvariety being blown up) and introduces an exceptional divisor over the center
- The exceptional divisor is a projective space or a bundle over the center of the blowup, depending on whether the center is a point or a subvariety
- For example, blowing up a point in a surface introduces an exceptional divisor isomorphic to $\mathbb{P}^1$
- Blowing up a curve in a threefold introduces an exceptional divisor that is a $\mathbb{P}^1$ -bundle over the curve

Local structure after blowing up

Effect on smooth and singular points

- Blowing up a smooth point on a variety results in a smooth variety, with the exceptional divisor being a projective space of codimension one
- Blowing up a singular point can improve the singularity, making it less severe or even resolving it completely
- The multiplicity of the singularity decreases after blowing up
- For example, blowing up an ordinary double point on a surface (a node) results in a smooth surface with an exceptional divisor of self-intersection -2

Intersection and self-intersection

- The exceptional divisor introduced by the blowup intersects the strict transform (the proper transform) of the original variety, and the intersection multiplicity provides information about the singularity
- The self-intersection number of the exceptional divisor is related to the type of singularity being blown up
- For example, blowing up an ordinary double point on a surface results in an exceptional divisor with self-intersection -2
- The self-intersection number decreases by 1 each time a point on the exceptional divisor is blown up

Successive blowups and local study

- Successive blowups can be used to study the local structure of a variety near a singularity, as each blowup provides a new model of the variety with a modified singularity
- The sequence of blowups and the configuration of the exceptional divisors encode information about the type of singularity
- For example, the resolution of a rational double point (ADE singularity) on a surface can be described by a sequence of blowups, with the final configuration of exceptional divisors forming a Dynkin diagram of the corresponding simple Lie algebra

Resolving singularities with blowups

Curves and surfaces

- Resolution of singularities is the process of finding a smooth variety birational to a given singular variety, achieved through a sequence of blowups
- For curves, resolution of singularities is always possible and can be achieved by a finite sequence of blowups at singular points
- The final result is a smooth curve birational to the original one
- For example, a curve with a node can be resolved by blowing up the node once, resulting in a smooth curve with an exceptional divisor of self-intersection -2
- For surfaces, resolution of singularities is also always possible, but the process may involve blowing up singular points and curves on the surface
- The sequence of blowups is not unique, but the final result is a smooth surface birational to the original one
- The resolution of rational double points (ADE singularities) on surfaces can be explicitly described using the Dynkin diagrams of the corresponding simple Lie algebras
- The intersection matrix of the exceptional divisors in the resolution of a surface singularity encodes important information about the singularity type

Higher dimensions and Hironaka's theorem

- Resolution of singularities in higher dimensions is a deep and active area of research, with significant progress made by Hironaka's theorem
- Hironaka's theorem proves the existence of resolution of singularities for varieties over fields of characteristic zero
- The theorem states that for any variety X over a field of characteristic zero, there exists a sequence of blowups along smooth centers that results in a smooth variety $\tilde{X}$ birational to X
- The proof of Hironaka's theorem is highly complex and involves intricate arguments from commutative algebra and algebraic geometry

Significance of resolution of singularities

Studying geometry and topology of varieties

- Resolution of singularities is a powerful tool for studying the geometry and topology of algebraic varieties, as it allows one to work with smooth models while preserving the birational equivalence class
- Many invariants and properties of varieties, such as the genus, Euler characteristic, and Hodge numbers, are preserved under birational equivalence and can be computed using a smooth model obtained by resolution of singularities
- For example, the genus of a curve can be computed using a smooth model obtained by resolving its singularities
- The Hodge numbers of a singular variety can be defined as the Hodge numbers of a smooth model obtained by resolution of singularities

Classification and minimal model program

- The existence of resolution of singularities has important consequences in the classification of algebraic varieties
- For example, it implies that every complex algebraic surface is birational to a smooth projective surface
- Resolution of singularities plays a crucial role in the minimal model program, which aims to find a "simplest" representative in each birational equivalence class of varieties
- The minimal models are obtained by performing a sequence of contractions and flips, starting from a smooth model obtained by resolution of singularities
- The minimal model program has been successfully completed for surfaces and threefolds, and it is an active area of research in higher dimensions

Connections with other areas of mathematics

- The study of singularities and their resolutions has deep connections with other areas of mathematics, such as topology, differential geometry, and representation theory
- The McKay correspondence relates the resolution of certain singularities (quotient singularities) to the representation theory of finite subgroups of $SL(2, \mathbb{C})$
- The exceptional divisors in the resolution of a quotient singularity correspond to the irreducible representations of the finite subgroup
- The intersection matrix of the exceptional divisors is related to the McKay quiver of the finite subgroup
- The study of singularities and their resolutions has applications in string theory and mathematical physics, particularly in the context of compactifications and dualities

Unit 6 – Hilbert's Nullstellensatz

6.1 Statement and proof of Hilbert's Nullstellensatz

Hilbert's Nullstellensatz is a game-changer in algebraic geometry. It connects the dots between algebraic varieties and ideals in polynomial rings, giving us a powerful tool to study geometric shapes using algebra.

This theorem is like a bridge between algebra and geometry. It lets us switch between looking at common zeros of polynomials and the ideals they generate, opening up new ways to solve tricky math problems.

Hilbert's Nullstellensatz

Statement and Significance

- Hilbert's Nullstellensatz, also known as Hilbert's zero-point theorem, establishes a correspondence between algebraic varieties and ideals in polynomial rings
- The theorem states that for any algebraically closed field k and polynomials $f_1, \dots, f_n \in k[x_1, \dots, x_n]$, the polynomials have a common zero in k^n if and only if the ideal generated by the polynomials is not the entire ring
- Provides a link between the geometric notion of an algebraic variety (set of common zeros of polynomials) and the algebraic concept of an ideal in a polynomial ring
- Allows for the study of geometric properties of varieties using algebraic techniques and vice versa
- Generalizes the fundamental theorem of algebra, which states that every non-constant polynomial over the complex numbers has a root
- Named after David Hilbert, who proved a version of the theorem in the late 19th century, although the current formulation is due to later mathematicians

Implications and Applications

- Has significant implications in algebraic geometry, enabling the use of algebraic techniques to study geometric properties of varieties and vice versa
- Establishes a correspondence between maximal ideals in $k[x_1, \dots, x_n]$ and points in affine space k^n
- Every maximal ideal is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some point $(a_1, \dots, a_n) \in k^n$
- Allows for the application of powerful algebraic tools, such as Gröbner bases, to solve systems of polynomial equations and study their solution sets
- Plays a crucial role in the development of modern algebraic geometry, including the study of schemes and sheaves

Proving Hilbert's Nullstellensatz

Key Algebraic Concepts

- The proof relies on several important algebraic concepts and techniques:
- Maximal ideals: An ideal I in a ring R is maximal if $I \neq R$ and there is no ideal J such that $I \subsetneq J \subsetneq R$

- Localization of rings: Constructing a new ring by inverting certain elements of the original ring, used to reduce the problem to the case of a local ring with a unique maximal ideal
- Rabinowitsch trick: A clever algebraic manipulation that introduces a new variable to transform a system of polynomial equations into a single equation
- The Hilbert basis theorem, which states that every ideal in a polynomial ring over a field is finitely generated, is also used in the proof

Proof Outline

- Show that if an ideal I in $k[x_1, \dots, x_n]$ is proper (not the entire ring), then there exists a maximal ideal containing I
- Use Zorn's lemma, which states that if every chain in a partially ordered set has an upper bound, then the set contains at least one maximal element
- Use localization of rings to reduce the problem to the case of a local ring with a unique maximal ideal
- Apply the Rabinowitsch trick to prove the "if" direction of the Nullstellensatz
- Show that if the polynomials have no common zero, then the ideal they generate must be the entire ring
- Combine these steps with the Hilbert basis theorem to complete the proof of Hilbert's Nullstellensatz

Ideals vs Varieties

Correspondence between Ideals and Varieties

- Hilbert's Nullstellensatz establishes a fundamental correspondence between ideals in polynomial rings and algebraic varieties
- Given polynomials $f_1, \dots, f_n \in k[x_1, \dots, x_n]$, the algebraic variety $V(f_1, \dots, f_n)$ is the set of all points in k^n that are common zeros of the polynomials
- The ideal generated by the polynomials, denoted by $(f_1, \dots, f_n)$, consists of all polynomials that can be expressed as linear combinations of the f_i with polynomial coefficients
- The Nullstellensatz states that the ideal $(f_1, \dots, f_n)$ is proper if and only if the variety $V(f_1, \dots, f_n)$ is non-empty

Studying Ideals and Varieties

- The correspondence allows for the study of geometric properties of varieties using algebraic techniques, such as:
 - Ideal operations (sum, product, intersection, quotient)
 - Gröbner bases, which provide a systematic way to solve systems of polynomial equations and study their solution sets
 - Conversely, it enables the study of algebraic properties of ideals using geometric intuition
 - Visualizing varieties as geometric objects in affine space
 - Using geometric arguments to prove algebraic statements about ideals

- The Nullstellensatz implies a one-to-one correspondence between maximal ideals in $k[x_1, \dots, x_n]$ and points in affine space k^n
- Every maximal ideal is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some point $(a_1, \dots, a_n) \in k^n$
- Every point in k^n corresponds to a unique maximal ideal

6.2 Consequences and applications

Hilbert's Nullstellensatz is a game-changer in algebraic geometry. It links algebra and geometry, showing how ideals in polynomial rings relate to geometric objects. This powerful tool helps us solve equations and understand the structure of algebraic sets.

The Nullstellensatz has far-reaching consequences. It's crucial for developing the Zariski topology, proving the Hilbert Basis Theorem, and advancing computational algebraic geometry. It even generalizes the Fundamental Theorem of Algebra to higher dimensions.

Hilbert's Nullstellensatz for Problem Solving

Applying the Nullstellensatz

- Hilbert's Nullstellensatz states that for any field k and any finitely generated k -algebra A , the maximal ideals of A correspond bijectively to the k -rational points of the affine scheme $\text{Spec} A$
- To apply the Nullstellensatz, one must first identify the field k and the finitely generated k -algebra A , then determine the maximal ideals of A and the k -rational points of $\text{Spec} A$
- The Nullstellensatz can be used to prove that a system of polynomial equations has no solution in an algebraically closed field if and only if the ideal generated by the polynomials is the entire ring
- The Nullstellensatz provides a correspondence between algebraic sets and radical ideals, allowing for the translation of geometric problems into algebraic ones and vice versa

Weak and Strong Nullstellensatz

- The weak Nullstellensatz asserts that if k is an algebraically closed field and I is a proper ideal in the polynomial ring $k[x_1, \dots, x_n]$, then the variety $V(I)$ is non-empty
- The strong Nullstellensatz states that if k is an algebraically closed field and I is an ideal in $k[x_1, \dots, x_n]$, then $I(V(I)) = \text{Rad}(I)$, where $\text{Rad}(I)$ denotes the radical of the ideal I
- The weak Nullstellensatz guarantees the existence of solutions to polynomial equations over algebraically closed fields, while the strong Nullstellensatz establishes a one-to-one correspondence between algebraic sets and radical ideals
- The strong Nullstellensatz is a more powerful result that allows for the complete characterization of the relationship between ideals and their corresponding algebraic sets

Implications of Hilbert's Nullstellensatz

Commutative Algebra and Algebraic Geometry

- The Nullstellensatz has significant implications in commutative algebra, as it relates the geometric notion of algebraic sets to the algebraic concept of ideals

- In algebraic geometry, the Nullstellensatz is used to establish a correspondence between affine varieties and prime ideals, forming the foundation for the study of schemes
- The Nullstellensatz plays a crucial role in the development of the Zariski topology, which is defined on the spectrum of a ring and provides a framework for studying geometric properties algebraically
- The Nullstellensatz is used in the proof of the Hilbert Basis Theorem, which states that every ideal in a polynomial ring over a field is finitely generated

Real Algebraic Geometry and Computational Applications

- The Nullstellensatz has applications in real algebraic geometry, where it is used to study the real solutions of polynomial equations and the properties of real algebraic sets
- In computational algebraic geometry, the Nullstellensatz is used to develop algorithms for solving systems of polynomial equations and for computing the radical of an ideal
- The Nullstellensatz provides a theoretical foundation for the development of effective methods for solving polynomial systems and for studying the geometry of algebraic varieties
- The computational aspects of the Nullstellensatz have led to the development of computer algebra systems (Mathematica, Maple) that can efficiently manipulate polynomial equations and ideals

Nullstellensatz vs Fundamental Theorem of Algebra

Generalization of the Fundamental Theorem of Algebra

- The fundamental theorem of algebra states that every non-constant polynomial with complex coefficients has at least one complex root
- Hilbert's Nullstellensatz can be seen as a generalization of the fundamental theorem of algebra to higher dimensions and more general fields
- In the case of a single variable polynomial over the complex numbers, the Nullstellensatz reduces to the fundamental theorem of algebra
- The Nullstellensatz extends the ideas of the fundamental theorem of algebra to multivariate polynomials and provides a more general framework for studying the relationship between polynomials and their roots

Connecting Algebra and Geometry

- Both the Nullstellensatz and the fundamental theorem of algebra establish a connection between the algebraic structure of polynomials and the geometric properties of their solution sets
- The fundamental theorem of algebra can be proved using the Nullstellensatz by considering the ideal generated by a polynomial in $\mathbb{C}[X]$ and showing that its variety is non-empty
- The Nullstellensatz provides a deeper understanding of the interplay between algebraic and geometric concepts, extending the insights gained from the fundamental theorem of algebra
- The connection between the Nullstellensatz and the fundamental theorem of algebra highlights the importance of understanding the relationship between polynomials and their roots in various mathematical contexts

6.3 Radical ideals and the Zariski topology

Radical ideals and the Zariski topology are key concepts in algebraic geometry. They provide a powerful framework for understanding the relationship between algebraic structures and geometric objects, connecting polynomial equations to their solution sets.

These ideas are crucial for grasping Hilbert's Nullstellensatz, which establishes a fundamental link between algebra and geometry. By relating radical ideals to algebraic sets, we can translate between algebraic problems and geometric ones, opening up new avenues for analysis and problem-solving.

