

Functional Analysis

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Export date: 2026-10-05T19:26:30.244Z

Contents

- Unit 1 – Normed Linear and Banach Spaces Intro - 4 guides
- Unit 2 – Linear Operators in Normed Spaces - 4 guides
- Unit 3 – Hahn-Banach Theorem: Key Implications - 3 guides
- Unit 4 – Open and Closed Graph Theorems - 3 guides
- Unit 5 – Hilbert Spaces and Orthonormal Bases - 4 guides
- Unit 6 – Bounded Linear Operators in Hilbert Spaces - 4 guides
- Unit 7 – Compact Operators: Properties and Analysis - 4 guides
- Unit 8 – Spectral Theory of Bounded Operators - 4 guides
- Unit 9 – Weak and Weak* Topologies - 4 guides
- Unit 10 – Duality Theory and Reflexive Spaces - 4 guides
- Unit 11 – Unbounded and Closed Linear Operators - 4 guides
- Unit 12 – Differential Equations & Quantum Mechanics - 4 guides
- Unit 13 – Optimization in Banach Spaces - 4 guides
- Unit 14 – Recent Developments in Functional Analysis - 4 guides

Unit 1 – Normed Linear and Banach Spaces Intro

1.1 Definition and properties of normed linear spaces

Normed linear spaces are vector spaces with a special function that measures vector size. This function, called a norm, must follow specific rules like being non-negative and obeying the triangle inequality.

These spaces are crucial in functional analysis. They allow us to measure distances between vectors, study convergence of sequences, and analyze continuity of functions in abstract settings beyond just regular Euclidean space.

Normed Linear Spaces

Definition of normed linear spaces

- Vector space V over a scalar field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) equipped with a norm function $\|\cdot\| : V \rightarrow \mathbb{R}$
- Assigns a non-negative real number to each vector in the space, measuring its "length" or "size"
- Induces a metric d on the space, defined by $d(x, y) = \|x - y\|$ for all $x, y \in V$, measuring the "distance" between vectors
- Must satisfy the following properties for all $x, y \in V$ and $\alpha \in \mathbb{F}$:
 - Positive definiteness: $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$, ensuring zero vector has zero norm and non-zero vectors have positive norms
 - Homogeneity: $\|\alpha x\| = |\alpha| \|x\|$, scaling a vector by a scalar scales its norm by the absolute value of the scalar
 - Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$, the norm of the sum of two vectors is less than or equal to the sum of their norms

Properties of norm functions

- Positive definiteness:
 - $\|x\| \geq 0$ for all $x \in V$, norms are non-negative
 - $\|x\| = 0$ if and only if $x = 0$, only the zero vector has zero norm
- Homogeneity:
 - $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in V$ and $\alpha \in \mathbb{F}$, scaling a vector by a scalar scales its norm by the absolute value of the scalar
- Ensures norm is compatible with scalar multiplication in the vector space
- Triangle inequality:
 - $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$, the norm of the sum of two vectors is less than or equal to the sum of their norms
- Often proven using properties of the absolute value and the scalar field

- Geometrically, the shortest path between two points is a straight line

Examples of normed linear spaces

- Euclidean space $\mathbb{R}^n$ with the Euclidean norm:

- For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$\|x\| = \sqrt{\sum_{i=1}^n |x_i|^2}$$

- Measures the "length" of a vector in $\mathbb{R}^n$
- Space of continuous functions $C[a, b]$ with the supremum norm:
- For $f \in C[a, b]$, $\|f\| = \sup_{x \in [a, b]} |f(x)|$
- Measures the maximum absolute value of a function on the interval $[a, b]$
- Space of square-summable sequences ℓ^2 with the ℓ^2 -norm:
- For $x = (x_1, x_2, \dots) \in \ell^2$,

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$$

- Measures the "size" of a sequence based on the sum of the squares of its terms

Geometric interpretation of norms

- Norm measures the "length" or "size" of vectors in the space
- Induces a metric d on the space, defined by $d(x, y) = \|x - y\|$ for all $x, y \in V$
- Measures the "distance" between two vectors in the space
- Satisfies the following properties for all $x, y, z \in V$:
 1. Non-negativity: $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
 2. Symmetry: $d(x, y) = d(y, x)$
 3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$
- Allows for the study of geometric properties of the space
- Convergence of sequences: a sequence (x_n) in V converges to $x \in V$ if $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$
- Continuity of functions: a function $f: V \rightarrow W$ between normed linear spaces is continuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(x) - f(y)\| < \varepsilon$ whenever $\|x - y\| < \delta$

1.2 Convergence and completeness in normed spaces

Convergence in normed linear spaces is a key concept in functional analysis. It defines how sequences of elements approach a limit point, using the norm to measure distance between elements.

Uniqueness of limits and Cauchy sequences are crucial ideas in this context. These concepts help us understand completeness, which ensures that every Cauchy sequence converges within the space, leaving no "gaps."

Convergence in normed linear spaces

Convergence in normed linear spaces

- Defines convergence of a sequence (x_n) in a normed linear space $(X, \|\cdot\|)$ as $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ for some $x \in X$
- Denotes convergence as $x_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$
- Illustrates convergence using the space of continuous functions $C[0, 1]$ with the supremum norm $\|f\|_\infty = \sup_{x \in [0, 1]} |f(x)|$
- Defines a sequence of functions (f_n) by $f_n(x) = x^n$ for $x \in [0, 1]$
- Shows (f_n) converges to the function $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$

Uniqueness of limits

- Assumes (x_n) converges to both x and y in a normed space $(X, \|\cdot\|)$
- Applies the triangle inequality to obtain $\|x - y\| \leq \|x - x_n\| + \|x_n - y\|$
- Takes the limit as $n \rightarrow \infty$, yielding $\lim_{n \rightarrow \infty} \|x - x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - y\| = 0$
- Concludes $\|x - y\| \leq 0 + 0 = 0$, implying $x = y$
- Proves the uniqueness of limits in normed spaces

Cauchy sequences and completeness

- Defines a Cauchy sequence (x_n) in a normed space $(X, \|\cdot\|)$ as one where for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq N$
- Explains intuitively that the terms of a Cauchy sequence become arbitrarily close to each other as the sequence progresses
- Relates Cauchy sequences to completeness by stating a normed space $(X, \|\cdot\|)$ is complete if every Cauchy sequence in X converges to an element of X
- Emphasizes completeness ensures there are no "gaps" in the space, and every Cauchy sequence has a limit within the space

Completeness via Cauchy criterion

- Outlines proving a normed space $(X, \|\cdot\|)$ is complete by showing every Cauchy sequence in X converges to an element of X
- Demonstrates completeness of $(\mathbb{R}, |\cdot|)$ as an example 1. Lets (x_n) be a Cauchy sequence in $\mathbb{R}$ 2. Applies the Cauchy criterion to obtain for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq N$ 3. Uses the completeness of $\mathbb{R}$ as an ordered field to show (x_n) converges to a real number x
- Concludes $(\mathbb{R}, |\cdot|)$ is complete

1.3 Banach spaces and their characteristics

Banach spaces are the backbone of functional analysis, combining vector spaces with a complete norm. They provide a powerful framework for studying infinite-dimensional spaces, crucial for solving complex mathematical problems in various fields.

These spaces offer key tools like the fixed-point theorem and uniform boundedness principle. Understanding Banach spaces is essential for grasping advanced concepts in functional analysis and their applications in physics and engineering.

Banach Spaces

Definition of Banach spaces

- A vector space V over a field $\mathbb{F}$ (usually $\mathbb{R}$ or $\mathbb{C}$) equipped with a norm $\|\cdot\|$ which satisfies the following properties for all $x, y \in V$ and $\alpha \in \mathbb{F}$:
 - Positivity: $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$
 - Homogeneity: $\|\alpha x\| = |\alpha| \|x\|$
 - Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$
 - Completeness property: every Cauchy sequence in V converges to an element in V with respect to the norm $\|\cdot\|$
- A sequence (x_n) is Cauchy if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for all $n, m \geq N$
- Examples of Banach spaces include $\mathbb{R}^n$ with Euclidean norm and $C([a, b])$ with supremum norm

Convergence in Banach spaces

- A normed linear space V is a Banach space if and only if every absolutely convergent series converges in V
- Forward implication: if V is a Banach space and

$$\sum_{n=1}^{\infty} \|x_n\| < \infty$$

, then

$$\sum_{n=1}^{\infty} x_n$$

converges in V

- Define partial sums

$$s_n = \sum_{k=1}^n x_k$$

and show that (s_n) is Cauchy using the completeness of $\mathbb{F}$ and the triangle inequality

- By completeness of V , (s_n) converges to some $s \in V$
- Backward implication: if every absolutely convergent series in V converges, then V is a Banach space

- Consider a Cauchy sequence (x_n) in V and show that

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| < \infty$$

using the Cauchy property

- By hypothesis,

$$\sum_{n=1}^{\infty} (x_{n+1} - x_n)$$

converges to some $x \in V$

- Prove that $x_n \rightarrow x$ using the triangle inequality and the convergence of the series

Properties of Banach spaces

- Banach fixed-point theorem (contraction mapping theorem): if $T : V \rightarrow V$ is a contraction mapping on a Banach space V , then T has a unique fixed point
- A mapping T is a contraction if there exists a constant $0 \leq c < 1$ such that $\|T(x) - T(y)\| \leq c\|x - y\|$ for all $x, y \in V$
- The fixed point can be found by iterating T on any initial point $x_0 \in V$
- Uniform boundedness principle: if $\mathcal{F}$ is a collection of bounded linear operators from a Banach space V to a normed linear space W such that $\sup_{T \in \mathcal{F}} \|T(x)\| < \infty$ for each $x \in V$, then $\sup_{T \in \mathcal{F}} \|T\| < \infty$
- The operator norm is defined as $\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|$
- Consequence: a pointwise bounded family of continuous linear operators on a Banach space is uniformly bounded

Significance in functional analysis

- Banach spaces provide a general framework for studying infinite-dimensional linear spaces with a notion of distance (norm) and convergence
- Many important function spaces are Banach spaces, such as:
 - L^p spaces (Lebesgue spaces of p -integrable functions)
 - $C(K)$ (continuous functions on a compact Hausdorff space K)
 - ℓ^p (sequences with p -summable absolute values)
- Banach spaces are used in the study of differential and integral equations, optimization problems, and quantum mechanics
- The solution space of a linear differential equation can be viewed as a Banach space
- The theory of bounded linear operators between Banach spaces is a central topic in functional analysis
- Includes the study of dual spaces, adjoint operators, and spectral theory

1.4 Examples and counterexamples of normed and Banach spaces

Banach spaces are complete normed vector spaces. They're crucial in functional analysis, providing a solid foundation for studying linear operators and continuous functions.

We'll look at examples of Banach spaces like bounded sequences and continuous functions on compact sets. We'll also explore counterexamples and compare Banach spaces to general normed spaces.

Examples of Banach spaces

- Space of bounded sequences $(l^\infty, \|\cdot\|_\infty)$
- Contains all sequences of real or complex numbers that are bounded
- Norm is the supremum of the absolute values of the sequence elements: $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$
- Forms a Banach space as it is complete under this norm
- Example: The sequence $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ belongs to l^∞
- Space of continuous functions on a compact set $(C(K), \|\cdot\|_\infty)$
- K represents a compact subset of a metric space
- Includes all continuous functions $f: K \rightarrow \mathbb{R}$ or $\mathbb{C}$
- Norm is the maximum absolute value of the function over its domain: $\|f\|_\infty = \sup_{x \in K} |f(x)|$
- Forms a Banach space as it is complete under this norm
- Example: The space of continuous functions on the closed interval $[0, 1]$ is a Banach space

Counterexamples to Banach spaces

- Space of continuous functions on an open interval $(C(a, b), \|\cdot\|_\infty)$
- Contains all continuous functions $f: (a, b) \rightarrow \mathbb{R}$ or $\mathbb{C}$
- Norm is the maximum absolute value of the function over its domain: $\|f\|_\infty = \sup_{x \in (a, b)} |f(x)|$
- Does not form a Banach space as it is not complete under this norm
- Counterexample: The sequence of functions $f_n(x) = x^n$ on $(0, 1)$ is Cauchy but does not converge to a continuous function on $(0, 1)$
- Space of polynomials $(P, \|\cdot\|)$
- Includes all polynomials with real or complex coefficients
- Norm can be defined in various ways, such as $\|p\| = \sup_{x \in [0, 1]} |p(x)|$
- Does not form a Banach space as it is not complete under any norm
- Counterexample: The sequence of polynomials

$$p_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

is Cauchy but converges to the exponential function e^x , which is not a polynomial

Completeness in specific spaces

- Space of polynomials $(P, \|\cdot\|)$
- Not complete under any norm, as demonstrated by the previous counterexample
- Space of rational functions $(R, \|\cdot\|)$
- Contains all rational functions of the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials and $q \neq 0$
- Norm can be defined in various ways, such as $\|f\| = \sup_{x \in [0, 1]} |f(x)|$
- Does not form a Banach space as it is not complete under any norm
- Counterexample: The sequence of rational functions $f_n(x) = \frac{1}{1 + nx^2}$ is Cauchy but converges to the function

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

, which is not a rational function

Comparison of normed vs Banach spaces

- Completeness
- Banach spaces are always complete under their respective norms
- Not all normed spaces are complete, as shown by the counterexamples
- Dimension
- Banach spaces can be finite-dimensional ($\mathbb{R}^n$ with any norm) or infinite-dimensional (l^∞ and $C(K)$)
- Normed spaces can also be either finite-dimensional or infinite-dimensional
- Separability
- Some Banach spaces are separable ($C([0, 1])$ and l^p for $1 \leq p < \infty$), while others are not (l^∞ and $L^\infty([0, 1])$)
- Separability is not directly related to completeness, as both Banach and non-Banach spaces can be separable or non-separable
- Duality
- The dual space of a normed space is always a Banach space
- The dual of a Banach space is also a Banach space
- The dual of a non-Banach space may not be complete under the operator norm

Unit 2 – Linear Operators in Normed Spaces

2.1 Bounded linear operators and their properties

Bounded linear operators are crucial in functional analysis, mapping between normed spaces while preserving structure. They're defined by their ability to transform bounded sets into bounded sets, with the operator norm quantifying this property.

Understanding bounded linear operators is key to grasping how functions behave in infinite-dimensional spaces. Their properties, like linearity and boundedness, form the foundation for studying more complex operators and functional analysis concepts.