Radical Ideals in Polynomial Rings

Definition and Characterization

- A radical ideal is an ideal I in a ring R such that for any element x in R , if some power of x is in I , then x itself is in I
- For example, in the ring of integers $\mathbb{Z}$, the ideal $(4) = \{0, \pm 4, \pm 8, \dots\}$ is not a radical ideal because $2^2 = 4 \in (4)$, but $2 \notin (4)$
- On the other hand, the ideal $(2) = \{0, \pm 2, \pm 4, \dots\}$ is a radical ideal in $\mathbb{Z}$
- In a polynomial ring $k[X_1, \dots, X_n]$ over a field k , an ideal I is radical if and only if I is the intersection of prime ideals
- For instance, in the ring $\mathbb{Q}[x]$, the ideal $(x^2 - 1)$ is radical because it is the intersection of the prime ideals $(x - 1)$ and $(x + 1)$

Radical and Nilradical

- The radical of an ideal I , denoted by $\sqrt{I}$ or $\text{rad}(I)$, is the smallest radical ideal containing I
- It consists of all elements x in the ring such that some power of x is in I
- For example, in the ring $\mathbb{Z}$, the radical of the ideal (4) is (2) , i.e., $\sqrt{(4)} = (2)$
- The radical of an ideal can be computed using the formula $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$
- The nilradical of a ring R , denoted by $\text{nil}(R)$, is the intersection of all prime ideals in R
- It is the set of all nilpotent elements in R (elements x such that $x^n = 0$ for some $n \in \mathbb{N}$)
- In the ring $\mathbb{Z}/12\mathbb{Z}$, the nilradical is $\{0, 4, 8\}$ because these are the elements that are annihilated by some power of themselves
- In a Noetherian ring, an ideal is radical if and only if it is the intersection of the prime ideals that contain it

Zariski Topology and Radical Ideals

Zariski Topology on Algebraic Varieties

- The Zariski topology on an affine variety $V \subseteq A^n$ over an algebraically closed field k is defined by taking the closed sets to be the algebraic subsets of V
- Algebraic subsets are sets of the form $V(I) = \{x \in V \mid f(x) = 0 \text{ for all } f \in I\}$ for some ideal $I \subseteq k[X_1, \dots, X_n]$

- For example, in A^2 , the algebraic subset $V(xy - 1)$ is the hyperbola $xy = 1$
- The Zariski topology on a projective variety is defined similarly, with closed sets being the projective algebraic subsets
- The Zariski topology is a T_1 topology, meaning that every singleton set is closed
- However, it is not Hausdorff, as distinct points may not have disjoint neighborhoods
- For instance, in A^1 , the points 0 and 1 do not have disjoint neighborhoods in the Zariski topology

Closure and Irreducibility

- In the Zariski topology, the closure of a set S is the smallest closed set containing S , given by $V(I(S))$, where $I(S)$ is the ideal of all polynomials vanishing on S
- For example, in A^2 , the closure of the set $\{(1, 1), (2, 2)\}$ is the line $V(x - y)$
- Irreducible closed sets in the Zariski topology correspond to prime ideals in the coordinate ring
- A closed set $V(I)$ is irreducible if and only if I is a prime ideal
- Maximal irreducible closed sets correspond to maximal ideals
- In A^2 , the line $V(x - y)$ is irreducible because the ideal $(x - y)$ is prime

Applications of Radical Ideals and the Zariski Topology

Computations and Problem Solving

- Determine whether a given ideal in a polynomial ring is radical by checking if it is the intersection of prime ideals or using the radical ideal formula
- For example, in $\mathbb{Q}[x, y]$, the ideal $(x^2 - y^3)$ is radical because it is prime, while $(x^2 - y^2) = (x - y)(x + y)$ is not radical
- Compute the radical of an ideal using the formula $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{N}\}$
- In $\mathbb{Q}[x]$, the radical of the ideal $(x^3 - x)$ is $(x^2 - 1)$ because $\sqrt{(x^3 - x)} = \{f \in \mathbb{Q}[x] \mid f^n \in (x^3 - x) \text{ for some } n \in \mathbb{N}\} = (x^2 - 1)$
- Find the closure of a set in the Zariski topology by computing the ideal of polynomials vanishing on the set and then finding the corresponding algebraic subset
- For the set $\{(1, 1), (-1, 1)\}$ in A^2 , $I(\{(1, 1), (-1, 1)\}) = (y - 1)$, so the closure is $V(y - 1)$, the line $y = 1$

Geometry of Algebraic Varieties

- Determine the irreducible components of an algebraic set by finding the minimal prime ideals containing the ideal of the algebraic set
- In A^2 , the algebraic set $V(xy)$ has two irreducible components: $V(x)$ (the y -axis) and $V(y)$ (the x -axis), corresponding to the minimal prime ideals (x) and (y)
- Use the correspondence between irreducible closed sets and prime ideals to study the geometry of algebraic varieties
- For instance, the dimension of an irreducible algebraic set is equal to the Krull dimension of its coordinate ring, which is the supremum of the lengths of chains of prime ideals

- Apply the concepts of dimension, tangent spaces, and singularities in the context of the Zariski topology and radical ideals
- The dimension of an algebraic set $V(I)$ is the Krull dimension of the quotient ring $k[X_1, \dots, X_n]/I$
- The tangent space at a point p of an algebraic variety V is the kernel of the Jacobian matrix of the defining equations of V evaluated at p
- A point p on an algebraic variety V is singular if the dimension of the tangent space at p is greater than the dimension of V

Hilbert's Nullstellensatz vs Radical Ideals and the Zariski Topology

Hilbert's Nullstellensatz

- Hilbert's Nullstellensatz states that if k is an algebraically closed field and $I \subseteq k[X_1, \dots, X_n]$ is an ideal, then $I(V(I)) = \sqrt{I}$, where $I(V(I))$ is the ideal of all polynomials vanishing on the algebraic set $V(I)$
- In other words, for any $f \in k[X_1, \dots, X_n]$, f vanishes on $V(I)$ if and only if some power of f is in I
- For example, in $\mathbb{C}[x, y]$, if $I = (x^2 + y^2 - 1)$, then $V(I)$ is the unit circle, and $I(V(I)) = (x^2 + y^2 - 1) = \sqrt{I}$
- The Nullstellensatz establishes a correspondence between radical ideals and algebraic subsets of affine space, which is the foundation of the Zariski topology
- Every algebraic subset $V(I)$ is uniquely determined by its radical ideal $I(V(I)) = \sqrt{I}$
- Conversely, every radical ideal I is the ideal of an algebraic subset, namely $V(I)$

Implications and Applications

- The Nullstellensatz implies that every maximal ideal in $k[X_1, \dots, X_n]$ is of the form $(X_1 - a_1, \dots, X_n - a_n)$ for some $(a_1, \dots, a_n) \in k^n$
- This follows from the fact that maximal ideals correspond to points in the algebraic closure of k
- Moreover, every prime ideal is the intersection of maximal ideals containing it
- The Nullstellensatz can be used to prove that the Zariski topology on an affine variety is T_1 and that irreducible closed sets correspond to prime ideals
- If p and q are distinct points in k^n , then the maximal ideals $(X_1 - p_1, \dots, X_n - p_n)$ and $(X_1 - q_1, \dots, X_n - q_n)$ are distinct, so $\{p\}$ and $\{q\}$ are closed in the Zariski topology
- If $V(I)$ is an irreducible closed set, then $I(V(I)) = \sqrt{I}$ is a prime ideal, and conversely, if I is a prime ideal, then $V(I)$ is irreducible
- The Nullstellensatz provides a bridge between the algebraic structure of polynomial rings and the geometric structure of algebraic varieties
- It enables the use of algebraic techniques to study geometric problems and vice versa
- For instance, the dimension of an algebraic variety can be computed as the Krull dimension of its coordinate ring, and the singular points of a variety can be found by studying the localization of its coordinate ring at maximal ideals

Unit 7 – Dimension Theory

7.1 Krull dimension and geometric dimension

Krull dimension and geometric dimension are key concepts in algebraic geometry. They measure the complexity of algebraic structures and spaces. Understanding these ideas helps us grasp the fundamental properties of rings, varieties, and their relationships.

These concepts connect abstract algebra with geometry. By studying prime ideals and chains, we can determine the dimension of algebraic objects. This knowledge is crucial for analyzing the structure and behavior of algebraic varieties and their coordinate rings.

Krull dimension for rings

Definition and properties

- The Krull dimension of a ring R is the supremum of the lengths of all chains of prime ideals in R , where the length of a chain is the number of strict inclusions
- For example, if $R = k[x, y]$ and $P_0 \subsetneq P_1 \subsetneq P_2$ is a chain of prime ideals, then the length of this chain is 2
- For an integral domain R , the Krull dimension is equal to the transcendence degree of its field of fractions over its prime subfield
- For instance, the field of fractions of $\mathbb{Q}[x, y]$ is $\mathbb{Q}(x, y)$, which has transcendence degree 2 over $\mathbb{Q}$, so the Krull dimension of $\mathbb{Q}[x, y]$ is 2
- The Krull dimension of a noetherian ring is finite, and it coincides with the maximum length of a chain of prime ideals in the ring
- A ring is noetherian if it satisfies the ascending chain condition on ideals, meaning that any ascending chain of ideals stabilizes (becomes constant) after finitely many steps

Relationship to algebraic varieties

- The Krull dimension of a finitely generated algebra over a field k is equal to the dimension of the corresponding algebraic variety defined over k
- For example, the polynomial ring $k[x, y, z]/(x^2 + y^2 - z^2)$ corresponds to the affine variety $V(x^2 + y^2 - z^2)$ in $\mathbb{A}^3$, which is a quadric surface of dimension 2
- For an affine algebraic variety V , the Krull dimension of its coordinate ring is equal to the dimension of V as a topological space
- The coordinate ring of an affine variety $V \subseteq \mathbb{A}^n$ is the quotient ring $k[x_1, \dots, x_n]/I(V)$, where $I(V)$ is the ideal of polynomials vanishing on V

Computing Krull dimension

Common rings

- The Krull dimension of a field is 0, as the only prime ideal is the zero ideal
- For instance, $\dim_{\text{Krull}}(\mathbb{Q}) = 0$ and $\dim_{\text{Krull}}(\mathbb{C}) = 0$

- The Krull dimension of the polynomial ring $k[x_1, \dots, x_n]$ over a field k is n , as the longest chain of prime ideals corresponds to the chain of irreducible subvarieties
- For example, $\dim_{\text{Krull}}(\mathbb{R}[x, y, z]) = 3$
- The Krull dimension of the ring of integers $\mathbb{Z}$ is 1, as the prime ideals are (0) and (p) for each prime number p
- The chain $(0) \subsetneq (p)$ has length 1, and there are no longer chains of prime ideals in $\mathbb{Z}$
- The Krull dimension of a product of rings $R_1 \times \dots \times R_n$ is the maximum of the Krull dimensions of the individual rings R_i
- For instance, $\dim_{\text{Krull}}(\mathbb{Z} \times \mathbb{Q}[x]) = \max\{1, 1\} = 1$

Affine varieties

- For an affine variety V defined by a prime ideal I in $k[x_1, \dots, x_n]$, the Krull dimension of V is equal to $n - \text{ht}(I)$, where $\text{ht}(I)$ is the height of the ideal I
- The height of a prime ideal I is the supremum of the lengths of chains of prime ideals contained in I
- For example, if $V = V(xy - 1) \subseteq \mathbb{A}^2$, then $I(V) = (xy - 1)$ has height 1, so $\dim_{\text{Krull}}(V) = 2 - 1 = 1$

Krull dimension properties

Localization and quotients

- For a ring R and a multiplicative subset S , the Krull dimension of the localization $S^{-1}R$ is equal to the Krull dimension of R
- This follows from the correspondence between prime ideals in R and prime ideals in $S^{-1}R$ that do not intersect S
- For example, $\dim_{\text{Krull}}(\mathbb{Z}_{(p)}) = \dim_{\text{Krull}}(\mathbb{Z}) = 1$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal (p)
- For a ring R and an ideal I , the Krull dimension of the quotient ring R/I is less than or equal to the Krull dimension of R
- This inequality becomes an equality if I is a minimal prime ideal
- For instance, $\dim_{\text{Krull}}(\mathbb{Z}/6\mathbb{Z}) = 0 \leq \dim_{\text{Krull}}(\mathbb{Z}) = 1$

Noetherian integral domains

- If R is a finitely generated algebra over a field k and P is a prime ideal of R , then $\dim(R/P) + \text{ht}(P) = \dim(R)$, where $\dim$ denotes the Krull dimension
- This property relates the dimension of a quotient ring to the height of the corresponding prime ideal
- For a noetherian integral domain R , the Krull dimension of R is equal to the supremum of the heights of maximal ideals in R
- A maximal ideal is a proper ideal that is not contained in any other proper ideal

Krull dimension vs transcendence degree

Finitely generated field extensions

- For a finitely generated field extension K/k , the transcendence degree of K over k is equal to the Krull dimension of any finitely generated k -subalgebra R of K such that the field of fractions of R is K
- The transcendence degree of a field extension K/k is the maximum number of algebraically independent elements in K over k
- For example, if $K = k(x, y)$ and $R = k[x, y]$, then $\dim_{\text{Krull}}(R) = \text{tr. deg}_k(K) = 2$

Integral domains and function fields

- If R is an integral domain with field of fractions K , then the Krull dimension of R is equal to the transcendence degree of K over the prime subfield of R
- The prime subfield of a ring is the smallest subfield contained in the ring (e.g., $\mathbb{Q}$ for $\mathbb{Z}$ or $\mathbb{Q}[x]$)
- For an affine variety V over a field k , the dimension of V is equal to the transcendence degree of the function field $k(V)$ over k
- The function field $k(V)$ is the field of rational functions on V , which consists of quotients of polynomials in the coordinate ring of V

Polynomial rings and rational function fields

- The Krull dimension of a finitely generated algebra R over a field k is equal to the transcendence degree of the field of fractions of R over k
- For instance, if $R = k[x, y, z]/(x^2 + y^2 - z^2)$, then its field of fractions has transcendence degree 2 over k , so $\dim_{\text{Krull}}(R) = 2$
- The Krull dimension of a polynomial ring $k[x_1, \dots, x_n]$ is equal to the transcendence degree of the field of rational functions $k(x_1, \dots, x_n)$ over k
- The field of rational functions $k(x_1, \dots, x_n)$ consists of quotients of polynomials in $k[x_1, \dots, x_n]$

7.2 Dimension of varieties and coordinate rings

Dimension is a crucial concept in algebraic geometry, measuring the "size" of varieties. It's defined as the Krull dimension of a variety's coordinate ring, which is the length of the longest chain of prime ideals.

Understanding dimension helps us classify varieties and study their properties. It's related to tangent spaces, behaves predictably under morphisms, and plays a key role in analyzing curves, surfaces, and higher-dimensional varieties.

Dimension of varieties

Determining dimension using coordinate rings

- The dimension of an affine or projective variety X is defined as the Krull dimension of its coordinate ring $A(X)$
- For an affine variety X , the coordinate ring $A(X)$ is the quotient ring $k[x_1, \dots, x_n]/I(X)$, where $I(X)$ is the ideal of polynomials vanishing on X

- For a projective variety X , the coordinate ring $A(X)$ is the homogeneous coordinate ring $k[x_0, \dots, x_n]/I(X)$, where $I(X)$ is the homogeneous ideal of polynomials vanishing on X
- The Krull dimension of a ring R is the supremum of the lengths of all chains of prime ideals in R
- To determine the dimension of a variety, one computes the Krull dimension of its coordinate ring by finding the longest chain of prime ideals
- The dimension of an irreducible variety is equal to the transcendence degree of its function field over the base field k

Examples of determining dimension

- Consider the affine variety X defined by the equation $x^2 + y^2 = 1$ in A^2 . The coordinate ring $A(X)$ is $k[x, y]/(x^2 + y^2 - 1)$, and the longest chain of prime ideals is $(0) \subset (x - 1, y)$, so $\dim(X) = 1$
- For the projective variety Y defined by the homogeneous equation $x^3 + y^3 + z^3 = 0$ in $\mathbb{P}^2$, the coordinate ring $A(Y)$ is $k[x, y, z]/(x^3 + y^3 + z^3)$, and the longest chain of prime ideals is $(0) \subset (x, y)$, so $\dim(Y) = 1$
- The affine variety Z defined by the equations $x^2 + y^2 = 1$ and $z = 0$ in A^3 has coordinate ring $k[x, y, z]/(x^2 + y^2 - 1, z)$, and the longest chain of prime ideals is $(0) \subset (x - 1, y, z)$, so $\dim(Z) = 0$

Dimension and coordinate ring

Relating dimension to Krull dimension

- The dimension of an affine or projective variety X is equal to the Krull dimension of its coordinate ring $A(X)$
- The Krull dimension of a ring R is the supremum of the lengths of all chains of prime ideals in R
- For an affine variety X , the coordinate ring $A(X)$ is the quotient ring $k[x_1, \dots, x_n]/I(X)$, where $I(X)$ is the ideal of polynomials vanishing on X
- For a projective variety X , the coordinate ring $A(X)$ is the homogeneous coordinate ring $k[x_0, \dots, x_n]/I(X)$, where $I(X)$ is the homogeneous ideal of polynomials vanishing on X
- The dimension of an irreducible variety is equal to the transcendence degree of its function field over the base field k

Dimension and tangent spaces

- The dimension of a variety X is also equal to the dimension of the tangent space at any smooth point of X
- The tangent space at a point $x \in X$ is the vector space of derivations of the local ring $\mathcal{O}_{X,x}$
- For an affine variety $X \subset A^n$, the tangent space at a smooth point x is isomorphic to the kernel of the Jacobian matrix of the defining equations evaluated at x
- The dimension of the tangent space at a singular point may be larger than the dimension of the variety

Dimension under morphisms

Behavior of dimension under morphisms

- A morphism of varieties $f: X \rightarrow Y$ is a continuous map that is locally given by polynomials

- The dimension of the image of a morphism $f: X \rightarrow Y$ is always less than or equal to the dimension of X
- If $f: X \rightarrow Y$ is a surjective morphism, then $\dim(X) \geq \dim(Y)$. If f is also finite, then $\dim(X) = \dim(Y)$
- The dimension of the fibers of a morphism can vary, but for a flat morphism, the dimension of the fibers is constant

Fibrations and dimension

- A fibration is a morphism $f: X \rightarrow Y$ such that for every point $y \in Y$, the fiber $f^{-1}(y)$ is a variety of constant dimension
- For a fibration $f: X \rightarrow Y$, the dimension formula holds: $\dim(X) = \dim(Y) + \dim(f^{-1}(y))$, where y is any point in Y
- Examples of fibrations include projections of product varieties ($X \times Y \rightarrow Y$) and bundle maps ($E \rightarrow B$, where E is a vector bundle over B)
- The dimension of the total space of a fibration is the sum of the dimensions of the base and the fiber

Dimension for geometric study

Dimension of specific varieties

- A curve is a variety of dimension 1. Examples include lines, conics, and elliptic curves
- The genus of a smooth projective curve C is related to its dimension by the formula $\dim H^0(C, \Omega^1) = g$, where Ω^1 is the sheaf of differential forms on C
- A surface is a variety of dimension 2. Examples include planes, quadrics, and K3 surfaces
- The Hodge diamond of a smooth projective surface encodes its Hodge numbers, which are related to its dimension and geometric properties
- A hypersurface is a variety defined by a single polynomial equation. The dimension of a hypersurface in $\mathbb{P}^n$ is $n-1$
- The degree of a hypersurface is related to its dimension and the degree of its defining polynomial

Dimension and singularities

- The dimension of the singular locus of a variety is always strictly less than the dimension of the variety itself
- For example, the singular locus of a curve consists of isolated points (dimension 0), while the singular locus of a surface can contain curves (dimension 1)
- The dimension of the intersection of two varieties X and Y in a smooth ambient space is related to their individual dimensions by the formula $\dim(X \cap Y) \geq \dim(X) + \dim(Y) - \dim(\text{ambient space})$, with equality holding if the intersection is transverse
- Transverse intersections occur when the tangent spaces of X and Y span the tangent space of the ambient variety at every point of intersection

7.3 Dimension theory for projective varieties

Dimension theory for projective varieties builds on concepts from affine varieties, extending them to the projective setting. It explores how to measure the "size" of these geometric objects, connecting ideas from algebra and geometry.

This topic dives into the relationships between projective varieties, their affine cones, and homogeneous coordinate rings. It also covers how dimensions behave under projective morphisms and rational maps, tying everything together in a cohesive framework.

Dimension Theory for Projective Varieties

Extending Dimension Theory to Projective Varieties and Graded Rings

- The dimension of a projective variety X is defined as the dimension of any affine open subset U of X , which is equal to the dimension of the affine variety U as an affine variety
- For a graded ring S , the dimension of $\text{Proj}(S)$ is equal to the Krull dimension of S minus 1
- The dimension of a projective variety X is equal to the transcendence degree of its function field over the base field
- The function field of X is the field of rational functions on X , which is the quotient field of any affine open subset U of X
- The transcendence degree measures the number of algebraically independent elements needed to generate the function field over the base field
- The dimension of a projective variety X is also equal to the maximal length of a chain of irreducible closed subvarieties of X
- A chain of subvarieties is a sequence of subvarieties $X_0 \subset X_1 \subset \dots \subset X_n = X$, where each X_i is a proper subvariety of X_{i+1}
- The length of the chain is the number of strict inclusions in the sequence
- The dimension of a projective variety X is invariant under projective isomorphisms
- A projective isomorphism is a morphism $f: X \rightarrow Y$ between projective varieties that has an inverse morphism $g: Y \rightarrow X$ such that $f \circ g$ and $g \circ f$ are identity morphisms

Dimension of Projective Varieties vs Affine Cones

Relationship Between Projective Varieties and Their Affine Cones

- For a projective variety X in P^n , its affine cone $C(X)$ is the affine variety in A^{n+1} defined by the same homogeneous polynomials that define X
- The affine cone $C(X)$ is obtained by considering the homogeneous coordinates of X as affine coordinates in A^{n+1}
- The origin of A^{n+1} is always included in $C(X)$
- The dimension of the affine cone $C(X)$ is equal to the dimension of the projective variety X plus 1
- This is because the affine cone has one additional dimension corresponding to the scaling of the homogeneous coordinates

- The projective variety X can be recovered from its affine cone $C(X)$ by taking the quotient of $C(X)$ minus the origin by the action of the multiplicative group of the base field
- The multiplicative group acts by scaling the homogeneous coordinates, which corresponds to the equivalence relation defining the projective space
- The affine cone $C(X)$ is irreducible if and only if the projective variety X is irreducible
- Irreducibility means that the variety cannot be written as the union of two proper closed subvarieties
- The correspondence between X and $C(X)$ preserves irreducibility

Dimension Computation in Homogeneous Rings

Using Homogeneous Coordinate Rings to Compute Dimension

- The homogeneous coordinate ring of a projective variety X in P^n is the graded ring $S = k[x_0, \dots, x_n]/I$, where I is the homogeneous ideal defining X
- The variables $x_0, \dots, x_n$ correspond to the homogeneous coordinates of P^n
- The ideal I consists of all homogeneous polynomials that vanish on X
- The dimension of X is equal to the Krull dimension of S minus 1
- The Krull dimension of a ring is the maximal length of a chain of prime ideals in the ring
- The minus 1 accounts for the grading of S , which adds an extra dimension
- The Krull dimension of S can be computed using the maximal length of a chain of prime ideals in S
- A prime ideal is an ideal P such that if $ab \in P$, then either $a \in P$ or $b \in P$
- The chain of prime ideals corresponds to a chain of irreducible closed subvarieties of X
- The dimension of X can also be computed using the Hilbert polynomial of S , which encodes information about the dimensions of the graded pieces of S
- The Hilbert polynomial is a polynomial $P(t)$ such that $P(n)$ equals the dimension of the n -th graded piece of S for sufficiently large n
- The degree of the Hilbert polynomial equals the dimension of X

Hilbert Functions and Polynomials

- The Hilbert function of a graded ring S is the function $H(n)$ that gives the dimension of the n -th graded piece of S
- The n -th graded piece of S consists of all homogeneous elements of S of degree n
- The dimension is taken as a vector space over the base field
- The Hilbert polynomial $P(t)$ of S is a polynomial such that $P(n) = H(n)$ for sufficiently large n
- The existence of the Hilbert polynomial follows from the fact that $H(n)$ agrees with a polynomial for large n
- The Hilbert polynomial encodes asymptotic information about the growth of the graded pieces of S
- The degree of the Hilbert polynomial $P(t)$ equals the dimension of $\text{Proj}(S)$

- This provides a way to compute the dimension of a projective variety from its homogeneous coordinate ring
- The leading coefficient of $P(t)$ is related to the degree of the projective variety

Dimension Under Projective Morphisms

Projective Morphisms and Fiber Dimensions

- A projective morphism $f: X \rightarrow Y$ between projective varieties is a morphism that can be represented by homogeneous polynomials of the same degree in the homogeneous coordinate rings of X and Y
- The homogeneous polynomials defining f must be compatible with the grading of the coordinate rings
- Projective morphisms are well-defined on the level of projective varieties, independent of the choice of homogeneous coordinates
- The dimension of the fibers of a projective morphism $f: X \rightarrow Y$ is upper semicontinuous, meaning that for any d , the set $\{y \in Y \mid \dim(f^{-1}(y)) \geq d\}$ is closed in Y
- The fiber of f over a point $y \in Y$ is the preimage $f^{-1}(y)$, which is a projective subvariety of X
- Upper semicontinuity implies that the dimension of the fibers can only jump up on closed subsets of Y
- If $f: X \rightarrow Y$ is a projective morphism between irreducible projective varieties, then $\dim(X) \geq \dim(Y)$, and equality holds if and only if f is surjective and the general fiber of f is finite
- Surjectivity means that every point of Y is in the image of f
- The general fiber refers to the fiber over a general point, meaning a point in a dense open subset of Y
- Finiteness of the general fiber means that it consists of a finite number of points

Dimension and Dominant Rational Maps

- A rational map $f: X \dashrightarrow Y$ between projective varieties is a morphism defined on an open dense subset U of X
- Rational maps allow for maps that are not defined everywhere, but are defined on a dense open subset
- The closure of the image of f is a closed subvariety of Y
- The dimension of the closure of the image of f is at most the dimension of X
- This follows from the fact that f is defined on an open dense subset of X , and the image of f is contained in Y
- If $f: X \dashrightarrow Y$ is a dominant rational map between irreducible projective varieties (meaning the image of f is dense in Y), then $\dim(X) \geq \dim(Y)$, and equality holds if and only if the general fiber of f is finite
- Dominance implies that the image of f is dense in Y , so the closure of the image is equal to Y
- The general fiber of f refers to the fiber over a general point in the open dense subset where f is defined
- Finiteness of the general fiber means that it consists of a finite number of points

- The inequality $\dim(X) \geq \dim(Y)$ follows from the fact that f is defined on an open dense subset of X , and the general fiber has dimension zero

Unit 8 – Singularities

8.1 Regular and singular points

Regular and singular points are crucial concepts in algebraic geometry. They help us understand the smoothness and behavior of curves and surfaces at specific points. This knowledge is essential for analyzing the overall structure and properties of geometric objects.

Identifying regular and singular points involves examining the tangent lines or planes at each point. We use tools like the Jacobian matrix and partial derivatives to determine regularity. Understanding these concepts is key to grasping the broader topic of singularities in algebraic geometry.