Bounded Linear Operators

Definition of bounded linear operators

- A linear operator $T : X \rightarrow Y$ between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is bounded if there exists a constant $M \geq 0$ such that $\|Tx\|_Y \leq M\|x\|_X$ for all $x \in X$
- The smallest such M is called the operator norm of T , denoted as $\|T\|$
- Intuitively, a bounded linear operator maps bounded sets in X to bounded sets in Y
- Examples of bounded linear operators:
 - Identity operator $I : X \rightarrow X$ defined by $Ix = x$ for all $x \in X$ is bounded with $\|I\| = 1$
 - Differentiation operator $D : C^1[a, b] \rightarrow C[a, b]$ defined by $(Df)(x) = f'(x)$ for all $f \in C^1[a, b]$ is bounded
 - Integration operator $J : C[a, b] \rightarrow C[a, b]$ defined by $(Jf)(x) = \int_a^x f(t)dt$ for all $f \in C[a, b]$ is bounded

Properties of bounded linear operators

- Linearity: Let $T : X \rightarrow Y$ be a bounded linear operator between normed spaces X and Y . For any $x_1, x_2 \in X$ and scalars α, β , we have:
 - $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$
- This property ensures that bounded linear operators preserve the vector space structure
- Boundedness: Let $T : X \rightarrow Y$ be a bounded linear operator with operator norm $\|T\|$. For any $x \in X$, we have:
 - $\|Tx\|_Y \leq \|T\|\|x\|_X$
 - Proof: By definition of the operator norm, $\|Tx\|_Y \leq \|T\|\|x\|_X$ for all $x \in X$
- This property ensures that the image of a bounded set under a bounded linear operator remains bounded

Boundedness and operator norm

- Boundedness test: A linear operator $T : X \rightarrow Y$ is bounded if and only if there exists a constant $M \geq 0$ such that $\|Tx\|_Y \leq M\|x\|_X$ for all $x \in X$
- This test provides a necessary and sufficient condition for a linear operator to be bounded
- Finding the operator norm:
 - The operator norm of a bounded linear operator $T : X \rightarrow Y$ is defined as:
$$\|T\| = \sup \{ \|Tx\|_Y : \|x\|_X \leq 1 \} = \sup \{ \|Tx\|_Y / \|x\|_X : x \neq 0 \}$$
 - Intuitively, the operator norm is the smallest upper bound for the "stretching" of vectors under the operator
 - Equivalently, the operator norm can be found using:
$$\|T\| = \inf \{ M \geq 0 : \|Tx\|_Y \leq M\|x\|_X \text{ for all } x \in X \}$$
- This definition provides an alternative way to compute the operator norm

Space of bounded linear operators

- The space of bounded linear operators from X to Y , denoted as $B(X, Y)$, is a normed space with the operator norm
- This space contains all bounded linear operators between X and Y
- Linearity of $B(X, Y)$: For any $T_1, T_2 \in B(X, Y)$ and scalars α, β , we have:
 - $(\alpha T_1 + \beta T_2)(x) = \alpha T_1(x) + \beta T_2(x)$ for all $x \in X$
 - $\|\alpha T_1 + \beta T_2\| \leq |\alpha| \|T_1\| + |\beta| \|T_2\|$
- These properties ensure that $B(X, Y)$ is a vector space and the operator norm is a norm on this space
- Completeness of $B(X, Y)$: If Y is a Banach space, then $B(X, Y)$ is also a Banach space
- Proof: Let (T_n) be a Cauchy sequence in $B(X, Y)$. Define $T : X \rightarrow Y$ by $Tx = \lim_{n \rightarrow \infty} T_n x$ for each $x \in X$. Then $T \in B(X, Y)$ and $\lim_{n \rightarrow \infty} \|T_n - T\| = 0$
- This property ensures that $B(X, Y)$ is complete with respect to the operator norm, making it a suitable space for analysis

2.2 Linear functionals and dual spaces

Linear functionals are key players in functional analysis, mapping elements from normed spaces to their scalar fields. They're like special measuring tools, giving us a single number for each element in a space.

The dual space, made up of all bounded linear functionals, is a powerful concept. It helps us understand the original space better, often revealing hidden structures and properties we might have missed otherwise.

Linear Functionals

Linear functionals on normed spaces

- Linear functionals map elements from a normed space X to its underlying scalar field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)
- Satisfy additivity property: $f(x + y) = f(x) + f(y)$ for all $x, y \in X$
- Satisfy homogeneity property: $f(\alpha x) = \alpha f(x)$ for all $x \in X$ and $\alpha \in \mathbb{F}$
- Example: On $C[a, b]$, the evaluation functional $f_t(x) = x(t)$ for fixed $t \in [a, b]$
- Example: On ℓ^p , the coordinate functional $f_n(x) = x_n$ for fixed $n \in \mathbb{N}$
- Example: On $\mathcal{P}$, the derivative functional $f(p) = p'(0)$

Dual space of normed spaces

- Dual space of a normed space X , denoted X^* , consists of all bounded linear functionals on X
- X^* is a normed space with the norm: $\|f\| = \sup \{|f(x)| : x \in X, \|x\| \leq 1\}$
- X^* is a Banach space (complete normed space) if X is a Banach space
- If X is a Hilbert space, X^* is isometrically isomorphic to X
- Dual space of a finite-dimensional normed space is isomorphic to the space itself

Boundedness and norm of functionals

- A linear functional f on a normed space X is bounded if there exists $M \geq 0$ such that: $|f(x)| \leq M\|x\|$ for all $x \in X$
- Norm of a bounded linear functional f is defined as: $\|f\| = \inf \{M \geq 0 : |f(x)| \leq M\|x\| \text{ for all } x \in X\}$
- Equivalent definition: $\|f\| = \sup \{|f(x)| : x \in X, \|x\| \leq 1\}$
- To find the norm of a linear functional: 1. Find an upper bound M for $|f(x)|$ in terms of $\|x\|$ 2. Prove the upper bound is sharp by finding a sequence (x_n) in X with $\|x_n\| = 1$ such that $|f(x_n)| \rightarrow M$ as $n \rightarrow \infty$

Normed spaces vs dual spaces

- Canonical isomorphism $J : X \rightarrow X^{**}$ (dual of the dual space) defined by: $J(x)(f) = f(x)$ for all $x \in X$ and $f \in X^*$
- J is a linear isometry: $\|J(x)\| = \|x\|$ for all $x \in X$
- J is injective: if $J(x) = J(y)$, then $x = y$
- For a Banach space X , J is an isometric isomorphism if and only if X is reflexive (J is surjective)
- Canonical isomorphism allows studying the relationship between a normed space and its dual
- Plays a crucial role in the theory of reflexive spaces

2.3 Operator norms and continuity

Operator norms are crucial tools in functional analysis, measuring the "size" of linear operators between normed spaces. They possess key properties like positivity, homogeneity, and subadditivity, which make them invaluable for analyzing operator behavior.

These norms connect to broader concepts like continuity and boundedness of linear operators. They also play a role in convergence of operator sequences and series, providing a framework for understanding limits in operator spaces.

Operator Norms

Properties of operator norms

- Positivity ensures operator norms are non-negative and zero only for the zero operator
- Homogeneity scales the operator norm by the absolute value of a scalar multiple
- Subadditivity (Triangle inequality) bounds the operator norm of a sum by the sum of operator norms
- Submultiplicativity bounds the operator norm of a composition by the product of operator norms
- Equivalence of operator norms means any two operator norms are within constant factors of each other (on the same space)

Continuity of bounded linear operators

- Theorem establishes the equivalence between continuity and boundedness for linear operators between normed spaces
- Bounded implies Lipschitz continuous with the operator norm as the Lipschitz constant
- Continuous implies bounded by considering the continuity condition at the origin with $\varepsilon = 1$

Calculation of operator norms

- ℓ^1 operator $(Tx)_n = \frac{x_n}{n}$ has operator norm at most 1 by comparing ℓ^1 norms of Tx and x
- $C[0, 1]$ integral operator $(Tf)(x) = \int_0^x f(t)dt$ has operator norm at most 1 by bounding the supremum of $|Tf|$ by the supremum of $|f|$

Operator norms and sequence convergence

- Theorem relates convergence in operator norm to pointwise convergence and boundedness of the limit operator
- Convergence in operator norm implies pointwise convergence and boundedness of the limit
- Corollary provides a sufficient condition for a series of operators to converge in operator norm
- Summable operator norms imply convergence to a bounded operator by completeness of the operator space

2.4 Adjoint operators and their properties

Adjoint operators are key players in functional analysis, linking operators between Hilbert spaces. They're defined by a special property that connects inner products, making them crucial for understanding operator behavior.

Adjoint operators have unique properties like conjugate linearity and norm preservation. They're used to classify operators as self-adjoint, normal, or unitary, which has big implications in quantum mechanics and other fields using Hilbert spaces.

Adjoint Operators

Definition of adjoint operator

- Let H_1 and H_2 be Hilbert spaces and $T : H_1 \rightarrow H_2$ be a bounded linear operator
- The adjoint operator of T , denoted by T^* , is a bounded linear operator from H_2 to H_1
- Satisfies the adjoint property $\langle Tx, y \rangle_{H_2} = \langle x, T^*y \rangle_{H_1}$ for all $x \in H_1$ and $y \in H_2$
- Also known as the defining property of the adjoint operator
- Allows for the study of the relationship between an operator and its adjoint (T and T^*)
- Plays a crucial role in the theory of bounded linear operators on Hilbert spaces (L^2, ℓ^2)

Existence and uniqueness of adjoint

- Existence:
- For each $y \in H_2$, define a linear functional $\phi_y : H_1 \rightarrow \mathbb{C}$ by $\phi_y(x) = \langle Tx, y \rangle_{H_2}$
- By the Riesz Representation Theorem, there exists a unique $z \in H_1$ such that $\phi_y(x) = \langle x, z \rangle_{H_1}$ for all $x \in H_1$
- This theorem guarantees the existence of a unique element in H_1 representing the linear functional ϕ_y
- Define $T^*y = z$, then $\langle Tx, y \rangle_{H_2} = \langle x, T^*y \rangle_{H_1}$ for all $x \in H_1$ and $y \in H_2$
- Uniqueness:
- Suppose $S : H_2 \rightarrow H_1$ is another operator satisfying the adjoint property
- Then, $\langle x, T^*y \rangle_{H_1} = \langle Tx, y \rangle_{H_2} = \langle x, Sy \rangle_{H_1}$ for all $x \in H_1$ and $y \in H_2$
- By the uniqueness part of the Riesz Representation Theorem, $T^*y = Sy$ for all $y \in H_2$, implying $T^* = S$
- Properties:
- Linearity: $(aT + bS)^* = \bar{a}T^* + \bar{b}S^*$ for all bounded linear operators T, S and scalars a, b
- The adjoint operator preserves linear combinations, with complex conjugates of the scalars
- Boundedness: $\|T^*\| = \|T\|$
- The adjoint operator has the same operator norm as the original operator

- Double adjoint: $(T^*)^* = T$
- Taking the adjoint of the adjoint operator yields the original operator

Calculation of adjoint operators

- Example 1: Let $T: L^2[0, 1] \rightarrow L^2[0, 1]$ be defined by $(Tf)(x) = \int_0^x f(t)dt$. Then,
 $(T^*g)(x) = \int_x^1 g(t)dt$
- The adjoint of the integral operator from 0 to x is the integral operator from x to 1
- Example 2: Let $T: \ell^2 \rightarrow \ell^2$ be defined by $(Tx)_n = \frac{x_n}{n}$. Then, $(T^*y)_n = \frac{y_n}{n}$
- The adjoint of the operator dividing each component by its index is the same operator
- Example 3: Let $T: \mathbb{C}^n \rightarrow \mathbb{C}^m$ be a matrix operator. Then, T^* is the conjugate transpose of the matrix representing T
- The adjoint of a matrix operator is the conjugate transpose of the matrix (A^* or A^H)

Operator vs adjoint relationships

- Self-adjointness:
- An operator T is self-adjoint if $T = T^*$
- The operator is equal to its adjoint
- Equivalently, $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all x, y in the Hilbert space
- The inner product is symmetric with respect to the operator
- Self-adjoint operators have real eigenvalues and orthogonal eigenvectors
- Spectral properties are similar to real symmetric matrices
- Normality:
- An operator T is normal if $TT^* = T^*T$
- The operator commutes with its adjoint
- Self-adjoint operators are always normal
- Self-adjointness is a stronger condition than normality
- Normal operators can be diagonalized by a unitary operator
- They have an orthonormal basis of eigenvectors (U^*TU is diagonal, U unitary)

Properties and Applications

Prove the existence and uniqueness of the adjoint operator and its properties

- Conjugate linearity: $\langle Tx, y \rangle = \overline{\langle x, T^*y \rangle}$

- The inner product with the adjoint is the complex conjugate of the inner product with the original operator
- Adjoint of the adjoint: $(T^*)^* = T$
- Taking the adjoint twice yields the original operator
- Adjoint of the inverse: $(T^{-1})^* = (T^*)^{-1}$, if T is invertible
- The adjoint of the inverse is the inverse of the adjoint
- Adjoint of the composition: $(ST)^* = T^* S^*$
- The adjoint of a composition is the composition of the adjoints in reverse order

Analyze the relationship between an operator and its adjoint, such as self-adjointness and normality

- Positive operators:
 - An operator T is positive if $\langle TX, x \rangle \geq 0$ for all x in the Hilbert space
 - The inner product with the operator is non-negative
 - Positive operators are always self-adjoint
 - Positivity implies self-adjointness
 - The spectrum of a positive operator is a subset of $[0, \infty)$
 - Eigenvalues of positive operators are non-negative real numbers
- Unitary operators:
 - An operator U is unitary if $UU^* = U^*U = I$
 - The operator and its adjoint are inverses of each other
 - Unitary operators preserve inner products: $\langle Ux, Uy \rangle = \langle x, y \rangle$
 - The inner product is invariant under unitary transformations
 - The spectrum of a unitary operator is a subset of the unit circle in the complex plane
 - Eigenvalues of unitary operators have modulus 1
- Applications:
 - Quantum mechanics: observables are represented by self-adjoint operators
 - Ensures real eigenvalues (measurable quantities) and orthogonal eigenvectors (states)
 - Fourier analysis: the Fourier transform is a unitary operator on $L^2(\mathbb{R})$
 - Preserves the L^2 norm (Parseval's identity) and has an inverse transform
 - Sturm-Liouville theory: eigenvalue problems for self-adjoint differential operators
 - Arises in the study of vibrating strings, heat conduction, and quantum mechanics

Unit 3 – Hahn-Banach Theorem: Key Implications

3.1 Statement and proof of the Hahn-Banach Theorem

The Hahn-Banach theorem is a cornerstone of functional analysis. It allows us to extend bounded linear functionals from subspaces to entire normed spaces, keeping their norms intact. This powerful tool opens doors to studying dual spaces and weak topologies.

The theorem's proof uses Zorn's lemma, a heavy-duty tool in set theory. It constructs a maximal extension of the original functional, showing that this extension must cover the whole space. This approach gives us the extension's existence without explicitly building it.