Regular vs Singular Points

Defining Regular and Singular Points

- A point on an algebraic curve or surface is regular if the curve or surface is smooth at that point, meaning it has a well-defined tangent line (for curves) or plane (for surfaces)
- A point on an algebraic curve or surface is singular if the curve or surface is not smooth at that point, meaning it does not have a well-defined tangent line or plane
- Regular points are also called simple or non-singular points, while singular points are also called multiple points or singularities

Determining Regularity using the Jacobian Matrix

- The Jacobian matrix of partial derivatives can be used to determine the regularity of a point on an algebraic curve or surface
- If the Jacobian matrix has full rank at a point, then the point is regular
- If the Jacobian matrix does not have full rank at a point, then the point is singular
- The set of all singular points on an algebraic curve or surface is called the singular locus

Singularities: Geometric Properties

Types of Singularities

- A node is a point where two or more branches of the curve intersect transversally, forming distinct tangent lines (e.g., the origin in the curve $y^2 = x^2(x + 1)$)
- A cusp is a point where two branches of the curve meet tangentially, forming a sharp point with a single tangent line (e.g., the origin in the curve $y^2 = x^3$)
- A tacnode is a point where two branches of the curve meet with the same tangent line and have at least second-order contact (e.g., the origin in the curve $y^2 = x^4$)
- An ordinary multiple point is a point where several branches of the curve intersect, each with its own distinct tangent line (e.g., the origin in the curve $y^2 = x^2(x - 1)^2$)

Special Cases of Singularities

- An isolated point or acnode is a point that lies on the curve but has no other points of the curve in its neighborhood (e.g., the origin in the curve $x^2 + y^2 = 0$)
- A self-intersection point is a point where a curve crosses itself, forming a node or a tacnode (e.g., the origin in the curve $y^2 = x^2(x - 1)$)
- A degenerate singularity is a point where the curve has a more complicated structure, such as a higher-order cusp or a point with an infinite number of tangent lines (e.g., the origin in the curve $y^2 = x^5$)

Identifying Regularity: Algebraic Techniques

Plane Algebraic Curves

- For a plane algebraic curve defined by a polynomial equation $f(x, y) = 0$, a point (a, b) is singular if and only if the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ are both zero
- Example: For the curve $y^2 = x^3$, the origin $(0, 0)$ is singular because $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$

Space Algebraic Curves

- For a space algebraic curve defined by the intersection of two polynomial equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$, a point (a, b, c) is singular if and only if the Jacobian matrix $[f_x f_y f_z; g_x g_y g_z]$ evaluated at (a, b, c) has rank less than 2
- Example: For the curve defined by $x^2 + y^2 - z^2 = 0$ and $x^2 - y^2 = 0$, the origin $(0, 0, 0)$ is singular because the Jacobian matrix at $(0, 0, 0)$ has rank 1

Algebraic Surfaces

- For an algebraic surface defined by a polynomial equation $f(x, y, z) = 0$, a point (a, b, c) is singular if and only if the gradient vector $\nabla f(a, b, c) = (f_x(a, b, c), f_y(a, b, c), f_z(a, b, c))$ is the zero vector
- Example: For the surface $x^2 + y^2 - z^2 = 0$, the origin $(0, 0, 0)$ is singular because $\nabla f(0, 0, 0) = (0, 0, 0)$

Advanced Techniques

- The multiplicity of a singular point can be determined by the order of vanishing of the defining polynomials and their partial derivatives at that point
- Blow-up techniques can be used to resolve singularities and study their local structure by introducing new coordinates and transforming the equation of the curve or surface

8.2 Tangent spaces and Jacobian criterion

Tangent spaces are crucial for understanding the local geometry of algebraic varieties. They provide a linear approximation of the variety at a point, helping us analyze smoothness and detect singularities. This concept is fundamental in algebraic geometry.

The Jacobian criterion is a powerful tool for detecting and classifying singularities. By examining the rank of the Jacobian matrix, we can determine whether a point is smooth or singular, shedding light on the variety's local structure and overall geometry.

Tangent spaces on varieties

Computing tangent spaces

- The tangent space at a point P on an algebraic variety V is the vector space of all tangent vectors to V at P
- For an affine variety V defined by polynomials $f_1, \dots, f_k$ in n variables, the tangent space at a point P is the kernel of the Jacobian matrix $J(f_1, \dots, f_k)$ evaluated at P
- The Jacobian matrix contains the partial derivatives of the defining polynomials with respect to each variable
- The kernel of the Jacobian matrix represents the directions in which the variety is "flat" or has zero curvature at P
- The tangent space can be computed by finding the nullspace of the Jacobian matrix using techniques from linear algebra, such as Gaussian elimination or singular value decomposition

Properties of tangent spaces

- The dimension of the tangent space at a smooth point is equal to the dimension of the variety
- A smooth point is a point where the variety is locally a manifold, without any singularities or self-intersections
- For a projective variety, the tangent space is defined in the same way, but using homogeneous coordinates
- Homogeneous coordinates allow the representation of points at infinity and provide a consistent framework for studying projective varieties
- The tangent space provides a linear approximation to the variety near the point P , which is useful for studying the local geometry and singularities of the variety

Singularity detection with the Jacobian

Jacobian criterion for singularities

- A point P on an algebraic variety V is singular if the dimension of the tangent space at P is greater than the dimension of V
- Singular points are points where the variety is not locally a manifold, such as self-intersections, cusps, or double points
- The Jacobian criterion states that a point P on an affine variety V defined by polynomials $f_1, \dots, f_k$ is singular if and only if the rank of the Jacobian matrix $J(f_1, \dots, f_k)$ evaluated at P is less than the codimension of V ($n - \dim(V)$)
- The codimension of V is the difference between the ambient space dimension n and the dimension of the variety $\dim(V)$
- For a hypersurface defined by a single polynomial f , a point P is singular if and only if all partial derivatives of f vanish at P
- A hypersurface is a variety of codimension 1, such as a curve in a plane or a surface in 3-dimensional space

Classifying singularities

- The Jacobian criterion can be used to classify the types of singularities, such as nodes, cusps, and double points, based on the rank of the Jacobian matrix and the local geometry of the variety near the singular point
- A node is a singular point where two branches of the variety intersect transversely (example: the origin of the curve $y^2 = x^2(x + 1)$)
- A cusp is a singular point where two branches of the variety meet tangentially (example: the origin of the curve $y^2 = x^3$)
- A double point is a singular point where the variety intersects itself with multiplicity 2 (example: the origin of the curve $y^2 = x^2$)
- The classification of singularities helps understand the local structure of the variety and is important in the study of resolution of singularities and the geometry of algebraic curves and surfaces

Jacobian matrix: Geometry and rank

Geometric interpretation of the Jacobian matrix

- The Jacobian matrix $J(f_1, \dots, f_k)$ encodes the first-order approximation of the algebraic variety V defined by polynomials $f_1, \dots, f_k$ near a point P
- The first-order approximation is the linear part of the Taylor expansion of the defining polynomials at the point P
- The kernel of the Jacobian matrix at a point P corresponds to the directions in which the variety is "flat" or has zero curvature at P
- These directions form the tangent space to the variety at P
- The image of the Jacobian matrix at a point P corresponds to the directions in which the variety is "curved" or has non-zero curvature at P
- These directions are orthogonal to the tangent space and indicate how the variety bends or twists near P

Rank of the Jacobian matrix and local geometry

- The rank of the Jacobian matrix at a point P determines the dimension of the tangent space and the local geometry of the variety near P
- The rank is the number of linearly independent rows or columns of the Jacobian matrix
- If the rank of the Jacobian matrix is equal to the codimension of V , then the point P is a smooth point, and the variety is locally a manifold near P
- At a smooth point, the tangent space has the same dimension as the variety, and the variety has no singularities or self-intersections
- If the rank of the Jacobian matrix is less than the codimension of V , then the point P is a singular point, and the variety has a more complicated local structure, such as self-intersections or cusps
- At a singular point, the tangent space has a higher dimension than the variety, indicating the presence of multiple tangent directions or a degenerate structure

- The rank of the Jacobian matrix can be computed using techniques from linear algebra, such as Gaussian elimination or singular value decomposition, and provides crucial information about the singularities and the local geometry of the variety

8.3 Classification of singularities

Singularities in algebraic geometry can be tricky, but don't worry! We'll break down the main types like nodes, cusps, and tacnodes. These quirky points on curves where things get weird are key to understanding the shape and behavior of algebraic curves.

We'll also look at ways to measure how complex singularities are, like the delta invariant and Milnor number. Plus, we'll explore techniques like blow-ups to analyze and resolve singularities, turning messy curves into smoother ones. It's like untangling mathematical knots!

Singularities Classification

Common Types of Singularities

- Singularity is a point on an algebraic curve where the curve is not smooth
- Has a sharp point, self-intersection, or other irregular behavior
- Node (ordinary double point) is a singularity where the curve crosses itself transversely
- Forms a point with two distinct tangent lines (crunode)
- Cusp is a singularity where the curve has a sharp point
- Two branches of the curve have the same tangent line at the singular point (spinode)
- Tacnode is a singularity where the curve touches itself
- Forms a point with two coincident tangent lines and a higher order of contact than a node
- Other types of singularities
- Triple points where three branches intersect at a point
- Ordinary multiple points where multiple branches intersect transversely
- Higher order cusps

Measuring Singularity Complexity

- Delta invariant measures the complexity of a singularity
- Related to the genus of the curve after resolving the singularity
- Higher delta invariant indicates a more complex singularity
- Milnor number is another invariant that measures the complexity of a singularity
- Defined as the degree of the gradient map of the curve at the singular point
- Provides information about the topological type of the singularity

Singularity Structure Analysis

Blow-up Technique

- Blow-up is a technique used to study the local structure of a singularity
- Transforms the curve into a new one with simpler singularities or no singularities
- Blow-up of a point on a curve replaces the point with the set of all tangent lines to the curve at that point
- Forms a projective line called the exceptional divisor
- Multiplicity of a singularity is the minimum number of blow-ups required to resolve the singularity
- Resolves into a smooth point or a collection of points

Analyzing Different Singularities with Blow-ups

- Blow-up of a node results in two smooth points
- Blow-up of a cusp results in a smooth point and a projective line tangent to the curve
- Blow-up of a tacnode may result in a node, a cusp, or a smooth point
- Depends on the nature of the tangency between the two branches
- Intersection multiplicity of two curves at a point can be determined by analyzing the exceptional divisors
- Obtained from successive blow-ups at the point

Singularity Resolution

Resolution Process

- Resolution of a singularity transforms the singular curve into a smooth curve or a collection of smooth curves
- Achieved through a sequence of blow-ups
- Minimal resolution of a singularity is the resolution obtained with the least number of blow-ups
- Equal to the multiplicity of the singularity
- Resolution graph is a tree-like diagram that represents the exceptional divisors and their intersections
- Obtained from the blow-up process

Computing Invariants from the Resolution

- Self-intersection number of an exceptional divisor in the resolution graph
- Negative of the number of other exceptional divisors it intersects
- Genus of the resolved curve can be computed using the genus formula
- Involves the delta invariants of the singularities and the self-intersection numbers of the exceptional divisors

- Canonical divisor of the resolved curve can be expressed in terms of the exceptional divisors
- Also involves the pullback of the canonical divisor of the original curve

Unit 9 – Algebraic Curves

9.1 Plane curves and their singularities

Plane curves are the building blocks of algebraic geometry, defined by polynomial equations in two variables. They come in various degrees and can be smooth or have singularities. Understanding their properties is key to grasping more complex geometric concepts.

Singularities on curves are special points where the curve behaves unusually. These can be nodes, cusps, or more complex types. Analyzing singularities helps us understand the curve's shape and properties, and is crucial for classifying and studying algebraic curves.

Plane algebraic curves

Definition and basic properties

- A plane algebraic curve is the set of points in the affine or projective plane whose coordinates satisfy a polynomial equation in two variables
- The degree of a plane curve is the degree of the defining polynomial equation
- Curves of degree 1 are lines, degree 2 are conics (circles, ellipses, parabolas, hyperbolas), degree 3 are cubics, and so on
- A plane curve is irreducible if its defining polynomial cannot be factored into lower degree polynomials
- An irreducible plane curve is also called a plane curve
- The intersection of two plane curves is a finite set of points
- The number of intersection points (counting multiplicity) is equal to the product of the degrees of the curves by Bézout's theorem

Rationality and smoothness

- A plane curve is rational if it can be parameterized by rational functions
- Every line and conic is rational, but not all curves of higher degree are rational
- For example, the cubic curve $y^2 = x^3 - x$ (elliptic curve) is not rational
- A plane curve is smooth or non-singular if it has no singular points
- Otherwise, it is called singular
- For example, the curve $y^2 = x^2(x + 1)$ has a singular point at $(0, 0)$

Singularities on curves

Types of singularities

- A point on a plane curve is singular if the partial derivatives of the defining polynomial vanish at that point
- A node or ordinary double point is a singular point where the curve crosses itself transversally

- The tangent lines at a node are distinct
- For example, the curve $y^2 = x^2(x + 1)$ has a node at $(0, 0)$
- A cusp is a singular point where the curve crosses itself tangentially
- The tangent lines at a cusp coincide
- For example, the curve $y^2 = x^3$ has a cusp at $(0, 0)$
- A tacnode is a singular point where the curve crosses itself with multiplicity 2, meaning it looks like two branches of the curve touching each other
- For example, the curve $y^2 = x^4$ has a tacnode at $(0, 0)$
- An ordinary multiple point of multiplicity m is a singular point where m smooth branches of the curve intersect transversally
- For example, the curve $(y^2 - x^2)(y - x^2) = 0$ has an ordinary triple point at $(0, 0)$

Higher order singularities

- Other higher order singularities can occur, such as triple points, inflection points, and more complicated singularities
- These singularities can be analyzed using more advanced techniques such as blow-ups and resolution of singularities
- The classification of higher order singularities is a deep and active area of research in algebraic geometry

Local behavior of curves

Tangent cones and multiplicities

- The tangent cone of a curve at a point is the set of tangent lines to the curve at that point
- It can be computed by taking the lowest degree terms of the Taylor expansion of the curve at the point
- For a node, the tangent cone consists of two distinct lines
- For a cusp, the tangent cone is a double line
- For a tacnode, the tangent cone is a double line
- The multiplicity of a singular point can be determined by the degree of the lowest degree terms in the Taylor expansion of the curve at the point

Puiseux series and branches

- Puiseux series can be used to analyze the local behavior of a curve near a singular point
- They provide a parametrization of the curve in terms of fractional power series
- For example, the curve $y^2 = x^3$ can be parametrized near $(0, 0)$ by $x = t^2, y = t^3$
- The branches of a curve at a singular point correspond to the Puiseux series expansions at that point
- The number of branches is equal to the number of distinct Puiseux series

- The intersection multiplicity of two branches at a singular point can be computed using the Puiseux series expansions of the branches
- For example, the curve $y^2 = x^2(x + 1)$ has two branches at $(0, 0)$ with intersection multiplicity 2

Multiplicity of singular points

Definition and computation

- The multiplicity of a point on a plane curve is the order of vanishing of the defining polynomial at that point
- It is the lowest degree of a monomial in the Taylor expansion of the polynomial at the point
- For a singular point, the multiplicity is at least 2
- For a non-singular point, the multiplicity is 1
- The multiplicity of a point can be computed by successively taking derivatives of the defining polynomial and evaluating at the point until a nonzero value is obtained
- For example, for the curve $y^2 = x^3$, the multiplicity of $(0, 0)$ is 2 since $f(0, 0) = f_x(0, 0) = 0$ but $f_y(0, 0) \neq 0$
- The multiplicity of a singular point is equal to the degree of the tangent cone at that point

Relations to other invariants

- The sum of the multiplicities of all singular points on a plane curve is bounded by the genus of the curve
- The genus can be computed using the degree-genus formula $g = \frac{(d-1)(d-2)}{2} - \sum_P \frac{m_P(m_P-1)}{2}$, where d is the degree and m_P is the multiplicity of a singular point P
- The delta invariant of a singular point is the difference between its multiplicity and the number of branches passing through it
- It measures the complexity of the singularity
- For example, a node has delta invariant 1, a cusp has delta invariant 1, and a tacnode has delta invariant 2

9.2 Genus and Riemann-Roch theorem

Genus and the Riemann-Roch theorem are key concepts in understanding algebraic curves. They help measure a curve's complexity and provide tools for analyzing its properties. These ideas are crucial for classifying curves and studying their behavior.