Hahn-Banach Theorem

Hahn-Banach theorem for normed spaces

- Assumes X is a normed linear space contains a subspace Y
- Considers a bounded linear functional f defined on the subspace Y
- Guarantees the existence of a bounded linear functional F defined on the entire space X that extends f
- Preserves the values of f on Y , so $F(y) = f(y)$ for all $y \in Y$
- Maintains the norm of f , ensuring $\|F\| = \|f\|$

Proof using Zorn's lemma

- Defines the set $\mathcal{F}$ consisting of pairs (Z, g)
- Z represents a subspace of X containing the original subspace Y
- g is a bounded linear functional on Z that extends the original functional f
- Introduces a partial order on $\mathcal{F}$ where $(Z_1, g_1) \leq (Z_2, g_2)$ if Z_1 is a subset of Z_2 and g_2 restricted to Z_1 equals g_1
- Demonstrates that every chain in $\mathcal{F}$ has an upper bound
- Constructs the upper bound (Z, g) by taking the union of subspaces Z_α and defining $g(z) = g_\alpha(z)$ for $z \in Z_\alpha$
- Verifies that (Z, g) is indeed an upper bound for the chain
- Applies Zorn's Lemma to conclude that $\mathcal{F}$ has a maximal element (Z_0, g_0)
- Argues that Z_0 must be equal to the entire space X , completing the proof of the theorem

Significance in functional extension

- Enables the extension of bounded linear functionals from a subspace to the entire normed linear space

- Ensures that the extended functional preserves the norm of the original functional
- Provides a powerful tool for studying dual spaces and weak topologies in functional analysis
- Guarantees the existence of the extension without explicitly constructing it

Applications to normed spaces

- Applies to the space $C[0, 1]$ of continuous functions on the interval $[0, 1]$ with the supremum norm
- Considers the subspace Y of polynomials and defines $f(p) = p(0)$ for polynomials $p \in Y$
- Asserts the existence of a bounded linear functional F on $C[0, 1]$ that extends f
- Extends to the space ℓ^1 of absolutely summable sequences with the ℓ^1 norm
- Takes the subspace Y of sequences with only finitely many non-zero terms
- Defines a functional

$$f((x_n)) = \sum_{n=1}^{\infty} a_n x_n$$

for sequences $(x_n) \in Y$, where (a_n) is a sequence in ℓ^∞

- Confirms the existence of an extension of f to a bounded linear functional on the entire space ℓ^1

3.2 Geometric interpretations and applications

The Hahn-Banach Theorem is a powerful tool in functional analysis, allowing us to separate convex sets with hyperplanes. It's like finding an invisible wall between two groups of objects that don't overlap.

This theorem has wide-ranging applications, from optimization problems to mind-bending paradoxes. It helps us find optimal solutions in linear programming and even plays a role in the bizarre Banach-Tarski paradox.

Geometric Interpretations of the Hahn-Banach Theorem

Geometric interpretation of Hahn-Banach theorem

- States if A and B are disjoint nonempty convex subsets of a real vector space, then there exists a nonzero linear functional f and a real number α such that $f(x) \leq \alpha$ for all $x \in A$ and $f(y) \geq \alpha$ for all $y \in B$
- Implies existence of a hyperplane $H = \{x : f(x) = \alpha\}$ that separates A and B , called a separating hyperplane
- In finite-dimensional spaces $(\mathbb{R}^n)$, a hyperplane is an $(n - 1)$ -dimensional affine subspace (subspace plus a translation)
- Guarantees existence of a separating hyperplane between any two disjoint convex sets in a real vector space (Euclidean space, Hilbert space)

Supporting hyperplanes for convex sets

- Supporting hyperplane of a convex set C at a point $x_0 \in C$ contains x_0 and has C lying entirely on one side of it

- To prove existence at a boundary point x_0 of C : 1. Consider singleton set $\{x_0\}$ and convex set $C \setminus \{x_0\}$ 2. By Hahn-Banach Theorem, there exists a separating hyperplane H between these disjoint convex sets 3. Since $x_0 \in H$ and $C \setminus \{x_0\}$ lies on one side of H , H is a supporting hyperplane of C at x_0

- Implies every boundary point of a convex set has a supporting hyperplane (polytopes, balls, ellipsoids)

Applications of the Hahn-Banach Theorem

Applications in optimization problems

- Powerful tool in optimization theory, particularly linear programming and convex optimization
- In linear programming, used to prove existence of optimal solutions and derive optimality conditions
- Proves existence of separating hyperplane between feasible region and objective function level sets
- In convex optimization, derives necessary and sufficient conditions for optimality (Karush-Kuhn-Tucker conditions)
- KKT conditions involve Lagrange multipliers, interpretable as coefficients of supporting hyperplane at optimal solution
- Plays role in duality theory, relating primal problem to dual problem
- Proves strong duality: optimal values of primal and dual problems are equal under certain conditions (Slater's condition)

Connection to Banach-Tarski paradox

- Banach-Tarski paradox: solid ball in 3D can be decomposed into finite disjoint subsets, reassembled to form two identical copies of original ball
- Seems to contradict intuitive notion of volume preservation under rigid motions and reassembly
- Hahn-Banach Theorem is key ingredient in proof
- Used to extend finitely additive measure (on certain subset of ball) to all subsets of ball
- Extension is non-unique, allows construction of paradoxical decomposition
- Paradox relies on Axiom of Choice, equivalent to generalized Hahn-Banach Theorem (real vector space version is special case)
- Highlights limitations of Axiom of Choice and Hahn-Banach Theorem for non-measurable sets and functions
- Demonstrates importance of countable additivity (vs. finite additivity) in measure theory to avoid counterintuitive results (Vitali sets, Hausdorff paradox)

3.3 Consequences for linear functionals and dual spaces

Dual spaces and the Hahn-Banach Theorem are key concepts in functional analysis. They provide powerful tools for understanding normed linear spaces through their linear functionals, extending bounded linear maps, and separating points and sets.

These ideas have far-reaching implications in analysis and optimization. They form the foundation for weak topologies, reflexivity of spaces, and the study of convex sets, which are crucial in many areas of

Dual Spaces and the Hahn-Banach Theorem

Existence of bounded linear functionals

- A normed linear space $(X, \|\cdot\|)$ consists of a vector space X equipped with a norm $\|\cdot\|$ that measures the "size" of elements in X
- The norm satisfies properties such as positive definiteness ($\|x\| = 0$ iff $x = 0$), scalar multiplication compatibility ($\|\alpha x\| = |\alpha| \|x\|$), and the triangle inequality ($\|x + y\| \leq \|x\| + \|y\|$)
- Linear functionals $f: X \rightarrow \mathbb{R}$ (or $\mathbb{C}$) are linear maps that preserve vector space operations ($f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$)
- Bounded linear functionals have their output controlled by the norm of the input ($|f(x)| \leq M \|x\|$ for some constant $M \geq 0$)
- The Hahn-Banach Theorem guarantees the existence of non-zero bounded linear functionals on any normed linear space (ℓ^p spaces, L^p spaces)

Dual space of normed spaces

- The dual space X^* of a normed linear space $(X, \|\cdot\|)$ contains all bounded linear functionals on X
- Dual spaces provide a way to study properties of the original space X through its linear functionals
- X^* forms a normed linear space with the operator norm $\|f\|_{X^*} = \sup \{|f(x)| : \|x\| \leq 1\}$
- If X is a Banach space (complete normed linear space), then X^* is also a Banach space
- The dual space separates points in X : for distinct $x, y \in X$, there exists $f \in X^*$ with $f(x) \neq f(y)$ (C_0 and ℓ^1)

Hahn-Banach Theorem for dual spaces

- The Hahn-Banach Theorem allows extending bounded linear functionals from a subspace to the whole space
- If Y is a subspace of a normed linear space X and $f: Y \rightarrow \mathbb{R}$ (or $\mathbb{C}$) is bounded, then f extends to a bounded linear functional $F: X \rightarrow \mathbb{R}$ (or $\mathbb{C}$) with the same norm
- Proof outline: 1. Use Zorn's Lemma to obtain a maximal extension of f 2. Show that the maximal extension has the same norm as f 3. Prove that the maximal extension is defined on the entire space X
- The theorem has significant implications in functional analysis and optimization (L^p spaces, $C[a, b]$)

Normed spaces vs second duals

- The second dual space X^{**} is the dual of the dual space X^*
- The canonical embedding $J: X \rightarrow X^{**}$ is defined by $J(x)(f) = f(x)$ for $x \in X$ and $f \in X^*$
- J preserves the norm ($\|J(x)\|_{X^{**}} = \|x\|_X$) and is injective (one-to-one)
- A normed space X is reflexive if J is surjective (onto), i.e., X is isometrically isomorphic to X^{**}
- Reflexive spaces include Hilbert spaces and L^p spaces for $1 < p < \infty$

- Non-reflexive spaces include L^1 , L^∞ , and C_0

Implications for weak topologies

- The weak topology on X is the coarsest topology making all functionals in X^* continuous
- A net (x_α) converges weakly to x in X iff $f(x_\alpha) \rightarrow f(x)$ for all $f \in X^*$
- The weak* topology on X^* is the coarsest topology making all evaluations $f \mapsto f(x)$ continuous for each $x \in X$
- A net (f_α) converges weak* to f in X^* iff $f_\alpha(x) \rightarrow f(x)$ for all $x \in X$
- The Banach-Alaoglu Theorem states that the closed unit ball of X^* is compact in the weak* topology
- The Hahn-Banach Theorem enables the separation of convex sets in locally convex spaces (L^p spaces, Hilbert spaces)
- Locally convex spaces are characterized by a family of seminorms and include all normed spaces

Unit 4 – Open and Closed Graph Theorems

4.1 Open Mapping Theorem: statement, proof, and applications

The Open Mapping Theorem is a cornerstone of functional analysis. It states that surjective bounded linear operators between Banach spaces map open sets to open sets, a powerful result with far-reaching implications.

This theorem's proof relies on the Baire Category Theorem and leads to important applications. It's crucial for establishing other key results like the Closed Graph Theorem and the Inverse Mapping Theorem, shaping our understanding of linear operators.

The Open Mapping Theorem

Open Mapping Theorem statement

- Asserts that a surjective bounded linear operator T between Banach spaces X and Y is an open map
- Maps open sets in X to open sets in Y
- Requires key assumptions:
 - X and Y are complete normed vector spaces (Banach spaces)
 - T is a continuous linear transformation (bounded linear operator)
 - T is onto, meaning for every $y \in Y$, there exists an $x \in X$ such that $T(x) = y$ (surjective)

Proof using Baire Category Theorem

- Proves by contradiction, assuming T is not an open map
- Supposes an open set U in X exists such that $T(U)$ is not open in Y
- Implies a point $y_0 \in T(U)$ exists where no open ball around y_0 is contained in $T(U)$
- Constructs a sequence of closed sets $F_n = \overline{T(nU)}$, where $nU = \{nx : x \in U\}$
- Each F_n is closed due to T 's continuity and nU 's closure
- Covers Y with the union of F_n ,

$$Y = \bigcup_{n=1}^{\infty} F_n$$

- T 's surjectivity ensures for every $y \in Y$, an $x \in X$ exists with $T(x) = y$
- Scaling x yields an n where $x \in nU$, implying $y \in T(nU) \subseteq F_n$
- Applies the Baire Category Theorem to conclude one F_n must have a non-empty interior
- Assumes F_N has a non-empty interior, with an open ball $B(y_1, r) \subseteq F_N$ contained in F_N
- Deduces $B(y_1, r) \subseteq T(nU)$, contradicting the assumption that no open ball around y_0 is contained in $T(U)$

- Concludes T must be an open map

Surjectivity of linear operators

- Establishes that an injective bounded linear operator $T : X \rightarrow Y$ between Banach spaces is surjective if and only if $T(X)$ is closed in Y
- Surjectivity implies $T(X) = Y$, which is closed
- $T(X)$'s closure, combined with the Open Mapping Theorem, implies T is an open map
- T 's injectivity makes it a bijection between X and $T(X)$
- $T(X)$'s closure makes it a Banach space, and the Inverse Mapping Theorem ensures T^{-1} is continuous
- T becomes a homeomorphism between X and $T(X) = Y$, implying surjectivity

Applications in functional analysis

- Proves the Closed Graph Theorem
- A linear operator $T : X \rightarrow Y$ between Banach spaces with a closed graph in $X \times Y$ is continuous
- Proves the Inverse Mapping Theorem
- The inverse of a bijective bounded linear operator $T : X \rightarrow Y$ between Banach spaces is also a bounded linear operator
- Proves the Closed Range Theorem
- For a bounded linear operator $T : X \rightarrow Y$ between Banach spaces, $\text{range}(T)$ is closed in Y if and only if $\text{range}(T^*)$ is closed in X^*
- Proves the completeness of function spaces like L^p spaces and Sobolev spaces

4.2 Closed Graph Theorem: statement, proof, and applications

The Closed Graph Theorem is a powerful tool in functional analysis. It links the continuity of linear operators between Banach spaces to the closure of their graphs, simplifying proofs of operator continuity in many cases.

This theorem has wide-ranging applications in functional analysis. It's used to prove the continuity of inverse operators, establish the Banach Isomorphism Theorem, and show continuity in various mathematical settings like Fourier transforms.

The Closed Graph Theorem

Closed Graph Theorem formulation

- States if X and Y are Banach spaces (complete normed vector spaces) and $T : X \rightarrow Y$ is a linear operator with a closed graph $G(T) = \{(x, Tx) : x \in X\}$ in the product topology of $X \times Y$, then T is continuous
- Requires X and Y to be Banach spaces (e.g., L^p spaces, $C[a, b]$) and T to be a linear operator between them (e.g., differentiation, integration)

- Graph of T , $G(T)$, must be closed in $X \times Y$ equipped with the product topology (e.g., pointwise convergence, uniform convergence)

Proof using Open Mapping Theorem

- Relies on the Open Mapping Theorem which states a surjective bounded linear operator $T : X \rightarrow Y$ between Banach spaces is an open map
- Defines a new linear, bounded, and surjective operator $S : G(T) \rightarrow X$ by $S(x, Tx) = x$
- Applies the Open Mapping Theorem to show S is an open map
- Uses the openness of S to prove T is continuous
 1. For any open set $U \subset X$, $S^{-1}(U) = \{(x, Tx) : x \in U\}$ is open in $G(T)$
 2. Since $G(T)$ is closed in $X \times Y$, $S^{-1}(U)$ is open in $X \times Y$
 3. The projection of $S^{-1}(U)$ onto Y is $T(U)$, which is open in Y
 4. Thus, T maps open sets to open sets, implying continuity

Continuity of linear operators

- Proves a linear operator $T : X \rightarrow Y$ is continuous by showing its graph $G(T)$ is closed
- Commonly used to prove continuity of the inverse operator T^{-1} when T is bijective by showing $G(T^{-1}) = \{(Tx, x) : x \in X\}$ is closed
- Establishes continuity of linear operators defined by specific properties (e.g., proving a linear functional $f : X \rightarrow \mathbb{R}$ satisfying a certain condition is continuous)

Implications in Banach spaces

- Provides a topological characterization of continuous linear operators between Banach spaces a linear operator is continuous if and only if its graph is closed
- Helps prove the Banach Isomorphism Theorem if $T : X \rightarrow Y$ is a bijective continuous linear operator between Banach spaces, then T^{-1} is also continuous
- Establishes continuity of linear operators in various settings (e.g., proving continuity of the Fourier transform on certain function spaces)
- Closely related to other fundamental results in functional analysis such as the Open Mapping Theorem and the Uniform Boundedness Principle

4.3 Uniform Boundedness Principle and its consequences

The Uniform Boundedness Principle is a key concept in functional analysis. It shows that if a family of operators is bounded at each point, it's actually bounded everywhere. This powerful result connects pointwise and uniform behavior of operators.