The genus relates to a curve's "holes" and degree, while Riemann-Roch connects meromorphic functions to divisors. Together, they offer powerful insights into curve structure and form the basis for deeper explorations in algebraic geometry.

Genus of Algebraic Curves

Definition and Geometric Interpretation

- The genus of an algebraic curve is a non-negative integer that measures the complexity or "holes" in the curve
- Geometrically, the genus represents the maximum number of non-intersecting simple closed curves that can be drawn on the surface without separating it
- The genus of a smooth, projective, irreducible algebraic curve over an algebraically closed field is a birational invariant
- The genus of a curve is related to its topological properties, such as its Euler characteristic and Betti numbers

Computing the Genus of Plane Curves

- For a smooth, projective, irreducible curve C of degree d in the complex projective plane, the genus g is given by the formula: $g = (d-1)(d-2)/2$
- This formula shows that the genus of a plane curve increases quadratically with its degree
- Curves of genus 0 are called rational curves and have a parametrization by rational functions (lines and conics)
- Curves of genus 1 are called elliptic curves and have a rich geometric and algebraic structure (smooth, irreducible plane curves of degree 3)

Riemann-Roch Theorem for Curves

Statement and Implications

- The Riemann-Roch theorem is a fundamental result in algebraic geometry that relates the dimension of the space of meromorphic functions on a curve to its genus and divisors
- For a smooth, projective, irreducible curve C over an algebraically closed field, and a divisor D on C , the Riemann-Roch theorem states: $l(D) - l(K - D) = \deg(D) - g + 1$, where $l(D)$ is the dimension of the space of meromorphic functions on C with poles bounded by D , K is the canonical divisor, and g is the genus of C
- The theorem provides a powerful tool for computing the dimension of linear systems on curves and understanding the behavior of meromorphic functions

Applications and Consequences

- The Riemann-Roch theorem has numerous applications in the study of algebraic curves, including the classification of curves, the computation of the gonality, and the study of special divisors such as Weierstrass points
- The theorem can be used to compute the dimension of the space of meromorphic functions on a curve with prescribed pole orders at specific points
- The Riemann-Roch theorem is a key ingredient in the proof of the Riemann-Hurwitz formula, which relates the genera of curves in a covering map

Dimension of Meromorphic Functions

Computing Dimensions using Riemann-Roch

- The dimension of the space of meromorphic functions on a curve C with poles bounded by a divisor D , denoted $l(D)$, can be computed using the Riemann-Roch theorem
- For a canonical divisor K and a divisor D on C , the dimension $l(D)$ is given by the formula:
 $l(D) = \deg(D) - g + 1 + l(K - D)$, where g is the genus of the curve
- In the case of the trivial divisor ($D = 0$), the dimension $l(0)$ equals 1, corresponding to the constant functions on the curve

Special Cases and Vanishing

- For a divisor D with degree greater than or equal to $2g-1$, where g is the genus, the dimension $l(D)$ is given by $l(D) = \deg(D) - g + 1$, as the term $l(K-D)$ vanishes
- The computation of $l(D)$ provides information about the number of linearly independent meromorphic functions on the curve with prescribed pole orders at specific points
- The vanishing of $l(K-D)$ for divisors of high degree is related to the Kodaira vanishing theorem in higher dimensions

Genus vs Degree of Plane Curves

Relationship and Classification

- For a smooth, projective, irreducible plane curve C of degree d in the complex projective plane, the genus g and the degree d are related by the formula: $g = (d-1)(d-2)/2$
- This formula demonstrates that the genus of a plane curve increases quadratically with its degree
- The relationship between genus and degree provides a way to classify and study algebraic curves based on their intrinsic geometric properties
- As the degree of a plane curve increases, so does its genus, leading to more complex geometric and algebraic properties

Examples and Special Cases

- Curves of genus 0 are called rational curves and have a parametrization by rational functions. The only smooth, irreducible plane curves of genus 0 are lines ($d=1$) and conics ($d=2$)
- Curves of genus 1 are called elliptic curves and have a rich geometric and algebraic structure. Smooth, irreducible plane curves of genus 1 have degree 3
- Hyperelliptic curves are curves of genus g that admit a degree 2 map to the projective line. They can be described by equations of the form $y^2 = f(x)$, where $f(x)$ is a polynomial of degree $2g + 1$ or $2g + 2$
- Plane curves of high degree and genus, such as curves of degree 4 (genus 3) and degree 5 (genus 6), exhibit increasingly complex geometric and algebraic properties, and are the subject of active research in algebraic geometry

9.3 Intersection theory for plane curves

Intersection theory for plane curves is a powerful tool in algebraic geometry. It helps us understand how curves intersect, counting their meeting points and measuring how they touch. This theory connects deeply to curve degrees and shapes.

Bézout's theorem is the star of intersection theory. It tells us that two curves' total intersections equal their degree product. This simple idea unlocks complex geometric problems and helps classify curves by their genus and singularities.

Intersection multiplicity of curves

Definition and properties

- The intersection multiplicity of two plane curves C and D at a point P is a non-negative integer that measures the degree of tangency between the curves at P
- If the curves C and D do not pass through P , the intersection multiplicity at P is 0
- If P is a simple point (transversal intersection) of both curves, the intersection multiplicity is 1
- Higher intersection multiplicities occur when the curves are tangent or have higher-order contact at P (osculating curves, curves with a common cusp)

Computation methods

- The intersection multiplicity can be computed using the dimensions of certain local ring quotients (local intersection number)
- The intersection multiplicity can also be determined by the order of vanishing of resultants (resultant-based approach)
- The sum of intersection multiplicities at all points of intersection of C and D is equal to the product of their degrees, assuming C and D have no common components (Bézout's theorem)

Bézout's theorem for intersections

Statement and proof

- Bézout's theorem states that if C and D are two plane curves of degrees m and n , respectively, with no common component, then the total number of intersections of C and D , counted with multiplicity, is equal to mn
- The proof of Bézout's theorem can be done using resultants, by considering the intersection of C and D as a divisor on the complex projective plane (divisor-based proof)
- Bézout's theorem is a fundamental result in intersection theory and has numerous applications in algebraic geometry and related fields (enumerative geometry, solving systems of polynomial equations)

Applications and examples

- Bézout's theorem can be used to determine the maximum number of intersections between two curves of given degrees (degree bounds)
- The theorem helps analyze the behavior of intersections under certain transformations, such as projective transformations or birational maps (invariance properties)

- Examples of Bézout's theorem include finding the number of intersections between a line and a conic (2 intersections), or between two conics (4 intersections, counting multiplicity)

Intersection theory in geometry

Solving geometric problems

- Intersection theory can be used to determine the number and nature of intersections between plane curves, which is useful in solving various geometric problems
- By applying Bézout's theorem, one can find the total number of intersections (real and complex) between two curves, given their degrees
- Intersection multiplicities can help identify tangency points, multiple points, or other special intersection points between curves (singularities, inflection points)

Studying families of curves

- Intersection theory can be used to study the geometry of families of curves, such as pencils of curves or linear systems (base points, multiplicities)
- Applications of intersection theory include finding common tangents to curves, studying singular points of curves, and analyzing the behavior of curves under certain transformations (projective, birational)
- Intersection theory helps classify special families of curves, such as rational curves, elliptic curves, or curves with prescribed singularities (cuspidal curves, nodal curves)

Intersection theory and curve genus

Genus and its properties

- The genus of a smooth, irreducible, projective plane curve C is a non-negative integer that measures the complexity of the curve's topology (number of holes or handles)
- The genus can be computed using the degree-genus formula: $g = (d-1)(d-2)/2$, where d is the degree of the curve C
- The degree-genus formula can be derived using intersection theory, specifically by considering the intersection of C with a line and applying the adjunction formula (canonical divisor, Riemann-Roch theorem)

Relation to singularities and classification

- The genus is related to the number and types of singularities (multiple points, cusps, etc.) that a curve possesses; intersection theory helps analyze these singularities (blow-ups, resolution of singularities)
- The genus is an important invariant in the classification of algebraic curves and surfaces, and intersection theory plays a crucial role in studying the geometry and arithmetic of curves (moduli spaces, Jacobians)
- Examples of curves with different genera include rational curves (genus 0), elliptic curves (genus 1), and hyperelliptic curves (genus ≥ 2)

Unit 10 – Algebraic Surfaces

10.1 Classification of algebraic surfaces

Algebraic surfaces are fascinating two-dimensional varieties defined by polynomial equations. They come in various types, like rational, ruled, and quadric surfaces, each with unique properties. Understanding these surfaces is key to grasping the broader world of algebraic geometry.

The classification of algebraic surfaces is a crucial aspect of their study. By examining invariants like degree, genus, and Kodaira dimension, we can categorize surfaces into distinct classes. This classification helps us understand their structure and properties more deeply.

Types of algebraic surfaces

Definition and characteristics

- Algebraic surfaces are two-dimensional varieties defined by polynomial equations in three or more variables over a field
- The dimension of an algebraic surface equals the dimension of the ambient space minus the number of defining equations
- Smooth algebraic surfaces have no singular points, while singular surfaces have at least one singular point where the tangent space is not well-defined

Examples of algebraic surfaces

- Rational surfaces are algebraic surfaces birationally equivalent to the projective plane, having a parametrization by rational functions (projective plane, Hirzebruch surfaces)
- Ruled surfaces are algebraic surfaces generated by the motion of a straight line in space (cylinders, cones, tangent developables)
- Quadric surfaces are degree two algebraic surfaces (ellipsoids, hyperboloids, paraboloids)
- Cubic surfaces are degree three algebraic surfaces, which can have up to 27 distinct lines lying on them

Classification of algebraic surfaces

Invariants and properties

- The degree of an algebraic surface is the degree of its defining polynomial equation
- The genus of an algebraic surface is a topological invariant related to the number of holes or handles in the surface
- Algebraic surfaces can be classified by their Kodaira dimension, which measures the growth rate of pluricanonical forms
- Surfaces with Kodaira dimension $-\infty$ include rational surfaces and ruled surfaces
- Surfaces with Kodaira dimension 0 include K3 surfaces, Enriques surfaces, and abelian surfaces
- Surfaces with Kodaira dimension 1 are elliptic surfaces, having a fibration by elliptic curves

- Surfaces with Kodaira dimension 2 are of general type, with ample canonical bundle

Enriques-Kodaira classification

- The Enriques-Kodaira classification categorizes algebraic surfaces based on their minimal models and numerical invariants
- Every algebraic surface is birationally equivalent to a minimal surface in one of ten classes
- Minimal rational surfaces are the projective plane and Hirzebruch surfaces
- Minimal surfaces of Kodaira dimension 0 include K3 surfaces, Enriques surfaces, and abelian surfaces

Degree and genus of surfaces

Degree-genus formula

- The degree-genus formula relates the degree d and genus g of a smooth algebraic surface:

$$g = (d - 1)(d - 2)/2$$
- For a degree d hypersurface in projective 3-space, the genus is given by $g = (d - 1)(d - 2)/2$

Related invariants

- The gonality of an algebraic surface is the minimum degree of a map from the surface to a curve, related to the genus
- The Noether inequality bounds the degree of a smooth algebraic surface in terms of its sectional genus and irregularity

Minimal models of surfaces

Definition and properties

- A minimal model of an algebraic surface is a smooth surface birationally equivalent to the original surface with no exceptional curves of the first kind ((-1)-curves)
- The minimal model program aims to find a minimal model for any given algebraic surface through a sequence of birational transformations

Criteria and examples

- Castelnuovo's criterion states that a smooth algebraic surface is rational if and only if its irregularity and second plurigenus are both zero
- Minimal rational surfaces are the projective plane and Hirzebruch surfaces
- Minimal surfaces of Kodaira dimension 0 include K3 surfaces, Enriques surfaces, and abelian surfaces

10.2 Ruled surfaces and rational surfaces

Ruled surfaces are like mathematical playgrounds, where every point hangs out on a straight line. They're the cool kids of geometry, showing up as cylinders, cones, and even fancy hyperboloids. These surfaces come in different flavors, from developable to skew, each with its own unique twist.

Rational surfaces take things up a notch, using fancy rational functions to describe themselves. They're like the VIP club of surfaces, including the projective plane and Hirzebruch surfaces. While all rational surfaces are ruled, not all ruled surfaces make the cut as rational.