This principle has far-reaching consequences. It's used to prove other important theorems, like the Banach-Steinhaus Theorem, and helps analyze operator sequences, adjoint operators, and spectral properties. It's a fundamental tool for understanding linear operators on Banach spaces.

The Uniform Boundedness Principle and Its Consequences

Uniform Boundedness Principle significance

- Fundamental result in functional analysis applies to a family of continuous linear operators between Banach spaces
- Establishes that if a family of continuous linear operators $\{T_\alpha\}_{\alpha \in A}$ from a Banach space X to a normed space Y is pointwise bounded, then it is uniformly bounded
- Pointwise bounded means for each $x \in X$, there exists $M_x > 0$ such that $\|T_\alpha(x)\| \leq M_x$ for all $\alpha \in A$
- Uniformly bounded means there exists $M > 0$ such that $\|T_\alpha\| \leq M$ for all $\alpha \in A$
- Allows for the extension of pointwise convergence to uniform convergence in certain cases (ℓ^p spaces)
- Provides a powerful tool for studying the behavior of families of operators (adjoint operators, operator algebras)
- Serves as a foundation for various important theorems in functional analysis (closed graph theorem, open mapping theorem)

Proof of Uniform Boundedness Principle

- Relies on the Baire Category Theorem, which states that a complete metric space cannot be expressed as a countable union of nowhere dense sets
- Define sets $E_n = \{x \in X : \sup_{\alpha \in A} \|T_\alpha(x)\| \leq n\}$ for each positive integer n
- Each E_n is closed due to the continuity of the operators and the pointwise boundedness condition
- Assume, for contradiction, that the family $\{T_\alpha\}_{\alpha \in A}$ is not uniformly bounded
- Implies

$$X = \bigcup_{n=1}^{\infty} E_n$$

, expressing X as a countable union of the sets E_n

- By the Baire Category Theorem, at least one E_n must have a non-empty interior
- Let x_0 be an interior point of some E_n , and let $r > 0$ be such that $B(x_0, r) \subset E_n$
- For any $x \in X$ with $\|x\| \leq r$, we have $x_0 + x \in B(x_0, r) \subset E_n$, implying $\sup_{\alpha \in A} \|T_\alpha(x_0 + x)\| \leq n$
- By the linearity of the operators, $\sup_{\alpha \in A} \|T_\alpha(x)\| \leq 2n$ for all $x \in X$ with $\|x\| \leq r$
- Scaling the result, we obtain $\sup_{\alpha \in A} \|T_\alpha(x)\| \leq \frac{2n}{r} \|x\|$ for all $x \in X$
- Implies uniform boundedness, contradicting the initial assumption
- Therefore, the family $\{T_\alpha\}_{\alpha \in A}$ must be uniformly bounded

Banach-Steinhaus Theorem derivation

- Also known as the Principle of Condensation of Singularities, is a direct consequence of the UBP

- Consider a sequence of continuous linear operators $\{T_n\}_{n=1}^{\infty}$ from a Banach space X to a normed space Y
- Suppose that for each $x \in X$, the sequence $\{T_n(x)\}_{n=1}^{\infty}$ converges in Y
- Define the pointwise limit operator $T : X \rightarrow Y$ by $T(x) = \lim_{n \rightarrow \infty} T_n(x)$ for each $x \in X$
- The pointwise convergence of $\{T_n(x)\}_{n=1}^{\infty}$ ensures that T is well-defined
- By the UBP, the sequence $\{T_n\}_{n=1}^{\infty}$ is uniformly bounded, i.e., there exists $M > 0$ such that $\|T_n\| \leq M$ for all n
- The Uniform Boundedness Principle implies that the pointwise limit operator T is continuous and $\|T\| \leq M$
- Follows from the Banach-Steinhaus Theorem, which states that a pointwise limit of a sequence of continuous linear operators is continuous if the sequence is pointwise bounded

Applications in operator theory

- Proving the continuity of pointwise limit operators (Banach-Steinhaus Theorem)
- Studying the behavior of adjoint operators by applying UBP to the family $\{T_{\alpha}^*\}_{\alpha \in A}$
- Investigating the convergence of operator sequences and series using pointwise convergence and boundedness
- Analyzing the properties of operator algebras (C*-algebras, von Neumann algebras)
- Developing the theory of spectral measures and spectral representations of linear operators
- Establishing the closed graph theorem and the open mapping theorem using UBP and Baire Category Theorem

Additional Topics Related to the Uniform Boundedness Principle

Relationship between Uniform Boundedness Principle and Principle of Condensation of Singularities

- The Principle of Condensation of Singularities, also known as the Banach-Steinhaus Theorem, is a direct consequence of the UBP
- States that if $\{T_n\}_{n=1}^{\infty}$ is a sequence of continuous linear operators from a Banach space X to a normed space Y such that $\lim_{n \rightarrow \infty} T_n(x)$ exists for each $x \in X$, then one of the following holds: 1. The sequence $\{T_n\}_{n=1}^{\infty}$ is uniformly bounded, and the pointwise limit operator $T(x) = \lim_{n \rightarrow \infty} T_n(x)$ is continuous 2. There exists a non-meager set $A \subset X$ such that for each $x \in A$, the sequence $\{T_n(x)\}_{n=1}^{\infty}$ is unbounded
- UBP is used to prove the first case, while the second case demonstrates the "condensation of singularities" phenomenon

Role of Baire Category Theorem in proof of Uniform Boundedness Principle

- The Baire Category Theorem is a crucial tool in the proof of the UBP

- States that a complete metric space cannot be expressed as a countable union of nowhere dense sets
- A set is nowhere dense if its closure has an empty interior
- In the proof of the UBP, the Banach space X is assumed to be the countable union of the sets $E_n = \{x \in X : \sup_{\alpha \in A} \|T_\alpha(x)\| \leq n\}$
- Each E_n is closed due to the continuity of the operators and the pointwise boundedness condition
- If the family $\{T_\alpha\}_{\alpha \in A}$ is not uniformly bounded, then

$$X = \bigcup_{n=1}^{\infty} E_n$$

- The Baire Category Theorem implies that at least one E_n must have a non-empty interior, leading to a contradiction
- This contradiction proves that the family $\{T_\alpha\}_{\alpha \in A}$ must be uniformly bounded

Unit 5 – Hilbert Spaces and Orthonormal Bases

5.1 Definition and properties of inner product spaces

Inner product spaces are vector spaces with a special function that pairs vectors, giving us a way to measure angles and lengths. This function, called the inner product, follows specific rules that make it behave nicely with vector operations.

The inner product lets us define norms, which measure vector lengths. It also gives us the Cauchy-Schwarz inequality, a powerful tool for bounding inner products. These concepts are key to understanding vector spaces more deeply.

Inner Product Spaces

Definition of inner product spaces

- Vector space V over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) equipped with an inner product function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ that assigns a scalar value to each pair of vectors
- Inner product satisfies the following axioms for all $x, y, z \in V$ and $\alpha \in \mathbb{F}$:
- Conjugate symmetry ensures $\langle x, y \rangle$ equals the complex conjugate of $\langle y, x \rangle$
- Linearity in the second argument distributes the inner product over vector addition and scalar multiplication $\langle x, \alpha y + z \rangle = \alpha \langle x, y \rangle + \langle x, z \rangle$
- Positive definiteness guarantees $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$, implying the inner product of a vector with itself is always non-negative and zero only for the zero vector

Properties of inner products

- Conjugate symmetry $\langle x, y \rangle = \overline{\langle y, x \rangle}$ holds for all $x, y \in V$ by the conjugate symmetry axiom
- Linearity in the first argument $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in V$ and $\alpha \in \mathbb{F}$ follows from conjugate symmetry and linearity in the second argument:
- Proof: $\langle \alpha x + y, z \rangle = \overline{\langle z, \alpha x + y \rangle} = \overline{\alpha \langle z, x \rangle + \langle z, y \rangle} = \overline{\alpha} \overline{\langle z, x \rangle} + \overline{\langle z, y \rangle} = \alpha \langle x, z \rangle + \langle y, z \rangle$
- Sesquilinearity combines linearity in one argument and conjugate linearity in the other, resulting from linearity in both arguments and conjugate symmetry

Norms from inner products

- Inner product $\langle \cdot, \cdot \rangle$ on vector space V induces a norm defined as $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in V$, measuring the length or magnitude of a vector
- Induced norm satisfies properties:
- Positivity $\|x\| \geq 0$ with $\|x\| = 0$ if and only if $x = 0$, ensuring non-negative lengths and zero length only for the zero vector

- Homogeneity $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{F}$ and $x \in V$, scaling the length by the absolute value of the scalar
- Triangle inequality $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$, bounding the length of a sum of vectors by the sum of their lengths
- Cauchy-Schwarz inequality $|\langle x, y \rangle| \leq \|x\| \|y\|$ for all $x, y \in V$ geometrically interprets the absolute value of the inner product as bounded by the product of the vector norms

Identification of inner products

- Verifying a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ is an inner product on vector space V requires checking the inner product axioms:
- Conjugate symmetry $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$
- Linearity in the second argument $\langle x, \alpha y + z \rangle = \alpha \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in V$ and $\alpha \in \mathbb{F}$
- Positive definiteness $\langle x, x \rangle \geq 0$ for all $x \in V$ with $\langle x, x \rangle = 0$ if and only if $x = 0$
- Common inner products include:
- Euclidean inner product on $\mathbb{R}^n$:

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i$$

for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ (dot product)

- Standard inner product on $\mathbb{C}^n$:

$$\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$$

for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ (Hermitian inner product)

- L^2 inner product on square-integrable functions: $\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$ for $f, g \in L^2([a, b])$ (integral of the product of one function with the complex conjugate of the other)

5.2 Hilbert spaces and their characteristics

Hilbert spaces are the powerhouses of functional analysis, combining completeness, norms, and inner products. They're like super-charged vector spaces that let us do amazing things with infinite dimensions and abstract functions.

These spaces are everywhere in math and physics. From quantum mechanics to signal processing, Hilbert spaces give us the tools to tackle complex problems. They're the perfect playground for studying operators, orthogonality, and approximation theory.

Hilbert Spaces

Definition of Hilbert spaces

- Hilbert spaces are complete normed vector spaces equipped with an inner product
- Completeness ensures every Cauchy sequence converges to an element within the space

- Normed vector spaces have a notion of distance between elements defined by a norm (Euclidean spaces, L^p spaces)
- Inner product generalizes the dot product and induces a norm $\|x\| = \sqrt{\langle x, x \rangle}$
- Inner product satisfies properties such as conjugate symmetry, linearity, and positive-definiteness
- Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- Linearity: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for scalars a, b
- Positive-definiteness: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$
- Distinguish Hilbert spaces from general inner product spaces by completeness
- General inner product spaces may not be complete (space of polynomials with L^2 inner product)
- Completeness is crucial for many important results and applications in functional analysis

Completeness of Hilbert spaces

- Prove Hilbert spaces are complete with respect to the norm induced by the inner product
- Consider a Cauchy sequence (x_n) in a Hilbert space H
- Show the sequence of inner products $(\langle x_n, x_m \rangle)$ is Cauchy in the underlying field ($\mathbb{R}$ or $\mathbb{C}$)
- Use the completeness of the field to conclude $(\langle x_n, x_m \rangle)$ converges to a limit L
- Define the limit element $x \in H$ using the limit L and properties of the inner product
- For any $y \in H$, define $\langle x, y \rangle := \lim_{n \rightarrow \infty} \langle x_n, y \rangle$
- Verify that x is well-defined and belongs to H using the boundedness of (x_n)
- Show that the Cauchy sequence (x_n) converges to the limit element x
- Use the Cauchy-Schwarz inequality and the convergence of inner products
- Conclude that every Cauchy sequence in H converges, proving completeness

Examples of Hilbert spaces

- L^2 spaces: square-integrable functions on a measure space (X, μ)
- Functions $f: X \rightarrow \mathbb{C}$ satisfying $\int_X |f|^2 d\mu < \infty$
- Inner product: $\langle f, g \rangle = \int_X f \bar{g} d\mu$
- Examples: $L^2(\mathbb{R})$ (functions on the real line), $L^2([0, 1])$ (functions on the unit interval)
- ℓ^2 spaces: square-summable sequences
- Sequences $(x_n)_{n=1}^\infty$ satisfying

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty$$

- Inner product:

$$\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

- Finite-dimensional Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$
- Inner product is the standard dot product:

$$\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$$

- Sobolev spaces $H^k(\Omega)$: functions with square-integrable weak derivatives up to order k

Cauchy-Schwarz inequality in Hilbert spaces

- Statement: $|\langle x, y \rangle| \leq \|x\| \|y\|$ for all $x, y \in H$
- Equality holds if and only if x and y are linearly dependent
- Geometric interpretation: relates the angle between vectors to their lengths
- For unit vectors, $|\langle x, y \rangle|$ is the cosine of the angle between them
- Cauchy-Schwarz inequality bounds the absolute value of the inner product by the product of norms
- Prove the Cauchy-Schwarz inequality using the properties of the inner product
- Consider the quadratic polynomial $p(t) = \langle x + ty, x + ty \rangle$ for $t \in \mathbb{R}$
- Use the positive-definiteness of the inner product to show $p(t) \geq 0$ for all t
- Conclude that the discriminant of $p(t)$ must be non-positive, yielding the inequality
- Applications and consequences of the Cauchy-Schwarz inequality
- Proving the triangle inequality for the norm: $\|x + y\| \leq \|x\| + \|y\|$
- Deriving uncertainty principles in quantum mechanics (Heisenberg's uncertainty principle)
- Studying the convergence of Fourier series in L^2 spaces

5.3 Orthogonality, projections, and Gram-Schmidt process

Orthogonality in vector spaces is all about perpendicular relationships between vectors. It's a key concept in inner product spaces and Hilbert spaces, where we can measure angles and distances between vectors using inner products.

Orthogonal projections are super useful for finding the closest point in a subspace to a given vector. We can compute these projections using inner products and orthonormal bases, which helps solve problems in many fields like signal processing and data analysis.

Orthogonality and Inner Product Spaces

Orthogonality in vector spaces

- Orthogonality is a geometric concept describing the perpendicular relationship between vectors in inner product spaces and Hilbert spaces
- Two vectors x and y are orthogonal if their inner product equals zero: $\langle x, y \rangle = 0$

- Inner product spaces are vector spaces with an inner product function assigning a scalar value to each pair of vectors
- Inner product satisfies properties for all vectors x, y, z and scalars a :
- Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- Linearity in the second argument: $\langle x, ay + z \rangle = a\langle x, y \rangle + \langle x, z \rangle$
- Positive definiteness: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$
- Hilbert spaces are complete inner product spaces where every Cauchy sequence converges to an element within the space
- Completeness allows application of powerful tools and theorems in functional analysis (L^2 spaces, Fourier series)

Computation of orthogonal projections

- Orthogonal projections find the closest point in a subspace to a given vector
- The orthogonal projection of vector x onto closed subspace M is the unique vector $P_M(x)$ in M minimizing distance between x and any vector in M
- Mathematically: $\|x - P_M(x)\| = \min_{y \in M} \|x - y\|$
- To compute the orthogonal projection of vector x onto subspace M spanned by orthonormal basis $\{e_1, e_2, \dots, e_n\}$:

•

$$P_M(x) = \sum_{i=1}^n \langle x, e_i \rangle e_i$$

- Vector $x - P_M(x)$ is orthogonal to every vector in M , representing the component of x perpendicular to the subspace (residual, error term)

Gram-Schmidt Process and Orthonormal Bases

Gram-Schmidt process for orthonormal bases

- The Gram-Schmidt process constructs an orthonormal basis from a linearly independent set of vectors
- Given linearly independent set $\{v_1, v_2, \dots, v_n\}$: 1. Define $e_1 = \frac{v_1}{\|v_1\|}$ 2. For $i = 2, 3, \dots, n$:
- Compute

$$u_i = v_i - \sum_{j=1}^{i-1} \langle v_i, e_j \rangle e_j$$

- Define $e_i = \frac{u_i}{\|u_i\|}$
- The resulting set $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis for the space spanned by the original linearly independent set
- Orthonormal bases simplify calculations and provide a convenient coordinate system (QR factorization, least squares)

Properties of orthogonal projections

- In Hilbert spaces, orthogonal projections onto closed subspaces always exist and are unique
- Existence proof:
 - Let M be a closed subspace of Hilbert space H and $x \in H$
 - Show set $\{x - y : y \in M\}$ contains a unique element of minimal norm using parallelogram law and completeness of M
 - This element is the orthogonal projection of x onto M
- Uniqueness proof:
 - Suppose two orthogonal projections $P_M(x)$ and $Q_M(x)$ of x onto M exist
 - Show $\|P_M(x) - Q_M(x)\|^2 = 0$ using properties of orthogonal projections and Pythagorean theorem
 - Conclude $P_M(x) = Q_M(x)$, proving uniqueness
- Existence and uniqueness of orthogonal projections underlie many applications (least squares approximation, signal processing)

5.4 Orthonormal bases and Fourier series

Orthonormal bases are the building blocks of Hilbert spaces. They're like a special set of tools that let us break down complex objects into simpler parts. These bases have two key features: the vectors are perpendicular to each other and have a length of 1.

Using orthonormal bases, we can represent any vector in the space as a unique combination of these basic elements. This idea extends to Fourier series, where we can express functions as sums of sines and cosines. It's a powerful way to analyze and solve problems in many areas of math and physics.

Orthonormal Bases

Orthonormal bases in Hilbert spaces

- Set of vectors in a Hilbert space satisfying two conditions:
 - Orthogonality: Inner product of any two distinct vectors equals zero (perpendicular)
 - Normality: Each vector has a norm (length) equal to 1 (unit vectors)
- Key properties of orthonormal bases:
 - Unique representation: Every vector in the Hilbert space can be expressed as a unique linear combination of the basis vectors (no ambiguity)
 - Parseval's identity: Sum of the squares of the coefficients in the basis expansion equals the square of the norm of the vector (energy conservation)
 - Completeness: Linear span of the orthonormal basis is dense in the Hilbert space (can approximate any vector arbitrarily well)

Linear combinations of orthonormal elements

- Given an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ in a Hilbert space H and a vector $x \in H$, express x as:

-

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$$

(infinite sum of basis vectors scaled by Fourier coefficients)

- Fourier coefficients $\langle x, e_n \rangle$ represent the contribution of each basis vector to the vector x
- Calculate Fourier coefficients using the inner product:
- $\langle x, e_n \rangle = \int_a^b x(t) \overline{e_n(t)} dt$ for functions in $L^2([a, b])$ (square-integrable functions on the interval $[a, b]$)

Fourier Series

Fourier series with orthonormal bases

- Expand a periodic function $f(x)$ as an infinite sum of sines and cosines:

-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega x) + b_n \sin(n\omega x))$$

(Fourier series)

- Fundamental frequency $\omega = \frac{2\pi}{T}$, where T is the period of $f(x)$ (number of cycles per unit length)
- Calculate coefficients a_n and b_n using the inner product with the orthonormal basis functions:
- $a_n = \frac{2}{T} \int_0^T f(x) \cos(n\omega x) dx$ (cosine coefficients)
- $b_n = \frac{2}{T} \int_0^T f(x) \sin(n\omega x) dx$ (sine coefficients)

Parseval's identity for Fourier series

- Parseval's identity for a function $f(x)$ with Fourier coefficients a_n and b_n :

-

$$\int_0^T |f(x)|^2 dx = \frac{T}{2} \left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right)$$

(energy conservation)

- Interpretation in the context of Fourier series:
- Left-hand side: Total energy of the function $f(x)$ over one period
- Right-hand side: Sum of the energies of the individual Fourier components
- Parseval's identity demonstrates that the energy of a function is conserved when decomposed into its Fourier components (no energy lost or gained)

Fourier series in boundary value problems

- Use Fourier series to solve boundary value problems (BVPs) in partial differential equations (PDEs)

- Steps to solve a BVP using Fourier series: 1. Assume a solution in the form of a Fourier series with unknown coefficients 2. Substitute the assumed solution into the PDE and boundary conditions 3. Use the orthogonality of the basis functions to determine the Fourier coefficients 4. Construct the final solution using the Fourier series with the calculated coefficients
- Example: Solving the heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(0, t) = u(L, t) = 0$ (insulated ends) and initial condition $u(x, 0) = f(x)$ (initial temperature distribution)

Unit 6 – Bounded Linear Operators in Hilbert Spaces

6.1 Adjoint operators in Hilbert spaces

Adjoint operators are crucial in Hilbert spaces, linking an operator's action to the inner product structure. They're defined by a unique property that swaps the operator's effect between inner product arguments, revealing deep connections in operator theory.

Adjoint operators exist for all bounded linear operators and have key properties like norm preservation. They're essential for studying self-adjoint, normal, and unitary operators, which are vital in functional analysis and quantum mechanics.

Adjoint Operators in Hilbert Spaces

Definition of adjoint operators

- Let H be a Hilbert space and $T : H \rightarrow H$ be a bounded linear operator
- The adjoint of T , denoted by T^* , is a unique bounded linear operator satisfying the property $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in H$
- This property relates the action of T on the first argument of the inner product to the action of T^* on the second argument
- The adjoint operator allows for the study of T through the inner product structure of the Hilbert space
- Properties of the adjoint operator include $\|T^*\| = \|T\|$, $(T^*)^* = T$, $(S + T)^* = S^* + T^*$ for bounded linear operators S and T , $(\alpha T)^* = \bar{\alpha}T^*$ for any scalar α , and $(ST)^* = T^*S^*$ for bounded linear operators S and T
- These properties demonstrate the close relationship between an operator and its adjoint

Existence and uniqueness of adjoints

- Existence of the adjoint operator T^* for a bounded linear operator $T : H \rightarrow H$ can be proven using the Riesz Representation Theorem 1. Define a linear functional $\phi_y : H \rightarrow \mathbb{C}$ by $\phi_y(x) = \langle Tx, y \rangle$ for each $y \in H$ 2. By the Riesz Representation Theorem, there exists a unique $z \in H$ such that $\phi_y(x) = \langle x, z \rangle$ for all $x \in H$ 3. Define $T^*y = z$, then $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in H$
- Uniqueness of the adjoint operator can be proven by assuming S and T are bounded linear operators satisfying $\langle Sx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$
- Then $\langle (S - T)x, y \rangle = 0$ for all $x, y \in H$
- Choosing $y = (S - T)x$ yields $\|(S - T)x\|^2 = 0$ for all $x \in H$, implying $S = T$ and proving uniqueness

Calculation of specific adjoints

- Identity operator I on a Hilbert space H is self-adjoint, meaning $I^* = I$
- This can be verified by observing $\langle Ix, y \rangle = \langle x, y \rangle = \langle x, Iy \rangle$ for all $x, y \in H$

- Multiplication operator M_f on the Hilbert space $L^2(\mu)$, defined by $(M_f\varphi)(x) = f(x)\varphi(x)$, has adjoint $(M_f)^* = M_{\bar{f}}$
- This can be shown by calculating $\langle M_f\varphi, \psi \rangle = \int f(x)\varphi(x)\overline{\psi(x)}d\mu = \int \varphi(x)\overline{f(x)\psi(x)}d\mu = \langle \varphi, M_{\bar{f}}\psi \rangle$ for $\varphi, \psi \in L^2(\mu)$
- Properties of adjoints can be verified for these specific operators, such as $\|I^*\| = \|I\| = 1$ and $(M_f)^{**} = (M_{\bar{f}})^* = M_f$

Operators vs adjoints in inner products

- The adjoint operator T^* is defined by the property $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in H$, which relates the action of T on the first argument of the inner product to the action of T^* on the second argument
- This property allows for the study of an operator T through the inner product structure of the Hilbert space
- Properties of the adjoint operator, such as $\|T^*\| = \|T\|$, demonstrate the close relationship between an operator and its adjoint
- The adjoint operator is a key concept in the study of self-adjoint $T = T^*$, normal $TT^* = T^*T$, and unitary $TT^* = T^*T = I$ operators in Hilbert spaces
- These special classes of operators have important properties and applications in functional analysis and quantum mechanics

6.2 Self-adjoint, unitary, and normal operators

Hilbert space operators play a crucial role in functional analysis and quantum mechanics. Self-adjoint, unitary, and normal operators each have unique properties that shape their behavior and applications in mathematical and physical contexts.

These operators are defined by specific relationships with their adjoints and possess distinct spectral properties. Understanding their characteristics is essential for analyzing quantum systems and solving complex mathematical problems in functional analysis.

Self-Adjoint, Unitary, and Normal Operators

Types of Hilbert space operators

- Self-adjoint (Hermitian) operators
- An operator T on a Hilbert space H is self-adjoint if it equals its own adjoint T^* , satisfying $T = T^*$
- The position operator $\hat{x}$ in quantum mechanics, defined by $(\hat{x}\psi)(x) = x\psi(x)$, serves as an example of a self-adjoint operator
- Unitary operators
- An operator U on a Hilbert space H is unitary if its product with its adjoint U^* equals the identity operator I , satisfying $UU^* = U^*U = I$
- The time-evolution operator $e^{-iHt/\hbar}$ in quantum mechanics, where H is the Hamiltonian, exemplifies a unitary operator
- Normal operators

- An operator N on a Hilbert space H is normal if it commutes with its adjoint, satisfying $NN^* = N^*N$
- Self-adjoint operators and unitary operators are both examples of normal operators
- The momentum operator $\hat{p} = -i\hbar \frac{d}{dx}$ in quantum mechanics also illustrates a normal operator

Spectral properties of operators

- Self-adjoint operators
 - The spectral theorem states that every self-adjoint operator on a Hilbert space possesses a real spectrum and can be represented as a multiplication operator on an appropriate function space
 - Eigenvectors corresponding to distinct eigenvalues of a self-adjoint operator are orthogonal to each other
- Unitary operators
 - The spectral theorem asserts that every unitary operator on a Hilbert space has a spectrum contained within the unit circle and can be represented as a multiplication operator on a suitable function space
 - Eigenvectors associated with distinct eigenvalues of a unitary operator are orthogonal
- Normal operators
 - The spectral theorem guarantees that every normal operator on a Hilbert space can be represented as a multiplication operator on an appropriate function space
 - Eigenvectors corresponding to distinct eigenvalues of a normal operator are orthogonal

Identification of operator types

- Checking for self-adjointness
 - Verify that the inner product $\langle TX, Y \rangle$ equals $\langle X, TY \rangle$ for all vectors X, Y in the Hilbert space H
 - Alternatively, check if the matrix representation of the operator is equal to its conjugate transpose
- Checking for unitarity
 - Verify that the inner product $\langle UX, UY \rangle$ equals $\langle X, Y \rangle$ for all vectors X, Y in the Hilbert space H
 - Alternatively, check if the product of the matrix representation of the operator and its conjugate transpose yields the identity matrix
- Checking for normality
 - Verify that the operator N commutes with all bounded linear operators T on the Hilbert space H , satisfying $NT = TN$ for all $T \in B(H)$
 - Alternatively, check if the matrix representation of the operator commutes with its conjugate transpose

Operators in quantum mechanics

- Self-adjoint operators
 - Represent observable quantities in quantum mechanics, such as position, momentum, and energy
 - Ensure real eigenvalues, which correspond to measurable values of the observable

- Eigenvectors form a complete orthonormal basis, allowing any state vector to be expanded in terms of these eigenvectors
- Unitary operators
- Represent symmetry transformations and time evolution in quantum systems
- Preserve inner products and norms of state vectors, ensuring conservation of probability
- Enable the description of quantum systems from different frames of reference
- Normal operators
- Provide a unified framework for studying both self-adjoint and unitary operators in quantum mechanics
- Allow for the spectral decomposition of operators, simplifying the analysis of quantum systems
- Eigenvectors form a complete orthonormal basis, enabling any state vector to be represented as a linear combination of the operator's eigenstates

6.3 Projection operators and their properties

Projection operators in Hilbert spaces are key tools for analyzing and manipulating vectors. They allow us to map elements onto specific subspaces, preserving important properties like orthogonality and minimizing distances.

These operators have unique characteristics like idempotence and self-adjointness. They're crucial for tasks such as orthogonal decomposition and least squares approximation, making them indispensable in many areas of mathematics and physics.

Projection Operators in Hilbert Spaces

Definition of projection operators

- Linear operator P on a Hilbert space H satisfies idempotence property $P^2 = P$
- Satisfies self-adjointness property $P^* = P$, where P^* denotes the adjoint operator of P
- Eigenvalues of projection operators are restricted to either 0 or 1 (no other values possible)
- Orthogonal projection operators have range and null space that are orthogonal to each other
- Identity operator can be expressed as the sum of orthogonal projection operators onto orthogonal subspaces

Construction of projection operators

- Projection operator P_M onto a closed subspace M of Hilbert space H maps any element $x \in H$ to the unique element $P_M(x)$ in M that minimizes the distance $\|x - P_M(x)\|$
- $P_M(x)$ is the closest point in M to x (minimizes the norm of the difference)
- Constructing P_M using an orthonormal basis $\{e_i\}_{i=1}^n$ for M :

$$P_M(x) = \sum_{i=1}^n \langle x, e_i \rangle e_i$$

- Inner products $\langle x, e_i \rangle$ give the coordinates of the projection in the basis

- Summing the basis vectors e_i scaled by these coordinates yields the projection $P_M(x)$
- Constructing P_M using the orthogonal complement $M^\perp$: $P_M = I - P_{M^\perp}$, where I is the identity operator
- Subtracting the projection onto the orthogonal complement from the identity gives the projection onto the original subspace

Properties of projection operators

- Orthogonality: For any $x \in H$, the difference $x - P_M(x)$ lies in the orthogonal complement $M^\perp$
- Proof: $\langle x - P_M(x), y \rangle = \langle x, y \rangle - \langle P_M(x), y \rangle = 0$ for any $y \in M$, since $P_M(x) \in M$
- Idempotence: For any projection operator P on H , applying it twice yields the same result as applying it once, i.e., $P^2 = P$
- Proof: $P^2(x) = P(P(x)) = P(x)$ for any $x \in H$, since $P(x)$ lies in the range of P where P acts as the identity

Applications of projection operators

- Orthogonal decomposition: Any $x \in H$ can be uniquely decomposed as $x = P_M(x) + P_{M^\perp}(x)$, where $P_M(x) \in M$ and $P_{M^\perp}(x) \in M^\perp$
- Components $P_M(x)$ and $P_{M^\perp}(x)$ are orthogonal, i.e., $\langle P_M(x), P_{M^\perp}(x) \rangle = 0$
- Useful for separating a vector into its components in a subspace and its orthogonal complement
- Least squares approximation: For a given $x \in H$ and a closed subspace M , the projection $P_M(x)$ is the best approximation of x in M in the least squares sense
- Minimizes the distance $\|x - P_M(x)\|$ over all elements in M
- Finds the closest point in M to the given vector x (useful for data fitting and regression problems)

6.4 Spectral theorem for compact self-adjoint operators

The Spectral Theorem for Compact Self-Adjoint Operators is a powerful tool in functional analysis. It shows that these operators have an orthonormal basis of eigenvectors, with real eigenvalues converging to zero if infinite.