Ruled Surfaces and Properties

Definition and Parametrization

- A ruled surface is a surface that can be swept out by a moving straight line, called a ruling or generator
- Every point on a ruled surface lies on a straight line that is contained in the surface
- Ruled surfaces can be parametrized by a curve and a family of lines, with the curve determining the direction of the ruling at each point
- Examples of ruled surfaces include:
 - Cylinders
 - Cones
 - Hyperboloids of one sheet
 - Hyperbolic paraboloids

Degree, Class, and Classification

- The degree of a ruled surface is the degree of the equation defining the surface, which is related to the number of rulings passing through a general point in space
- The class of a ruled surface is the degree of the dual surface, which is the set of all tangent planes to the surface
- Ruled surfaces can be classified as:
 - Developable surfaces: Tangent plane is constant along each ruling (cylinders and cones)
 - Non-developable (skew) surfaces: Tangent plane varies along each ruling (hyperboloids and hyperbolic paraboloids)

Rational vs Ruled Surfaces

Rational Surfaces

- A rational surface is a surface that can be parametrized by rational functions, i.e., the ratio of two polynomials in two variables
- Examples of rational surfaces:
 - Projective plane ($\mathbb{P}^2$)
 - Hirzebruch surfaces ($\mathbb{F}_n$)
- Rational surfaces can be obtained by:
 - Blowing up points on the projective plane
 - Blowing down exceptional curves on a surface

- The Picard number of a rational surface, which measures the rank of its Picard group (the group of divisors modulo linear equivalence), is related to the number of blowups and blowdowns performed

Relationship between Rational and Ruled Surfaces

- Every rational surface is a ruled surface, but not every ruled surface is rational
- The minimal model of a rational surface is the simplest surface birationally equivalent to it, obtained by contracting all exceptional curves

Birational Geometry of Rational Surfaces

Birational Equivalence and Maps

- Two surfaces are birationally equivalent if there exist rational maps between them that are inverses of each other outside of a lower-dimensional subset
- Birational maps preserve many geometric properties, such as:
 - Degree
 - Arithmetic genus
 - Kodaira dimension of a surface
- The birational geometry of rational surfaces is determined by the minimal model and the configuration of exceptional curves

Cremona Group and Factorization

- The Cremona group is the group of birational automorphisms of the projective plane, which is generated by:
 - Linear automorphisms
 - Standard quadratic transformations
- Birational maps between rational surfaces can be factored into a sequence of blowups and blowdowns, as described by the Castelnuovo-Enriques-Noether theorem

Minimal Ruled Surfaces

Definition and Properties

- A minimal ruled surface is a ruled surface that does not contain any exceptional curves, i.e., curves with self-intersection number -1
- Every ruled surface is birationally equivalent to a minimal ruled surface, obtained by contracting all exceptional curves
- The Picard number of a minimal ruled surface is 2, and its Picard group is generated by:
 - The class of a fiber
 - The class of a section

Hirzebruch Surfaces

- Minimal rational ruled surfaces are isomorphic to Hirzebruch surfaces, which are $\mathbb{P}^1$ -bundles over $\mathbb{P}^1$
- The Hirzebruch surface F_n is defined by the projectivization of the vector bundle $\mathcal{O}(0) \oplus \mathcal{O}(n)$ over $\mathbb{P}^1$, where n is a non-negative integer
- The self-intersection number of the minimal section (the section with the smallest self-intersection) on a Hirzebruch surface F_n is $-n$

10.3 Elliptic surfaces and K3 surfaces

Elliptic surfaces and K3 surfaces are fascinating objects in algebraic geometry. They combine complex geometry, topology, and arithmetic in unique ways, offering deep insights into the structure of algebraic surfaces.

These surfaces play a crucial role in the classification of algebraic surfaces. Elliptic surfaces provide a bridge between curves and higher-dimensional varieties, while K3 surfaces represent a key class of surfaces with Kodaira dimension zero.

Elliptic Surfaces and Elliptic Curves

Definition and Construction of Elliptic Surfaces

- An elliptic surface is a complex surface that admits a fibration by elliptic curves over a base curve
- The generic fiber of an elliptic surface is an elliptic curve, which is a smooth, projective algebraic curve of genus one with a specified base point
- Elliptic surfaces can be constructed by taking the product of an elliptic curve with another curve and then blowing up the singular points (resolution of singularities)
- The Weierstrass equation can be used to describe elliptic surfaces, with the coefficients being functions on the base curve
- For example, the Weierstrass equation $y^2 = x^3 + a(t)x + b(t)$ defines an elliptic surface over the base curve with coordinate t

Properties and Invariants of Elliptic Surfaces

- The j -invariant of an elliptic surface is a rational function on the base curve that encodes information about the isomorphism class of the fibers
- Two elliptic curves are isomorphic if and only if they have the same j -invariant
- Elliptic surfaces can have singular fibers, which are fibers that are not smooth elliptic curves
- Examples of singular fibers include nodal cubic curves, cuspidal cubic curves, and configurations of rational curves
- The Euler characteristic of an elliptic surface can be computed using the topological Euler characteristic of the base curve and the number and types of singular fibers
- Elliptic surfaces with non-constant j -invariant have a positive Euler characteristic, while those with constant j -invariant have Euler characteristic zero

Classification of Elliptic Surfaces

Kodaira Classification of Singular Fibers

- Kodaira classified the possible types of singular fibers that can occur on an elliptic surface
- Type I^n : a cycle of n rational curves (nodal cubic for $n = 1$, two rational curves meeting at two points for $n = 2$)
- Type II: a cuspidal cubic curve
- Type III: two rational curves meeting at a tacnode
- Type IV: three rational curves meeting at a single point
- Types I_n^* , II^* , III^* , IV^* : more complicated configurations of rational curves
- The types of singular fibers that appear on an elliptic surface determine its Euler characteristic and Kodaira dimension

Kodaira Dimension and Euler Characteristic

- The Kodaira dimension of an elliptic surface is determined by its Euler characteristic and the number of singular fibers
- Kodaira dimension $-\infty$: ruled surfaces (Euler characteristic < 0)
- Kodaira dimension 0: K3 surfaces, abelian surfaces, Enriques surfaces (Euler characteristic $= 0$)
- Kodaira dimension 1: properly elliptic surfaces (Euler characteristic > 0)
- Elliptic surfaces with non-constant j -invariant have a positive Euler characteristic, while those with constant j -invariant have Euler characteristic zero
- For example, a rational elliptic surface (base curve is $\mathbb{P}^1$) with 12 singular fibers of type I^1 has Euler characteristic 12

Geometric and Arithmetic Properties of K3 Surfaces

Geometric Properties of K3 Surfaces

- K3 surfaces are complex surfaces with trivial canonical bundle and irregularity zero
- The canonical bundle is the top exterior power of the cotangent bundle, and its triviality means that K3 surfaces have a unique non-vanishing holomorphic 2-form (up to scaling)
- Irregularity is the dimension of the first cohomology group with coefficients in the structure sheaf, and its vanishing implies that K3 surfaces have no non-trivial 1-forms
- K3 surfaces are simply connected and have a unique holomorphic 2-form up to scaling
- The second cohomology group of a K3 surface is a 22-dimensional lattice with a specific intersection form
- The intersection form is even, unimodular, and of signature $(3, 19)$
- The lattice is isomorphic to the direct sum of three copies of the hyperbolic plane and two copies of the E^8 lattice

Arithmetic Properties of K3 Surfaces

- The Picard group of a K3 surface is a discrete subgroup of the second cohomology group and describes the algebraic curves on the surface
- The rank of the Picard group (Picard number) is between 0 and 20
- K3 surfaces with Picard number 20 are called singular K3 surfaces and have interesting arithmetic properties
- K3 surfaces can be realized as double covers of the projective plane branched over a sextic curve or as quartic surfaces in projective 3-space
- The sextic curve and the quartic surface must satisfy certain conditions to ensure that the double cover or the quartic is a K3 surface
- Elliptic K3 surfaces are K3 surfaces that admit an elliptic fibration, and they have a rich arithmetic structure related to the Mordell-Weil group of sections
- The Mordell-Weil group is the group of rational points on the generic fiber of the elliptic fibration
- The rank of the Mordell-Weil group is related to the Picard number of the K3 surface

Moduli Spaces of Elliptic vs K3 Surfaces

Moduli Spaces of Elliptic Surfaces

- The moduli space of elliptic surfaces with a fixed base curve is a complex analytic space that parametrizes isomorphism classes of elliptic surfaces
- The dimension of the moduli space depends on the genus of the base curve and the number and types of singular fibers
- The moduli space of Jacobian elliptic surfaces, which are elliptic surfaces with a section, is related to the moduli space of stable curves
- A section of an elliptic surface is a map from the base curve to the surface that is a right inverse to the fibration map
- The moduli space of stable curves is a compactification of the moduli space of smooth curves that allows certain types of singularities (nodes)

Moduli Spaces of K3 Surfaces

- The moduli space of K3 surfaces is a 20-dimensional quasi-projective variety that can be compactified using the theory of periods
- The period of a K3 surface is the line integral of the holomorphic 2-form over a basis of the second homology group
- The period domain is a 20-dimensional complex manifold that parametrizes Hodge structures on the second cohomology of a K3 surface
- The moduli space of polarized K3 surfaces, which are K3 surfaces with a chosen ample line bundle, is a 19-dimensional quasi-projective variety
- An ample line bundle is a line bundle whose sections define a projective embedding of the surface

- The degree of the polarization is the self-intersection number of the ample line bundle
- The moduli spaces of elliptic and K3 surfaces have interesting arithmetic properties, such as the existence of dense sets of rational points in certain cases
- For example, the moduli space of elliptic K3 surfaces with a section has a dense set of rational points corresponding to surfaces with Picard number 20

Unit 11 – Commutative Algebra

Connections

11.1 Graded rings and modules

Graded rings and modules are powerful tools in algebraic geometry. They provide a way to study geometric objects like projective varieties using algebraic techniques. This connection allows us to translate complex geometric problems into more manageable algebraic ones.

The concept of grading adds structure to rings and modules, making them easier to work with. It's particularly useful for understanding projective spaces, their subvarieties, and associated sheaves. This algebraic approach opens up new ways to tackle geometric questions.

Graded rings and modules

Definition and basic properties

- A graded ring is a ring R together with a direct sum decomposition $R = \bigoplus_{n=0} R_n$ (as abelian groups) such that $R_m \cdot R_n \subseteq R_{m+n}$ for all $m, n \geq 0$
- The elements of R_n are called homogeneous elements of degree n
- A graded R -module is an R -module M together with a direct sum decomposition $M = \bigoplus_{n=0} M_n$ (as abelian groups) satisfying $R_m \cdot M_n \subseteq M_{m+n}$ for all $m, n \geq 0$
- The elements of M_n are called homogeneous elements of degree n
- A homomorphism of graded rings (resp. modules) maps homogeneous elements to homogeneous elements of the same degree, preserving the grading structure
- The direct sum, tensor product, and Hom of graded modules inherit a natural grading
- Similarly, the kernel, image, and cokernel of a homomorphism of graded modules are also graded

Noetherian property

- A graded ring (resp. module) is called Noetherian if it satisfies the ascending chain condition on graded ideals (resp. submodules)
- This means that any ascending chain of graded ideals (resp. submodules) stabilizes after finitely many steps
- A graded ring is Noetherian if and only if it is finitely generated as an R_0 -algebra
- In other words, there exist finitely many homogeneous elements that generate the ring over R_0

Examples from algebraic geometry

Polynomial rings and projective varieties

- The polynomial ring $R = k[x_0, \dots, x_n]$ over a field k is a graded ring with the standard grading $\deg(x_i) = 1$ for all i

- The homogeneous elements of degree d are the homogeneous polynomials of degree d
- The homogeneous coordinate ring of a projective variety $X \subseteq \mathbb{P}^n$ is the graded ring $S(X) = k[x_0, \dots, x_n]/I(X)$, where $I(X)$ is the homogeneous ideal of X
- The degree d part of $S(X)$ corresponds to the global sections of the line bundle $\mathcal{O}_X(d)$ on X

Twisted sheaves and graded modules

- The twisted sheaf $\mathcal{O}_X(d)$ on a projective variety X is a graded $S(X)$ -module, where the degree n part is given by the global sections of $\mathcal{O}_X(d+n)$
- More generally, any coherent sheaf F on X gives rise to a graded $S(X)$ -module $\bigoplus_n H^0(X, F(n))$
- The graded module encodes information about the sheaf cohomology of F

Weighted projective spaces

- The homogeneous coordinate ring of a weighted projective space $P(a_0, \dots, a_n)$ is the graded ring $k[x_0, \dots, x_n]$ with the weighted grading $\deg(x_i) = a_i$
- Weighted projective spaces and their subvarieties provide examples of graded rings with non-standard gradings
- They arise naturally in the study of certain singular varieties and orbifolds

Applications in algebraic geometry

Hilbert functions and polynomials

- The Hilbert function of a graded module M is the function $HM(n) = \dim_k M_n$, measuring the dimension of each graded component
- For a projective variety X , the Hilbert function of $S(X)$ encodes important geometric information (dimension, degree)
- The Hilbert polynomial of a graded module M is the unique polynomial $PM(t)$ such that $PM(n) = HM(n)$ for all sufficiently large n
- The Hilbert polynomial of a projective variety X is a key invariant used to classify X and study its properties
- The Hilbert series of a graded module M is the generating function $\text{Hilb}M(t) = \sum_n (\dim_k M_n) t^n$, encoding the same information as the Hilbert function and polynomial in a compact form

Sheaf cohomology and regularity

- Graded modules can be used to study the sheaf cohomology of projective varieties
- For example, the graded module $\bigoplus_n H^i(X, \mathcal{O}_X(n))$ determines the cohomology groups $H^i(X, \mathcal{O}_X(n))$ for all i and n
- The Castelnuovo-Mumford regularity of a graded module M is the smallest integer r such that M_n is spanned by its elements of degree $\leq n+r$ for all n
- It measures the complexity of M and has applications to the computation of syzygies and free resolutions

- Regularity is a key invariant in the study of projective embeddings and the geometry of subvarieties

Graded rings vs projective varieties

Correspondence between graded rings and projective varieties

- The homogeneous coordinate ring $S(X)$ of a projective variety $X \subseteq \mathbb{P}^n$ determines X up to isomorphism
- Conversely, every finitely generated graded k -algebra $R = \bigoplus_{n=0}^{\infty} R_n$ with $R_0 = k$ gives rise to a projective variety $\text{Proj}(R)$
- The functor Proj establishes a contravariant equivalence between the category of finitely generated graded k -algebras (modulo nilpotents) and the category of projective k -varieties
- This correspondence is a fundamental tool in algebraic geometry, allowing to study geometric properties using algebraic techniques

Subvarieties and graded ideals

- The closed subvarieties of $\text{Proj}(R)$ correspond to the graded prime ideals of R (except the irrelevant ideal $\bigoplus_{n=1}^{\infty} R_n$)
- The dimension of $\text{Proj}(R)$ is equal to the Krull dimension of R minus one
- The twisting sheaf $\mathcal{O}_{\text{Proj}(R)}(n)$ on $\text{Proj}(R)$ corresponds to the graded R -module $R(n)$, where $R(n)_m = R_{n+m}$
- More generally, coherent sheaves on $\text{Proj}(R)$ correspond to finitely generated graded R -modules (modulo torsion)

Serre-Grothendieck correspondence

- The Serre-Grothendieck correspondence establishes an equivalence between the category of coherent sheaves on $\text{Proj}(R)$ and the quotient category of finitely generated graded R -modules modulo torsion modules
- It allows to translate between sheaf-theoretic and module-theoretic properties
- Many geometric constructions (tensor products, pullbacks, pushforwards) can be studied algebraically using this correspondence

11.2 Primary decomposition and associated primes

Primary decomposition and associated primes are powerful tools in algebraic geometry. They help us break down complex ideals into simpler parts, giving us insights into the structure of algebraic sets and modules.