This theorem allows us to represent compact self-adjoint operators in a canonical form, diagonalize them, and solve eigenvalue problems. It's crucial for understanding operator behavior and properties in Hilbert spaces.

Spectral Theorem for Compact Self-Adjoint Operators

Spectral theorem for compact operators

- States that if T is a compact self-adjoint operator on a Hilbert space H , then H has an orthonormal basis consisting of eigenvectors of T
- Eigenvalues of T are real and converge to 0 if there are infinitely many $(\lambda_1, \lambda_2, \dots)$
- T can be represented as

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$$

, where λ_n are the eigenvalues and e_n are the corresponding eigenvectors

- Provides a canonical form for compact self-adjoint operators (

$$T = \sum_{n=1}^{\infty} \lambda_n \langle \cdot, e_n \rangle e_n$$

)

- Allows for the diagonalization of compact self-adjoint operators ($U^*TU = D$, where D is diagonal)
- Enables the solution of eigenvalue problems involving compact self-adjoint operators ($Tx = \lambda x$)
- Establishes a connection between the operator's properties and its eigenvalues and eigenvectors ($\lambda_n \rightarrow 0$, $\{e_n\}$ orthonormal basis)

Orthonormal basis of eigenvectors

- Let T be a compact self-adjoint operator on a Hilbert space H
- Consider the eigenvalue problem $Tx = \lambda x$
- Eigenvalues are real since T is self-adjoint ($\langle Tx, y \rangle = \langle x, Ty \rangle$)
- Eigenvectors corresponding to distinct eigenvalues are orthogonal ($\langle e_i, e_j \rangle = 0$ for $i \neq j$)
- If $\lambda \neq 0$ is an eigenvalue, then the eigenspace $E_\lambda = \{x \in H : Tx = \lambda x\}$ is finite-dimensional because $T - \lambda I$ is compact and has a finite-dimensional null space
- Choose an orthonormal basis for each eigenspace E_λ ($\{e_1, \dots, e_k\}$ for E_λ)
- The union of these bases forms an orthonormal set in H ($\{e_1, e_2, \dots\}$)
- If H is not spanned by the eigenvectors, consider the orthogonal complement of the span of eigenvectors ($H \ominus \text{span}\{e_1, e_2, \dots\}$)
- T restricted to this complement is compact and self-adjoint, so 0 is the only possible eigenvalue
- Conclude that the union of the orthonormal bases for the eigenspaces forms an orthonormal basis for H

Spectral decomposition of operators

- Let T be a compact self-adjoint operator on a Hilbert space H
- Find the eigenvalues λ_n and corresponding orthonormal eigenvectors e_n of T
- Express T as

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$$

, which is the spectral decomposition of T

- The spectral decomposition can be truncated if there are finitely many eigenvalues (

$$Tx = \sum_{n=1}^N \lambda_n \langle x, e_n \rangle e_n$$

)

- Provides a way to represent the operator in terms of its eigenvalues and eigenvectors (

$$T = \sum_{n=1}^{\infty} \lambda_n \langle \cdot, e_n \rangle e_n$$

)

- Useful for understanding the behavior and properties of the operator (

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$$

)

Applications of spectral theorem

- Given a compact self-adjoint operator T , use the spectral theorem to find the eigenvalues and eigenvectors of T and express T in terms of its spectral decomposition

- Diagonalization: 1. Let $\{\lambda_n\}$ be the eigenvalues and $\{e_n\}$ be the corresponding orthonormal eigenvectors of T 2. Define $U : H \rightarrow H$ by $Ue_n = e_n$ (identity on the basis) 3. Then $U^*TU = D$, where D is a diagonal operator with λ_n on the diagonal

- Solving eigenvalue problems: 1. Use the spectral decomposition to express the eigenvalue problem $Tx = \lambda x$ as

$$\sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n = \lambda x$$

2. Solve for the eigenvalues and eigenvectors using the properties of the spectral decomposition ($\lambda = \lambda_n, x = e_n$)

- Approximation of compact self-adjoint operators (

$$T \approx \sum_{n=1}^N \lambda_n \langle \cdot, e_n \rangle e_n$$

)

- Analysis of integral equations with compact self-adjoint integral operators ($Tx(s) = \int_a^b K(s, t)x(t)dt$)

Unit 7 – Compact Operators: Properties and Analysis

7.1 Definition and examples of compact operators

Compact operators are crucial in functional analysis, bridging finite and infinite dimensions. They map bounded sets to relatively compact ones, ensuring convergent subsequences. This property makes them invaluable for solving integral equations and studying spectral theory.

Examples include integral operators and finite-rank operators. Compact operators form a closed subspace of bounded linear operators. They're closely related to completely continuous operators, coinciding in reflexive Banach spaces.

Compact Operators

Definition of compact operators

- A bounded linear operator $T : X \rightarrow Y$ between Banach spaces (X and Y) is compact if it maps bounded sequences (x_n) in X to sequences (Tx_n) in Y that have convergent subsequences
- Compact operators transform bounded sets into relatively compact sets, meaning the closure of the image of the unit ball B_X under T is compact in Y
- For Hilbert spaces (H), a compact operator $T : H \rightarrow H$ is characterized by the property that every bounded sequence (x_n) in H has a subsequence (Tx_n) that converges in H

Examples of compact operators

- Integral operators on function spaces ($L^2([a, b])$) are often compact, such as the operator $(Tf)(x) = \int_a^b K(x, y)f(y)dy$ with a continuous kernel function K on $[a, b] \times [a, b]$
- The Arzelà-Ascoli theorem can be used to prove the compactness of such integral operators
- Finite-rank operators, which have a finite-dimensional range, are always compact
- These operators can be expressed as

$$Tx = \sum_{i=1}^n f_i(x)y_i$$

, where $f_i \in X^*$ and $y_i \in Y$ for $i = 1, \dots, n$

- The compactness of finite-rank operators follows from the compactness of the unit ball in finite-dimensional spaces
- The composition of a bounded linear operator with a compact operator results in a compact operator
- If $T : X \rightarrow Y$ is compact and $S : Y \rightarrow Z$ is bounded, then their composition $ST : X \rightarrow Z$ is compact

Compact operators as closed subspace

- The set of compact operators $\mathcal{K}(X, Y)$ forms a closed subspace of the bounded linear operators $\mathcal{B}(X, Y)$

- To prove this, we need to show that: 1. The sum $T + S$ and scalar multiple αT of compact operators $T, S \in \mathcal{K}(X, Y)$ are also compact for any scalar $\alpha \in \mathbb{C}$ 2. If a sequence of compact operators (T_n) converges to a bounded linear operator $T \in \mathcal{B}(X, Y)$, then T is also compact
- The first property follows from the linearity of limits and the definition of compact operators
- The second property can be proven using the characterization of compact operators in terms of relatively compact sets and the fact that the limit of a sequence of compact sets is compact

Compact vs completely continuous operators

- A bounded linear operator $T : X \rightarrow Y$ between Banach spaces is completely continuous if it maps weakly convergent sequences (x_n) to strongly convergent sequences (Tx_n)
- Every compact operator is completely continuous
- If T is compact and (x_n) converges weakly to x , then (Tx_n) has a convergent subsequence, and the limit of this subsequence must be Tx by the uniqueness of weak limits
- In reflexive Banach spaces, the concepts of compact operators and completely continuous operators coincide
- A bounded linear operator $T : X \rightarrow Y$ is compact if and only if it is completely continuous when X is reflexive

7.2 Properties and characterizations of compact operators

Compact operators are crucial in functional analysis, bridging finite and infinite-dimensional spaces. They generalize finite-rank operators, preserving many of their properties. Understanding compact operators is key to grasping advanced concepts in operator theory.

This section delves into the properties of compact operators, including their behavior under limits and adjoints. We'll explore how compactness interacts with operator products and examine the relationship between compact operators and precompact sets.

Properties of Compact Operators

Compactness of sequence limits

- Consider Banach spaces X and Y and a sequence of compact operators (T_n) from X to Y converging to an operator T in the operator norm
- To prove the compactness of T , take a bounded sequence (x_n) in X
- The compactness of T_1 guarantees a subsequence (x_{n_k}) such that $(T_1 x_{n_k})$ converges in Y
- Repeatedly apply this argument to (x_{n_k}) and subsequent operators $T_2, T_3, \dots$ to extract further subsequences
- This process yields a diagonal subsequence (x_{n_k}) for which $(T_n x_{n_k})$ converges for each n
- Demonstrate that (Tx_{n_k}) forms a Cauchy sequence in Y
- Given $\varepsilon > 0$, select N such that $\|T_n - T\| < \varepsilon/3$ for all $n \geq N$
- For sufficiently large indices k and l , ensure $\|T_N x_{n_k} - T_N x_{n_l}\| < \varepsilon/3$
- Apply the triangle inequality to show $\|Tx_{n_k} - Tx_{n_l}\| < \varepsilon$

- The convergence of (Tx_{n_k}) in Y establishes the compactness of T

Properties of compact operators

- An operator $T : X \rightarrow Y$ is compact if and only if $T(B_X)$ is precompact in Y , where B_X denotes the unit ball in X
- Precompact sets have compact closures
- For a compact operator T and a sequence (x_n) in B_X , the sequence (Tx_n) admits a convergent subsequence in Y
- This property implies the precompactness of $T(B_X)$
- Conversely, if $T(B_X)$ is precompact and (x_n) is a bounded sequence in X , then (x_n) lies within a multiple of B_X
- Consequently, (Tx_n) is contained in a multiple of the precompact set $T(B_X)$
- The existence of a convergent subsequence of (Tx_n) confirms the compactness of T

Compactness of adjoint operators

- Consider a compact operator $T : X \rightarrow Y$ between Banach spaces X and Y
- The adjoint operator $T^* : Y^* \rightarrow X^*$ is defined by $(T^* f)(x) = f(Tx)$ for all $f \in Y^*$ and $x \in X$
- To establish the compactness of T^* , consider a bounded sequence (f_n) in Y^*
- The sequence $(T^* f_n)$ is bounded in X^* due to the inequality $\|T^* f_n\| \leq \|T\| \|f_n\|$
- For each $x \in X$, the sequence $((T^* f_n)(x))$ is bounded in the scalar field $\mathbb{K}$
- The compactness of T ensures that (Tx) is precompact in Y
- The boundedness of (f_n) implies the boundedness of $((T^* f_n)(x))$
- The Banach-Alaoglu theorem guarantees the existence of a weak-* convergent subsequence of $(T^* f_n)$ in X^*
- The compactness of T^* follows from the above arguments

Compactness in operator products

- Let $S : X \rightarrow Y$ be a compact operator and $T : Y \rightarrow Z$ be a bounded linear operator between Banach spaces X , Y , and Z
- To verify the compactness of the product $TS : X \rightarrow Z$, consider a bounded sequence (x_n) in X
- The compactness of S implies the existence of a convergent subsequence (Sx_{n_k}) in Y
- The boundedness of T ensures the convergence of (TSx_{n_k}) in Z
- Specifically, $\|TSx_{n_k} - TSx_{n_l}\| \leq \|T\| \|Sx_{n_k} - Sx_{n_l}\|$
- The convergence of (TSx_{n_k}) establishes the compactness of the product operator TS

7.3 Spectral theory of compact operators

Compact self-adjoint operators on Hilbert spaces have a powerful spectral theorem. This theorem guarantees an orthonormal basis of eigenvectors and real eigenvalues converging to zero, allowing for a spectral representation of the operator.

The spectral theorem enables diagonalization of compact self-adjoint operators, simplifying calculations and analysis. It also provides insights into the operator's spectrum, which consists of eigenvalues and possibly zero, with nonzero eigenvalues having finite multiplicities.

Spectral Theory of Compact Self-Adjoint Operators

Spectral theorem for compact operators

- States that for a compact self-adjoint operator T on a Hilbert space H , there exists an orthonormal basis $\{e_n\}$ of H consisting of eigenvectors of T
- The corresponding eigenvalues $\{\lambda_n\}$ are real numbers that converge to 0 as n approaches infinity
- Provides a spectral representation of T as

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$$

for all $x \in H$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in H

- Proof involves showing eigenvalues are real, eigenvectors corresponding to distinct eigenvalues are orthogonal, eigenspaces are finite-dimensional, constructing orthonormal basis using Gram-Schmidt process, and establishing spectral representation using compactness of T

Diagonalization of self-adjoint operators

- Compact self-adjoint operators can be diagonalized using the spectral theorem
- Given orthonormal eigenvectors $\{e_n\}$ and corresponding eigenvalues $\{\lambda_n\}$, define diagonal operator D by

$$Dx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$$

- T can be expressed as $T = UDU^*$, where U is a unitary operator with columns $\{e_n\}$ and U^* is its adjoint (conjugate transpose)
- Diagonalization simplifies computations involving powers and functions of T (spectral calculus) and enables analysis of long-term behavior in dynamical systems governed by compact self-adjoint operators

Spectrum of compact operators

- The spectrum $\sigma(T)$ of a compact operator T consists of eigenvalues and possibly 0
- Nonzero eigenvalues have finite multiplicities (dimensions of corresponding eigenspaces)
- For self-adjoint T , the spectrum is real and eigenspaces corresponding to distinct eigenvalues are orthogonal
- Nonzero spectrum elements are eigenvalues, and eigenvectors span the range of T

- In the self-adjoint case, eigenvectors form an orthonormal basis of the Hilbert space

Orthonormal basis of eigenvectors

- The spectral theorem guarantees the existence of a countable orthonormal basis $\{e_n\}$ of eigenvectors for a compact self-adjoint operator T on a Hilbert space H
- The basis is countable due to finite-dimensional eigenspaces and eigenvalues converging to 0
- Hilbert space can be decomposed into a direct sum of eigenspaces

$$H = \bigoplus_{n=1}^{\infty} E_n$$

, where E_n is the eigenspace corresponding to eigenvalue λ_n

- Spectral representation

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$$

holds for all $x \in H$, enabling efficient computation of operator powers and functions (functional calculus)

7.4 Fredholm alternative and its applications

The Fredholm Alternative is a key theorem in functional analysis that helps determine when equations involving compact operators have solutions. It states that for an equation $(I - K)x = y$, where K is compact, either a unique solution exists for all y , or the homogeneous equation has non-zero solutions.