These concepts connect algebraic geometry to commutative algebra by linking geometric properties to algebraic ones. They're crucial for understanding the local behavior of varieties, their irreducible components, and singularities.

Primary Decomposition of Ideals and Modules

Definitions and Properties

- A proper ideal Q in a commutative ring R is called primary if for any $a, b \in R$ with $ab \in Q$, either $a \in Q$ or $b^n \in Q$ for some positive integer n
- The radical of a primary ideal Q , denoted $\sqrt{Q}$, is a prime ideal
- If $\sqrt{Q} = P$, then Q is called P -primary
- Every proper ideal in a Noetherian ring can be expressed as a finite intersection of primary ideals, called a primary decomposition
- A primary decomposition is called minimal if:
 - The prime ideals $\sqrt{Q_i}$ are all distinct
 - $Q_i \not\subseteq \bigcap_{j \neq i} Q_j$ for each i

Associated Primes and Modules

- The prime ideals $\sqrt{Q_i}$ in a minimal primary decomposition are called the associated primes of the ideal
- The concept of primary decomposition can be extended to modules over a Noetherian ring
- Submodules take the place of ideals in this context
- Associated primes play a key role in understanding the structure of modules
- The set of associated primes of a module M is denoted $\text{Ass}(M)$
- For an ideal I , $\text{Ass}(R/I)$ corresponds to the associated primes of I

Computing Primary Decompositions

Polynomial Rings and Gröbner Bases

- To find a primary decomposition of an ideal I in a polynomial ring $k[x_1, \dots, x_n]$: 1. Start by finding a decomposition $I = Q_1 \cap \dots \cap Q_r$, where the Q_i are primary ideals 2. The associated primes are then $P_i = \sqrt{Q_i}$ for each i
- The associated primes can be computed using techniques such as finding the minimal primes containing I
- Gröbner basis techniques can be used to compute primary decompositions in more complex cases
- Buchberger's algorithm for computing Gröbner bases is a key tool (Buchberger, 1965)

Simple Cases and Examples

- In simple cases, the primary decomposition can often be found by inspection or using the structure theorem for modules over a PID
- Example: In the ring $\mathbb{Z}$, the ideal (12) has primary decomposition $(12) = (2^2) \cap (3)$
- Monomial ideals in polynomial rings also have straightforward primary decompositions

- Example: In $k[x,y]$, the ideal (x^2, xy) has primary decomposition $(x^2, xy) = (x) \cap (x^2, y)$

Primary Decomposition for Algebraic Varieties

Irreducible Components and Dimension

- If I is an ideal in a polynomial ring $k[x_1, \dots, x_n]$, the algebraic set $V(I)$ is the set of points in k^n where all polynomials in I vanish
- The minimal associated primes of I correspond to the irreducible components of $V(I)$
- The dimension of each component is given by the codimension of the corresponding prime ideal
- Example: Let $I = (xy, xz)$ in $C[x,y,z]$. Then $V(I)$ has two irreducible components:
 - $V(x)$, which has codimension 1 and dimension 2
 - $V(y,z)$, which has codimension 2 and dimension 1

Embedded Components and Singularities

- The primary decomposition of I can be used to study the local structure of $V(I)$ at its singular points
- The embedded primes (associated primes that are not minimal) correspond to the embedded components of $V(I)$
- The multiplicity of a component can be determined from the length of the corresponding primary ideal in the local ring at a point
- Example: Let $I = (x^2, xy)$ in $C[x,y]$. The primary decomposition is $I = (x) \cap (x^2, y)$
- $V(I)$ has one irreducible component $V(x)$ of dimension 1
- $V(x^2, y)$ is an embedded component consisting of the origin, which is a singular point

Primary Decomposition vs Geometric Properties

Irreducible and Embedded Components

- The irreducible components of an algebraic set $V(I)$ correspond to the minimal associated primes of I
- These are the primary components of I of maximum dimension
- The embedded components of $V(I)$ correspond to the embedded primes of I
- These are the primary components of I of smaller dimension that are "embedded" in the larger components
- Example: In the previous example, $V(x)$ is the irreducible component and $V(x^2, y)$ is the embedded component

Local Rings and Support

- The local ring of $V(I)$ at a point P is obtained by localizing the coordinate ring $k[x_1, \dots, x_n]/I$ at the maximal ideal corresponding to P
- The primary decomposition of I determines the structure of this local ring

- The support of a module M over $k[x_1, \dots, x_n]/I$ is the set of points in $V(I)$ where the localization of M is non-zero
- This can be determined from the associated primes of M
- Example: Let $M = (k[x,y]/(xy))_x$ be the localization of $k[x,y]/(xy)$ at the prime ideal (x) . Then:
- $\text{Ass}(M) = \{(x), (y)\}$
- $\text{Supp}(M) = V(x) \cup V(y)$, the union of the x and y axes in k^2

Invariants of Algebraic Sets

- The primary decomposition of I can be used to compute invariants of $V(I)$ such as:
- Dimension: the maximum dimension of the irreducible components
- Degree: the sum of the degrees of the irreducible components, weighted by their multiplicities
- Arithmetic genus: a measure of the "complexity" of the singularities of $V(I)$
- These invariants provide key information about the geometry and topology of the algebraic set $V(I)$
- They can be used to classify algebraic varieties and study their moduli spaces

11.3 Depth, regular sequences, and Cohen-Macaulay rings

Depth, regular sequences, and Cohen-Macaulay rings are key concepts in commutative algebra that bridge algebraic and geometric properties. They measure how "nice" a ring or module is, with implications for smoothness, singularities, and cohomology of algebraic varieties.

These ideas connect algebra to geometry by relating the length of regular sequences to the codimension of supports. They're crucial for understanding singularities, computing cohomology, and studying properties of algebraic varieties like normality and rationality.

Module Depth and Geometry

Definition and Geometric Interpretation

- Define the depth of a module M over a local ring $(R, \mathfrak{m})$ as the length of a maximal M -regular sequence in the maximal ideal $\mathfrak{m}$
- Interpret the depth of a module geometrically as the codimension of the support of the module in the spectrum of the ring (affine algebraic variety, projective variety)
- Understand that for a finite module M over a local ring $(R, \mathfrak{m})$, the depth of M is always less than or equal to the dimension of M
- Recognize that if M is a finite module over a local ring $(R, \mathfrak{m})$, then $\text{depth}(M) = \dim(M)$ if and only if M is a Cohen-Macaulay module

Measuring Distance from Cohen-Macaulay Property

- Interpret the depth of a module as a measure of how far the module is from being Cohen-Macaulay
- Understand that the closer the depth of a module is to its dimension, the closer it is to being Cohen-Macaulay

- Recognize that if the depth of a module is strictly less than its dimension, it fails to be Cohen-Macaulay
- Use the difference between depth and dimension to quantify the deviation from the Cohen-Macaulay property (local rings, graded rings)

Regular Sequences in Commutative Algebra

Definition and Properties

- Define an M -regular sequence as a sequence of elements $x_1, \dots, x_n$ in a ring R such that x_i is a non-zero-divisor on $M/(x_1, \dots, x_{i-1})M$ for all $i = 1, \dots, n$
- Recognize the importance of regular sequences in the study of depth and Cohen-Macaulay rings and modules
- Understand that the length of a maximal regular sequence in an ideal I of a local ring $(R, \mathfrak{m})$ is equal to the grade of the ideal I
- Apply the property that if $x_1, \dots, x_n$ is an M -regular sequence, then M is a free module over $R/(x_1, \dots, x_n)$ if and only if $M/(x_1, \dots, x_n)M$ is a free $R/(x_1, \dots, x_n)$ -module

Applications in Homological Algebra

- Understand the role of regular sequences in defining the Koszul complex, a fundamental tool in homological algebra
- Use regular sequences to compute the depth of a module via the Koszul complex
- Apply the Auslander-Buchsbaum formula, which relates the projective dimension of a module to its depth using regular sequences
- Recognize the connection between regular sequences and the Tor functor in homological algebra (resolutions, spectral sequences)

Cohen-Macaulay Rings and Modules

Definition and Characterizations

- Define a local ring $(R, \mathfrak{m})$ as Cohen-Macaulay if $\text{depth}(R) = \dim(R)$
- Define a finite module M over a local ring $(R, \mathfrak{m})$ as Cohen-Macaulay if $\text{depth}(M) = \dim(M)$
- Understand equivalent characterizations of Cohen-Macaulay rings and modules, such as the existence of a maximal regular sequence generating the maximal ideal or the vanishing of certain local cohomology modules
- Recognize that regular local rings are always Cohen-Macaulay, but the converse is not true in general (singular Cohen-Macaulay rings)

Closure Properties

- Apply the property that if R is a Cohen-Macaulay local ring and x is a non-zero-divisor in R , then R/xR is also Cohen-Macaulay
- Understand that the Cohen-Macaulay property is preserved under localization and completion

- Recognize that the Cohen-Macaulay property is not necessarily preserved under taking quotients or extensions (non-flat homomorphisms, non-Cohen-Macaulay rings)
- Apply the result that if R is a Cohen-Macaulay ring and I is an ideal generated by a regular sequence, then R/I is also Cohen-Macaulay

Cohen-Macaulay Rings for Algebraic Geometry

Affine Algebraic Varieties

- Understand that an affine algebraic variety X over an algebraically closed field k is Cohen-Macaulay if and only if its coordinate ring $k[X]$ is a Cohen-Macaulay ring
- Apply the property that the Cohen-Macaulay property of an algebraic variety is preserved under taking open subsets and irreducible components
- Use the Cohen-Macaulay property of an algebraic variety to study its singularities and compute its cohomology groups (singular cohomology, étale cohomology)

Connections to Other Geometric Properties

- Relate the Cohen-Macaulay property to Serre's criterion for normality and Hartshorne's connectedness theorem
- Understand the connection between the Cohen-Macaulay property and the canonical module of an algebraic variety
- Apply the Hochster-Roberts theorem, which states that rings of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay
- Recognize the role of Cohen-Macaulay rings in the study of singularities, such as rational singularities and Du Bois singularities

Cohen-Macaulay Property vs Geometry

Relation to Smoothness

- Understand that a smooth algebraic variety over a field k is always Cohen-Macaulay
- Recognize that the converse is not true in general: there exist singular Cohen-Macaulay varieties (cones, monomial curves)
- Apply the property that if X is an equidimensional algebraic variety over a field k , then X is Cohen-Macaulay if and only if the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay for all x in X

Relation to Complete Intersections and Gorenstein Property

- Understand that the Cohen-Macaulay property is a necessary condition for an algebraic variety to be a complete intersection, but not sufficient
- Recognize that complete intersections are always Cohen-Macaulay, but not all Cohen-Macaulay varieties are complete intersections (determinantal varieties)
- Relate the Cohen-Macaulay property to the Gorenstein property, which is a stronger condition on the local rings of an algebraic variety

- Apply the result that Gorenstein rings are always Cohen-Macaulay, but the converse is not true in general (hypersurfaces, canonical singularities)

Unit 12 – Applications in Algebraic Geometry

12.1 Algebraic groups and group actions

Algebraic groups blend algebra and geometry, giving us powerful tools to study symmetries and transformations. They're like mathematical Swiss Army knives, helping us understand everything from simple shapes to complex geometric structures.

In this part, we'll see how these groups work and how they act on varieties. We'll explore their structure, learn about important examples, and see how they're used to solve real-world problems in math and beyond.