This theorem has wide-ranging applications, from integral equations to boundary value problems. It provides a powerful tool for analyzing the existence and uniqueness of solutions in various mathematical contexts, especially when dealing with compact operators in Banach spaces.

Fredholm Alternative

Fredholm alternative for compact operators

- Fundamental theorem in functional analysis characterizes solvability of equations with compact operators
- Let X be a Banach space and $K : X \rightarrow X$ a compact operator
- For equation $(I - K)x = y$, where I is identity operator and $y \in X$, exactly one holds:
- For every $y \in X$, equation $(I - K)x = y$ has unique solution $x \in X$
- Homogeneous equation $(I - K)x = 0$ has non-zero solution $x \in X$
- Proof outline:
- Show kernel (null space) of $(I - K)$ is finite-dimensional
- Prove range of $(I - K)$ is closed
- Use Riesz lemma to show either range of $(I - K)$ is all of X or codimension equals dimension of kernel

Applications to integral equations

- Fredholm integral equations of second kind: $u(x) - \lambda \int_a^b K(x, t)u(t)dt = f(x)$
- Integral operator K often compact, allows application of Fredholm alternative
- If λ not eigenvalue of K , equation has unique solution for every f
- Volterra integral equations: $u(x) - \lambda \int_a^x K(x, t)u(t)dt = f(x)$
- Fredholm alternative does not directly apply, but can transform into Fredholm equation
- Boundary value problems:
- Consider second-order linear differential equation with homogeneous boundary conditions
- Green's function $G(x, t)$ converts problem into Fredholm integral equation
- Integral operator associated with Green's function often compact
- Apply Fredholm alternative to determine existence and uniqueness of solutions

Solutions with compact operators

- Use Fredholm alternative to determine solvability of equations $(I - K)x = y$, where K compact operator
- If y in range of $(I - K)$, solution exists
- If kernel of $(I - K)$ trivial (only zero element), solution unique
- Eigenvalues and eigenfunctions of compact operators:
- If λ eigenvalue of compact operator K , equation $(I - \lambda K)x = 0$ has non-zero solution (eigenfunction)
- Fredholm alternative implies $(I - \lambda K)x = y$ has solution iff y orthogonal to all eigenfunctions corresponding to eigenvalue λ

Solvability in Banach spaces

- Consider linear equation $Ax = y$ in Banach space X , where $A : X \rightarrow X$ bounded linear operator
- If A compact operator, Fredholm alternative directly applies
- If A not compact, can still use Fredholm alternative by decomposing A into sum of compact operator and bounded invertible operator
- Write $A = I - K$, where K compact
- Solvability of $Ax = y$ equivalent to solvability of $(I - K)x = y$
- Riesz-Schauder theory:
- Extends Fredholm alternative to closed operators (not necessarily bounded) in Banach spaces
- Provides conditions for existence and uniqueness of solutions to linear equations with closed operators

Unit 8 – Spectral Theory of Bounded Operators

8.1 Spectrum and resolvent of bounded linear operators

Bounded linear operators on Hilbert spaces have a spectrum and resolvent set. The spectrum includes eigenvalues, continuous spectrum, and residual spectrum. Understanding these concepts is crucial for analyzing operator properties and solving related problems.

The resolvent set contains complex numbers where the operator minus that value times the identity is invertible. The spectrum is the complement of this set. These ideas help determine operator invertibility and behavior in various mathematical contexts.

Spectrum and Resolvent of Bounded Linear Operators

Spectrum and resolvent set definition

- Let H be a Hilbert space and $T : H \rightarrow H$ be a bounded linear operator maps elements of H to itself
- The resolvent set of T , denoted by $\rho(T)$, is the set of all complex numbers λ such that:
 - $T - \lambda I$ is invertible has an inverse that is also a bounded linear operator
 - $(T - \lambda I)^{-1}$ is bounded its norm is finite
 - $(T - \lambda I)^{-1}$ is defined on all of H maps every element of H to an element of H
- The spectrum of T , denoted by $\sigma(T)$, is the complement of the resolvent set in the complex plane $\mathbb{C}$, i.e., $\sigma(T) = \mathbb{C} \setminus \rho(T)$ contains all complex numbers not in the resolvent set
- The spectrum can be further classified into:
 - Point spectrum: $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective there exist nonzero $x \in H$ such that $(T - \lambda I)x = 0$ (eigenvalues)
 - Continuous spectrum: $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is injective, has dense range its image is dense in H , but is not surjective does not map onto all of H
 - Residual spectrum: $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is injective, but its range is not dense in H there exist $y \in H$ not in the closure of the image of $T - \lambda I$

Computation of spectrum examples

- Identity operator: $T = I$ maps each element to itself
- $\sigma(I) = \{1\}$ (point spectrum) only eigenvalue is 1
- $\rho(I) = \mathbb{C} \setminus \{1\}$ all complex numbers except 1
- Multiplication operator: $(Mf)(x) = x \cdot f(x)$ on $L^2([0, 1])$ multiplies each function by the independent variable x
- $\sigma(M) = [0, 1]$ (continuous spectrum) no eigenvalues, but range is dense
- $\rho(M) = \mathbb{C} \setminus [0, 1]$ all complex numbers outside $[0, 1]$

- Left shift operator: $(Lx)_n = x_{n+1}$ on $\ell^2(\mathbb{N})$ shifts sequence elements to the left
- $\sigma(L) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ unit disk in complex plane
- $\{0\}$ is the point spectrum only eigenvalue is 0
- $\{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$ is the residual spectrum open unit disk excluding 0
- $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is the continuous spectrum unit circle
- $\rho(L) = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$ outside the closed unit disk

Properties of spectrum and resolvent

- The resolvent set $\rho(T)$ is an open subset of $\mathbb{C}$ any open disk centered at a point in $\rho(T)$ is contained in $\rho(T)$
- For $\lambda_0 \in \rho(T)$, the resolvent operator $R(\lambda_0, T) = (T - \lambda_0 I)^{-1}$ is bounded by the inverse mapping theorem
- For λ close to λ_0 , the resolvent operator can be expressed as a convergent series:

$$R(\lambda, T) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, T)^{n+1}$$

Neumann series expansion

- This series converges for $|\lambda - \lambda_0| < \|R(\lambda_0, T)\|^{-1}$, proving that $\rho(T)$ is open contains an open disk around each of its points
- The spectrum $\sigma(T)$ is a nonempty, compact subset of $\mathbb{C}$ closed and bounded
- $\sigma(T)$ is nonempty: if $\sigma(T) = \emptyset$, then $\rho(T) = \mathbb{C}$, implying that $T - \lambda I$ is invertible for all $\lambda \in \mathbb{C}$, which is impossible for bounded operators on infinite-dimensional spaces by the Fredholm alternative
- $\sigma(T)$ is bounded: $\|T - \lambda I\| \geq |\lambda| - \|T\|$, so for $|\lambda| > \|T\|$, $T - \lambda I$ is invertible by the Neumann series, and thus $\lambda \in \rho(T)$
- $\sigma(T)$ is closed: as the complement of the open set $\rho(T)$ in $\mathbb{C}$

Spectrum vs operator invertibility

- T is invertible if and only if $0 \in \rho(T)$ zero is in the resolvent set
- If T is invertible, then $T - 0 \cdot I = T$ is invertible, so $0 \in \rho(T)$ by definition of the resolvent set
- If $0 \in \rho(T)$, then $T = T - 0 \cdot I$ is invertible since 0 is in the resolvent set
- T is not invertible if and only if $0 \in \sigma(T)$ zero is in the spectrum
- If T is not invertible, then $T - 0 \cdot I = T$ is not invertible, so $0 \in \sigma(T)$ by definition of the spectrum
- If $0 \in \sigma(T)$, then $T = T - 0 \cdot I$ is not invertible since 0 is in the spectrum
- The spectral radius of T is defined as $r(T) = \sup \{|\lambda| : \lambda \in \sigma(T)\}$ the supremum of the moduli of elements in the spectrum
- The spectral radius formula states that $r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$ can be computed as a limit involving the operator norm of powers of T

- If $r(T) < 1$, then

$$\sum_{n=0}^{\infty} T^n$$

converges, and $I - T$ is invertible by the geometric series formula

8.2 Spectral mapping theorem and functional calculus

The spectral mapping theorem connects an operator's spectrum to functions applied to it. It's a powerful tool for understanding how transformations affect an operator's properties, allowing us to analyze complex operators through simpler functions.

Functional calculus extends this idea, letting us apply continuous or even measurable functions to operators. This broadens our toolkit for studying operators, enabling us to construct and analyze more sophisticated mathematical objects in functional analysis.

Spectral Mapping Theorem

Spectral mapping theorem

Let T be a bounded linear operator on a Banach space X and f be a complex-valued function analytic on an open set containing $\sigma(T)$, the spectrum of T . The spectral mapping theorem states that $\sigma(f(T)) = f(\sigma(T))$, meaning the spectrum of $f(T)$ is the image of the spectrum of T under f . The proof involves showing that if $\lambda \in \sigma(T)$ and $z = f(\lambda)$, then $f(T) - zI$ is not invertible, implying $z \in \sigma(f(T))$. Conversely, if $z \in \sigma(f(T))$, there exists $\lambda \in \sigma(T)$ such that $f(\lambda) = z$, proving the equality $\sigma(f(T)) = f(\sigma(T))$.

Applications of spectral mapping

To find the spectrum of $f(T)$, where T is a bounded linear operator and f is an analytic function: 1. Compute the spectrum of T , denoted by $\sigma(T)$ 2. Apply the function f to each element of $\sigma(T)$ 3. The resulting set is the spectrum of $f(T)$, denoted by $\sigma(f(T))$

For example, let T be a bounded linear operator with $\sigma(T) = \{1, 2, 3\}$ and $f(z) = z^2$. Applying f to each element of $\sigma(T)$ yields $f(1) = 1$, $f(2) = 4$, and $f(3) = 9$. Therefore, $\sigma(f(T)) = \sigma(T^2) = \{1, 4, 9\}$

Functional Calculus

Continuous functional calculus

The continuous functional calculus extends the spectral mapping theorem to continuous functions. Let T be a bounded linear operator on a Banach space X with spectrum $\sigma(T)$ and f be a continuous function on $\sigma(T)$. The continuous functional calculus defines an operator $f(T)$ as follows: - If $\sigma(T)$ is real, then $f(T) = \int_{\sigma(T)} f(\lambda) dE(\lambda)$, where E is the spectral measure associated with T - If $\sigma(T)$ is complex, then $f(T) = \int_{\sigma(T)} f(z) dE(z)$, where the integral is taken over the complex plane

Properties of the continuous functional calculus include: - If f and g are continuous functions on $\sigma(T)$, then $(f + g)(T) = f(T) + g(T)$ and $(fg)(T) = f(T)g(T)$ - If f_n is a sequence of continuous functions converging uniformly to f on $\sigma(T)$, then $f_n(T)$ converges to $f(T)$ in the operator norm

Borel functional calculus

The Borel functional calculus extends the functional calculus to include unbounded Borel measurable functions. Let T be a bounded linear operator on a Banach space X with spectrum $\sigma(T)$ and f be a Borel measurable function on $\sigma(T)$. The Borel functional calculus defines an operator $f(T)$ as $f(T) = \int_{\sigma(T)} f(\lambda) dE(\lambda)$, where E is the spectral measure associated with T . The integral is interpreted as a Bochner integral, which extends the Lebesgue integral to Banach space-valued functions.

Properties of the Borel functional calculus include: - If f and g are Borel measurable functions on $\sigma(T)$, then $(f + g)(T) \subseteq f(T) + g(T)$ and $(fg)(T) \subseteq f(T)g(T)$ - If f_n is a sequence of Borel measurable functions converging pointwise to f on $\sigma(T)$, then $f_n(T)$ converges strongly to $f(T)$

The Borel functional calculus allows for the construction of operators such as the square root ($\sqrt{T}$), logarithm ($\log T$), and exponential (e^T) of a bounded linear operator

8.3 Spectral theorem for normal operators

Normal operators are a crucial class of linear operators in Hilbert spaces. They include self-adjoint and unitary operators, which are fundamental in quantum mechanics. Understanding normal operators is key to grasping the structure of linear transformations in complex vector spaces.

The spectral theorem for normal operators is a powerful tool in functional analysis. It allows us to decompose these operators into simpler parts, making it easier to study their properties and applications in various fields of mathematics and physics.

Normal Operators and the Spectral Theorem

Normal operators and examples

- Normal operators defined as linear operators on a Hilbert space that commute with their adjoint ($T^*T = TT^*$)
- Self-adjoint operators are normal operators equal to their own adjoint ($T = T^*$)
- Position operator Q and momentum operator P in quantum mechanics are self-adjoint
- Unitary operators are normal operators whose adjoint is their inverse ($U^*U = UU^* = I$)
- Preserve inner products and norms of vectors in the Hilbert space
- Time-evolution operator $U(t) = e^{-iHt/\hbar}$ in quantum mechanics is unitary (H is the Hamiltonian)

Spectral theorem for compact normals

- Compact normal operators on a Hilbert space H possess an orthonormal basis $\{e_n\}$ consisting of their eigenvectors
- Corresponding eigenvalues $\{\lambda_n\}$ of the compact normal operator converge to zero
- Proof outline: 1. Eigenvectors corresponding to distinct eigenvalues are orthogonal 2. Eigenspaces of the compact normal operator span the entire Hilbert space H 3. Eigenvalues converge to zero due to the compactness of the operator

Extension to bounded normals

- Bounded normal operators on a Hilbert space H have a unique resolution of the identity E

- Spectral decomposition: $T = \int_{\sigma(T)} \lambda dE(\lambda)$, where $\sigma(T)$ is the spectrum of T
- Functional calculus extends the spectral theorem to bounded normal operators
- For a bounded Borel function f on $\sigma(T)$, the operator $f(T) = \int_{\sigma(T)} f(\lambda) dE(\lambda)$
- Functional calculus properties: $(fg)(T) = f(T)g(T)$, $(\bar{f})(T) = f(T)^*$, $\|f(T)\| \leq \sup_{\lambda \in \sigma(T)} |f(\lambda)|$

Applications of spectral theorem

- Diagonalization of normal operators
- Normal operators are unitarily equivalent to multiplication operators
- Unitary operator U exists such that $T = U^* M U$, where M is the multiplication operator $M e_n = \lambda_n e_n$
- Computing spectra of normal operators
- Spectrum of a normal operator consists of its eigenvalues
- Compact normal operators have a countable spectrum with the only accumulation point at zero
- Bounded normal operators have a compact spectrum in the complex plane
- Quantum mechanics applications
- Observables are self-adjoint operators with real spectra and complete orthonormal eigenvectors
- Measurements are projections onto eigenvectors of the observable's self-adjoint operator
- Diagonalization of the unilateral shift operator S on $\ell^2(\mathbb{N})$
- $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ is a bounded normal operator
- Spectrum of S is $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$
- Spectral theorem diagonalizes S as a multiplication operator on a suitable Hilbert space

8.4 Applications to differential and integral operators

Differential operators like the Laplace operator play a crucial role in functional analysis. Their spectrum, including eigenvalues and continuous spectrum, helps us understand their behavior. The resolvent operator provides insights into the operator's properties and its inverse.