Algebraic groups and properties

Definition and basic structure

- An algebraic group is a variety G equipped with morphisms for multiplication $\mu: G \times G \rightarrow G$ and inversion $\iota: G \rightarrow G$, satisfying the group axioms
- The identity element of an algebraic group is a distinguished point e in G
- The group axioms (associativity, identity, inverses) hold in the category of varieties, meaning they are satisfied by the morphisms μ and ι
- The multiplication map μ and the inversion map ι are required to be morphisms of varieties, ensuring compatibility between the group structure and the algebraic geometry of G

Examples of algebraic groups

- The multiplicative group G_m is the variety $\mathbb{A}^1 \setminus \{0\}$ with multiplication $(x, y) \mapsto xy$ and inversion $x \mapsto x^{-1}$
- The additive group G_a is the variety $\mathbb{A}^1$ with addition $(x, y) \mapsto x + y$ and inversion $x \mapsto -x$
- The general linear group GL_n is the variety of invertible $n \times n$ matrices with matrix multiplication and inversion
- The special linear group SL_n is the subgroup of GL_n consisting of matrices with determinant 1
- Elliptic curves and abelian varieties are examples of projective algebraic groups

Structure of algebraic groups

Subgroups and quotients

- A subgroup H of an algebraic group G is a closed subvariety that is also a subgroup in the group-theoretic sense
- The identity component G_0 of an algebraic group G is the connected component containing the identity element e

- The quotient G/H of an algebraic group G by a normal subgroup H has a natural structure of an algebraic group induced by the group operations on G
- The quotient morphism $\pi: G \rightarrow G/H$ is a morphism of algebraic groups with kernel H

Lie algebras and the exponential map

- The Lie algebra $\mathfrak{g}$ of an algebraic group G is the tangent space at the identity $T_e G$, equipped with a Lie bracket operation $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$
- The exponential map $\exp: \mathfrak{g} \rightarrow G$ is a local isomorphism from a neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G , relating the Lie algebra to the algebraic group
- The differential of the multiplication map μ at (e, e) induces the Lie bracket on $\mathfrak{g}$, making it compatible with the group structure
- The adjoint representation $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$ describes the action of G on its Lie algebra by conjugation

Classification of algebraic groups

- An algebraic group is solvable if it has a composition series with solvable quotients (successive extensions by $\mathbb{G}_a$ or $\mathbb{G}_m$)
- An algebraic group is unipotent if it has a composition series with unipotent quotients (successive extensions by $\mathbb{G}_a$)
- An algebraic group is semisimple if it has no non-trivial solvable normal subgroups
- The Jordan decomposition expresses an algebraic group as an extension of a semisimple group by a solvable group
- The classification of semisimple algebraic groups is related to the classification of root systems and Dynkin diagrams

Group actions on varieties

Definition and orbits

- An action of an algebraic group G on a variety X is a morphism $\alpha: G \times X \rightarrow X$ satisfying the usual axioms: $\alpha(e, x) = x$ and $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$
- The orbit of a point x in X under the G -action is the set $G \cdot x = \{\alpha(g, x) \mid g \in G\}$, which is a locally closed subvariety of X
- Orbits partition X into disjoint subvarieties, each isomorphic to a quotient of G by a stabilizer subgroup
- The stabilizer subgroup G_x of a point x is the subgroup $\{g \in G \mid \alpha(g, x) = x\}$, which is a closed subgroup of G

Quotients and fibers

- There is a bijective correspondence between orbits $G \cdot x$ and cosets of stabilizers G/G_x , given by $gG_x \mapsto g \cdot x$
- A G -action is transitive if it has only one orbit, meaning every point can be reached from any other point by the action of G

- A G -action is free if all stabilizers are trivial, i.e., $G_x = \{e\}$ for all x in X
- A G -action is faithful if the map $G \rightarrow \text{Aut}(X)$ given by $g \mapsto (x \mapsto g \cdot x)$ is injective
- A geometric quotient of X by a G -action is a variety Y with a morphism $\pi: X \rightarrow Y$ constant on orbits, such that π induces a bijection between orbits and points of Y
- Fibers of the quotient map π are the orbits of the G -action, and Y parameterizes the set of orbits

Algebraic groups for geometry

Symmetries and automorphisms

- Algebraic groups can be used to study symmetries and automorphisms of algebraic varieties
- The automorphism group $\text{Aut}(X)$ of a variety X is an algebraic group, often with a rich structure
- Symmetries of X correspond to elements of $\text{Aut}(X)$ or its subgroups, and can be used to simplify the study of X
- The structure theory of algebraic groups helps classify certain types of varieties with large automorphism groups, e.g., toric varieties, spherical varieties, flag varieties

Homogeneous spaces and bundles

- Group actions encode the geometry of homogeneous spaces, fiber bundles, and principal bundles in algebraic geometry
- A homogeneous space is a variety X with a transitive action of an algebraic group G , e.g., projective spaces, Grassmannians, flag varieties
- A G -equivariant bundle over X is a variety E with a G -action and a G -invariant morphism $\pi: E \rightarrow X$, such that π is locally trivial with fibers isomorphic to a fixed G -variety F
- A principal G -bundle is a G -equivariant bundle where the fibers are isomorphic to G acting on itself by multiplication
- The quotient of E by the G -action is isomorphic to X , and E can be recovered from X and the cocycle defining the bundle

Representation theory and invariant theory

- Representation theory of algebraic groups provides tools for studying linear actions on vector spaces and sheaves
- A representation of an algebraic group G is a morphism of algebraic groups $\rho: G \rightarrow \text{GL}(V)$ for some finite-dimensional vector space V
- Representations can be used to construct G -equivariant sheaves and study their cohomology, which often has additional structure coming from the representation
- Invariant theory describes polynomials and rational functions invariant under a group action, and their relations to quotients
- The ring of invariants $\mathbb{C}[V]^G$ of a G -representation V is the subring of G -invariant polynomials, which is finitely generated by the Hilbert Basis Theorem

- The GIT quotient $V//G$ is the variety $\text{Spec}(\mathbb{C}[V]^G)$, which parameterizes closed orbits of the G -action on V

Applications over finite fields

- Algebraic groups over finite fields are used in coding theory, cryptography, and other applications
- The Lang-Steinberg theorem states that the fixed points of a Frobenius endomorphism on a connected algebraic group over a finite field form a finite subgroup
- Finite subgroups of algebraic groups over finite fields are used to construct error-correcting codes, such as Goppa codes and algebraic-geometric codes
- Cryptographic protocols based on the hardness of the discrete logarithm problem or the Diffie-Hellman problem can be formulated using algebraic groups over finite fields
- Algebraic groups over finite fields also appear in the study of zeta functions, L-functions, and other arithmetic invariants of varieties

12.2 Toric varieties

Toric varieties are fascinating geometric objects that blend algebra, geometry, and combinatorics. They're built from fans, which are collections of cones in a lattice, and contain an algebraic torus as a dense open subset.

These varieties have deep connections to convex polytopes and offer a wealth of examples in algebraic geometry. Their combinatorial nature makes them ideal for studying singularities, cohomology, and intersection theory, with applications ranging from mirror symmetry to moduli spaces.

Toric varieties from combinatorial data

Construction of toric varieties

- Toric varieties contain an algebraic torus as a dense open subset
- The action of the torus on itself extends to an action on the entire variety
- Toric varieties are constructed from fans
- Fans are collections of strongly convex rational polyhedral cones in a lattice
- Torus-invariant prime divisors of a toric variety correspond bijectively to the rays (1-dimensional cones) of the fan

Affine and projective toric varieties

- Affine toric varieties are in one-to-one correspondence with strongly convex rational polyhedral cones in a lattice
- Projective toric varieties are in one-to-one correspondence with complete fans
- Complete fans have cones that cover the entire space
- The gluing of affine toric varieties to form a toric variety is determined by the combinatorial data of the fan

Toric varieties and convex polytopes

Relationship between toric varieties and convex polytopes

- Toric varieties are closely related to convex polytopes
- Convex polytopes are the convex hulls of finitely many points in a lattice
- The normal fan of a convex polytope is a fan that encodes the combinatorial structure of the polytope and gives rise to a toric variety
- The toric variety associated with a convex polytope is projective if and only if the polytope is a lattice polytope
- Lattice polytopes have vertices with integer coordinates

Geometric and combinatorial properties

- The moment map of a projective toric variety is a map from the variety to the polytope
- Provides a geometric link between the two objects
- The faces of a polytope correspond to the torus-invariant subvarieties of the associated toric variety
- Combinatorial properties of a polytope can be studied using the associated toric variety
- Properties include the face lattice and Ehrhart polynomial

Geometric and algebraic properties of toric varieties

Singularities and homological properties

- Toric varieties are normal algebraic varieties
- Locally isomorphic to affine spaces
- The singularities of a toric variety can be characterized combinatorially in terms of the cones in the fan
- Toric varieties are Cohen-Macaulay
- Their local rings have a certain homological property

Cohomology and intersection theory

- The cohomology ring of a smooth projective toric variety can be described combinatorially in terms of the fan
- Toric varieties admit a natural action of the algebraic torus
- Leads to a decomposition into torus orbits
- Intersection theory on a toric variety can be studied using the combinatorics of the fan
- Includes the intersection numbers of torus-invariant divisors

Applications of toric varieties

Examples and counterexamples in algebraic geometry

- Toric varieties provide a rich source of examples and counterexamples in algebraic geometry
- Examples include Fano varieties, Calabi-Yau varieties, and varieties with prescribed singularities
- Toric geometry can be used to study the geometry of linear systems and the structure of the cone of effective divisors on a variety

Applications in computational algebraic geometry and beyond

- Toric varieties appear naturally in the study of Newton polytopes and sparse polynomial systems
- Have applications in computational algebraic geometry
- Toric varieties have been used to construct explicit examples of mirror symmetry
- Mirror symmetry is a duality between certain pairs of Calabi-Yau varieties
- Toric methods have been applied to the study of moduli spaces
- Examples include the moduli space of curves and the moduli space of stable vector bundles
- Toric varieties have connections to other areas of mathematics
- Areas include combinatorics, symplectic geometry, and mathematical physics

12.3 Introduction to schemes and modern algebraic geometry

Schemes revolutionize algebraic geometry by providing a unified framework for studying geometric spaces. They generalize algebraic varieties, allowing for the examination of geometric objects over arbitrary base rings and enabling the study of both local and global properties.

The theory of schemes introduces powerful tools like sheaf cohomology and the functor of points perspective. This modern approach opens up new avenues for research in areas such as arithmetic geometry, complex geometry, and mathematical physics.

Motivation for Schemes

Generalizing Algebraic Varieties

- Schemes generalize the notion of algebraic varieties and provide a unified framework for studying geometric spaces locally and globally
- Schemes are locally ringed spaces that are locally isomorphic to the spectrum of a commutative ring (affine schemes)
- Affine schemes are the building blocks of schemes
- Defined as the spectrum of a commutative ring R , denoted $\text{Spec}(R)$
- $\text{Spec}(R)$ is the set of all prime ideals of R equipped with the Zariski topology
- The structure sheaf on an affine scheme is determined by the ring of regular functions on open subsets

- Schemes are constructed by gluing together affine schemes along open subsets, similar to the construction of manifolds (gluing together open subsets of Euclidean space)

Morphisms and Categories

- Morphisms between schemes are defined as morphisms of locally ringed spaces
- Continuous maps that respect the structure sheaves
- The category of schemes is a natural generalization of the category of affine varieties
- Provides a rich setting for studying geometric objects and their properties
- Schemes allow for the study of geometric objects over arbitrary base rings, not just fields
- Enables the study of arithmetic and geometric properties of varieties over various base schemes (integers, finite fields)

Local and Global Properties of Schemes

Local Structure and Singularities

- The local structure of a scheme at a point is determined by the local ring at that point
- Obtained by localizing the structure sheaf at the point
- Encodes information about the scheme in a neighborhood of the point
- The dimension of a scheme at a point is defined as the Krull dimension of the local ring at that point
- The dimension of a scheme is the supremum of the dimensions at all points
- Schemes can have singularities, which are points where the local ring is not regular
- The study of singularities is a central topic in algebraic geometry
- Has applications in various areas of mathematics and physics

Global Properties and Cohomology

- Schemes can be reducible or irreducible
- Irreducible if they cannot be written as the union of two proper closed subschemes
- Irreducible schemes are the building blocks of schemes and play a crucial role in the study of their geometry
- The connectedness and connectivity of schemes can be studied using the Zariski topology
- A scheme is connected if it is not the disjoint union of two proper closed subschemes
- Global properties of schemes, such as cohomology and intersection theory, can be studied using the tools of sheaf theory and homological algebra
- Sheaves and sheaf cohomology are fundamental tools in the study of schemes
- Provide a way to study global properties of geometric objects by considering local data

Schemes vs Algebraic Varieties

Generalizing Algebraic Varieties

- Every algebraic variety over an algebraically closed field can be naturally associated with a scheme, called the associated scheme
- The associated scheme captures the intrinsic geometry of the variety and is independent of the choice of embedding
- Schemes can have non-reduced structure, which means that the local rings may have nilpotent elements
- Allows for the study of more general geometric objects (infinitesimal deformations, non-reduced fibers of morphisms)

Functor of Points Perspective

- The functor of points perspective provides a way to study schemes by considering their morphisms into other schemes
- Generalizes the classical approach of studying varieties via their points
- Schemes can be defined over arbitrary base rings, not just fields
- Allows for the study of arithmetic and geometric properties of varieties over various base schemes (integers, finite fields)

Key Concepts in Algebraic Geometry

Sheaves and Cohomology

- Sheaves and sheaf cohomology are fundamental tools in the study of schemes
- Provide a way to study global properties of geometric objects by considering local data
- Coherent sheaves are a special class of sheaves that generalize the notion of vector bundles on varieties
- Play a central role in the study of the geometry of schemes
- Derived categories and derived functors provide a powerful framework for studying the homological algebra of sheaves on schemes and the cohomology of algebraic varieties

Divisors, Line Bundles, and Flatness

- Divisors and line bundles are important tools for studying the birational geometry of schemes and the behavior of morphisms between them
- The theory of flatness and base change is crucial for understanding the behavior of schemes under morphisms and the properties of families of schemes
- Flat morphisms preserve important geometric properties and are essential in the study of moduli spaces and deformation theory

Schemes in Algebraic Geometry

Unifying and Generalizing Algebraic Geometry

- Schemes provide a unifying framework for studying various aspects of algebraic geometry
- Includes the theory of algebraic varieties, arithmetic geometry, and complex geometry
- The language of schemes allows for the generalization of many classical results in algebraic geometry
- Examples: Riemann-Roch theorem, Hirzebruch-Riemann-Roch theorem
- Generalized to more general settings using the theory of schemes

Applications and Further Developments

- Schemes have applications in various areas of mathematics
- Number theory, representation theory, mathematical physics
- Provide a common language for studying geometric objects across different fields
- The theory of moduli spaces, which studies families of geometric objects, relies heavily on the language of schemes and the tools of modern algebraic geometry
- The study of schemes has led to the development of new techniques and insights in algebraic geometry
- Theory of stacks and the geometry of derived schemes
- Opened up new avenues for research and applications