Self-adjoint operators have special properties, including an orthonormal basis of eigenfunctions. This allows us to solve boundary value problems using eigenfunction expansions, making them incredibly useful in physics and engineering applications.

Differential Operators

Spectrum of differential operators

- Laplace operator defined as

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

acts on functions in various spaces $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n), C_c^\infty(\mathbb{R}^n))$

- Spectrum of the Laplace operator consists of point spectrum (eigenvalues), continuous spectrum, and residual spectrum
- Resolvent of the Laplace operator $R_\lambda(T) = (T - \lambda I)^{-1}$ defined for λ not in the spectrum of T
- Resolvent set complements the spectrum
- Resolvent operator acts as a bounded linear operator for λ in the resolvent set

Spectral theorem for self-adjoint operators

- Self-adjoint operators satisfy $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all x, y in the domain of T (Laplace operator with appropriate boundary conditions)
- Spectral theorem for self-adjoint operators guarantees the existence of an orthonormal basis of eigenfunctions and decomposes the Hilbert space into a direct sum of eigenspaces
- Eigenvalues and eigenfunctions determined by the eigenvalue equation $Tx = \lambda x$, where eigenfunctions are non-zero solutions and eigenvalues are corresponding scalar values λ
- Applications to boundary value problems involve solving PDEs using eigenfunction expansions (heat equation with Dirichlet boundary conditions)

Integral Operators

Spectrum of integral operators

- Integral operators defined as $(Tf)(x) = \int_\Omega K(x, y)f(y)dy$, where K is the kernel function (Fredholm integral operator with a continuous kernel on a compact domain)
- Spectrum of integral operators includes eigenvalues, which are solutions to the homogeneous integral equation, and the essential spectrum, which consists of accumulation points of the spectrum
- Resolvent of integral operators involves the resolvent kernel, which is the solution to the resolvent equation, and the Neumann series expansion of the resolvent operator

Applications of spectral theory

- Bounded linear operators satisfy $\|Tx\| \leq M\|x\|$ for all x in the domain of T and some constant M (integral operators with bounded kernels)
- Spectral theory for bounded operators includes the Riesz-Schauder theorem, which characterizes the spectrum, and the spectral radius $r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$
- Solving integral equations involves the Fredholm alternative, which determines the existence and uniqueness of solutions, and numerical methods such as the Galerkin method and collocation method
- Applications to boundary value problems utilize Green's functions and integral representations of solutions (Poisson equation with Neumann boundary conditions)

Unit 9 – Weak and Weak* Topologies

9.1 Weak topology on normed spaces

The weak topology in normed spaces is a crucial concept in functional analysis. It's a less restrictive topology than the norm topology, allowing for more convergent sequences and compact sets. This makes it a powerful tool for studying linear functionals and operators.

Understanding the weak topology helps in analyzing convergence of sequences and series in infinite-dimensional spaces. It's particularly useful in applications to differential equations, optimization, and quantum mechanics, where weak convergence often occurs naturally.

Weak Topology on Normed Spaces

Weak topology in normed spaces

- Defines the coarsest topology on a normed space X that ensures continuity of all continuous linear functionals on X
- A subset $U \subset X$ is weakly open if for every $x \in U$, there exist continuous linear functionals $f_1, \dots, f_n \in X^*$ and $\varepsilon > 0$ such that the set $\{y \in X : |f_i(y) - f_i(x)| < \varepsilon, i = 1, \dots, n\}$ is contained in U
- The basis for the weak topology on X consists of sets of the form $\{x \in X : |f_i(x) - a_i| < \varepsilon, i = 1, \dots, n\}$, where $f_1, \dots, f_n \in X^*$, $a_1, \dots, a_n \in \mathbb{R}$, and $\varepsilon > 0$
- These basis sets are called weak neighborhoods (ℓ^p spaces, L^p spaces)

Comparison of weak vs norm topologies

- To prove the weak topology is coarser than the norm topology, show every weakly open set is open in the norm topology
- Let U be a weakly open set in X and $x \in U$
- There exist continuous linear functionals $f_1, \dots, f_n \in X^*$ and $\varepsilon > 0$ such that $\{y \in X : |f_i(y) - f_i(x)| < \varepsilon, i = 1, \dots, n\} \subset U$
- By continuity of linear functionals in the norm topology, there exists $\delta > 0$ such that $\|y - x\| < \delta$ implies $|f_i(y) - f_i(x)| < \varepsilon$ for all $i = 1, \dots, n$
- This means the open ball $B(x, \delta) = \{y \in X : \|y - x\| < \delta\}$ is contained in U ($C[0, 1], \ell^1$)
- Thus, every weakly open set is open in the norm topology, proving the weak topology is coarser than the norm topology

Hausdorff property of weak topology

- To show the weak topology is Hausdorff, prove for any two distinct points $x, y \in X$, there exist disjoint weakly open sets U and V such that $x \in U$ and $y \in V$
- By the Hahn-Banach theorem, there exists a continuous linear functional $f \in X^*$ such that $f(x) \neq f(y)$
- Let $a = f(x)$ and $b = f(y)$, and choose $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \cap (b - \varepsilon, b + \varepsilon) = \emptyset$
- Define weakly open sets $U = \{z \in X : |f(z) - a| < \varepsilon\}$ and $V = \{z \in X : |f(z) - b| < \varepsilon\}$

- $x \in U, y \in V$, and $U \cap V = \emptyset$ (ℓ^2, L^2)
- Therefore, the weak topology is Hausdorff

Characteristics of weakly closed sets

- A subset $C \subset X$ is weakly closed if and only if for every sequence (x_n) in C that converges weakly to $x \in X$, we have $x \in C$
- Equivalently, C is weakly closed if and only if it is closed in the weak topology (C_0, ℓ^∞)
- A subset $K \subset X$ is weakly compact if and only if every sequence in K has a subsequence that converges weakly to a point in K
- By the Eberlein-Šmulian theorem, K is weakly compact if and only if it is weakly sequentially compact
- In a reflexive Banach space, a subset is weakly compact if and only if it is bounded and weakly closed
- This follows from the Banach-Alaoglu theorem and the fact that the weak topology on a reflexive space coincides with the weak* topology on its dual (L^p spaces with $1 < p < \infty$)

9.2 Weak* topology on dual spaces

The weak topology on dual spaces is a crucial concept in functional analysis. It's defined on the dual space X using seminorms and allows for a more relaxed notion of convergence compared to the norm topology.

Weak topology is coarser than the weak topology on X , making it easier to work with in certain contexts. The Banach-Alaoglu theorem, which states that the closed unit ball in X is weak compact, is a key result in this area.

The Weak* Topology on Dual Spaces

Weak* topology in dual spaces

- Defined on the dual space X^* of a normed space X using the family of seminorms $\{p_x : x \in X\}$
- Each seminorm $p_x(f)$ evaluates the absolute value of the functional f at the point x , i.e., $p_x(f) = |f(x)|$
- Subbase for the weak* topology consists of sets $\{f \in X^* : |f(x) - f_0(x)| < \varepsilon\}$
- Determined by a point $x \in X$, a functional $f_0 \in X^*$, and a positive real number $\varepsilon > 0$
- Convergence in the weak* topology is equivalent to pointwise convergence on X
- A net (f_α) in X^* converges to $f \in X^*$ in the weak* topology if and only if $f_\alpha(x) \rightarrow f(x)$ for each $x \in X$

Weak* vs weak topology comparison

- Weak topology on X^* defined by seminorms $\{p_x : x \in X\}$ with $p_x(f) = \|f\| \|x\|$
- Every subbase element of the weak* topology is also a subbase element of the weak topology
- For $x \in X, f_0 \in X^*, \varepsilon > 0$, the set $\{f \in X^* : |f(x) - f_0(x)| < \varepsilon\}$ is open in the weak topology
- Follows from the inequality $|f(x) - f_0(x)| \leq \|f - f_0\| \|x\|$
- The weak* topology is coarser than the weak topology on X^*

- Every weak* open set is also weakly open, but not conversely

Weak* Closed and Compact Sets in Dual Spaces

Characteristics of weak* sets

- A subset $A \subset X^*$ is weak* closed if and only if it is closed under pointwise limits
- Equivalent to: for every net (f_α) in A that converges to $f \in X^*$ in the weak* topology, $f \in A$
- Banach-Alaoglu theorem: the closed unit ball $B_{X^*} = \{f \in X^* : \|f\| \leq 1\}$ is compact in the weak* topology
- A subset $A \subset X^*$ is weak compact if and only if it is weak closed and bounded in the norm topology
- Weak* compact sets in X^* have the following properties:
- Every net in A has a subnet that converges in the weak* topology to an element of A
- A is compact in the weak* topology if and only if every continuous linear functional on (X^*, weak^*) attains its maximum on A

Relationship of weak* and weak topologies

- If X is reflexive (canonical embedding $J : X \rightarrow X^{**}$ is surjective), the weak* and weak topologies on X^* coincide
- In this case, weak* closed (resp. compact) sets are the same as weakly closed (resp. compact) sets
- If X is not reflexive, the weak* topology is strictly coarser than the weak topology on X^*
- There exist subsets of X^* that are:
- Weak* closed but not weakly closed
- Weak* compact but not weakly compact

9.3 Weak and weak* convergence

Weak and weak* convergence are crucial concepts in functional analysis, offering alternative notions of convergence in normed spaces. They allow us to study sequences that don't converge in the usual norm topology but still exhibit useful convergence properties.

These convergence types are weaker than norm convergence but stronger than pointwise convergence. They're especially useful in infinite-dimensional spaces, where norm convergence can be too restrictive for many important sequences or series.

Weak and Weak* Convergence

Weak vs weak* convergence

- Weak convergence applies to sequences in a normed space X converging to an element in X
- Sequence (x_n) converges weakly to x if for every bounded linear functional f in the dual space X^* , the sequence of scalars $f(x_n)$ converges to $f(x)$
- Notation: $x_n \rightharpoonup x$

- Weak* convergence applies to sequences in the dual space X^* converging to an element in X^*
- Sequence (f_n) converges weak* to f if for every x in the original space X , the sequence of scalars $f_n(x)$ converges to $f(x)$
- Notation: $f_n \xrightarrow{*} f$
- Only applicable in the dual space, not in the original normed space

Boundedness from weak convergence

- Weakly convergent sequences in a normed space are always bounded
- Consequence of the Uniform Boundedness Principle
- If (x_n) converges weakly to x , then there exists $M > 0$ such that $\|x_n\| \leq M$ for all n
- If (x_n) were unbounded, there would exist a functional f such that $|f(x_n)|$ is unbounded, contradicting weak convergence
- Boundedness is a necessary condition for weak convergence, but not sufficient
- Bounded sequences need not converge weakly (ℓ^∞ with coordinate functionals)

Weak* and pointwise convergence

- Weak* convergence of a sequence of functionals implies pointwise convergence
- If (f_n) converges weak* to f in X^* , then for every fixed x in X , the sequence of scalars $f_n(x)$ converges to $f(x)$
- Pointwise convergence does not imply weak* convergence
- Pointwise convergence only considers convergence at each fixed point, while weak* convergence requires uniform convergence on bounded sets
- Weak* convergence is stronger than pointwise convergence
- Weak* convergence guarantees pointwise convergence, but not vice versa (ℓ^1 with coordinate functionals)

Examples of weak convergence types

- Weak convergence without norm convergence:
- Sequence (e_n) in ℓ^2 , where $e_n = (0, \dots, 0, 1, 0, \dots)$ with 1 in the n -th position
- Converges weakly to 0 since for any $f \in (\ell^2)^*$, $\lim_{n \rightarrow \infty} f(e_n) = 0$
- Does not converge in norm to 0 since $\|e_n\| = 1$ for all n
- Weak* convergence without norm convergence:
- Sequence (f_n) in ℓ^1 , where

$$f_n(x) = \sum_{k=1}^n x_k$$

for $x = (x_k) \in \ell^\infty$

- Converges weak* to

$$f(x) = \sum_{k=1}^{\infty} x_k$$

since for any $x \in \ell^{\infty}$, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$

- Does not converge in norm since $\|f_n\| = n$ for all n

9.4 Banach-Alaoglu Theorem and its applications

The Banach-Alaoglu Theorem is a game-changer in functional analysis. It shows that the closed unit ball in a dual space is compact in the weak* topology, which is super useful for solving optimization problems.

This theorem helps us find minimizers in infinite-dimensional spaces and proves the existence of weak* convergent subsequences. It's a powerful tool with applications in operator theory, measure theory, and approximation theory.

The Banach-Alaoglu Theorem

State and prove the Banach-Alaoglu Theorem for the weak* compactness of the unit ball in the dual space

- Banach-Alaoglu Theorem asserts closed unit ball of dual space of normed vector space is compact in weak* topology
- X denotes normed vector space and X^* its dual space
- Closed unit ball of X^* defined as $B_{X^*} = \{f \in X^* : \|f\| \leq 1\}$
- Proving theorem involves following steps: 1. Equip B_{X^*} with weak topology 2. Demonstrate B_{X^*} is closed subset of product space $\prod_{x \in X} \overline{B(0, \|x\|)}$, where $\overline{B(0, \|x\|)}$ represents closed ball in $\mathbb{C}$ or $\mathbb{R}$ centered at 0 with radius $\|x\|$ 3. Tychonoff's theorem implies product space is compact 4. B_{X^*} being closed subset of compact space, it is also compact in weak topology

Applications in optimization problems

- Banach-Alaoglu Theorem proves existence of minimizers for optimization problems in infinite-dimensional spaces
- Consider continuous functional $F : X^* \rightarrow \mathbb{R}$, with X^* being dual space of normed vector space X
- Restrict F to closed unit ball B_{X^*}
- Banach-Alaoglu Theorem ensures B_{X^*} is weak* compact
- Continuous F and compact B_{X^*} imply F attains minimum on B_{X^*} by extreme value theorem
- Approach applicable to various optimization problems:
 - Minimal norm problems
 - Variational problems
 - Optimal control problems

Applications of the Banach-Alaoglu Theorem

Weak* convergent subsequences

- Banach-Alaoglu Theorem implies existence of weak* convergent subsequences for bounded sequences in dual space
- (f_n) represents bounded sequence in dual space X^* of normed vector space X
- Definition states there exists $M > 0$ such that $\|f_n\| \leq M$ for all n
- Consider closed ball $B_{X^*}(0, M) = \{f \in X^* : \|f\| \leq M\}$
- Banach-Alaoglu Theorem ensures $B_{X^*}(0, M)$ is weak* compact
- (f_n) contained in $B_{X^*}(0, M)$ implies existence of subsequence (f_{n_k}) converging to some $f \in B_{X^*}(0, M)$ in weak* topology
- For every $x \in X$, $\lim_{k \rightarrow \infty} f_{n_k}(x) = f(x)$

Significance in functional analysis

- Banach-Alaoglu Theorem is fundamental result in functional analysis with numerous applications:
- Operator theory
- Proves existence of adjoints for bounded linear operators
- Establishes weak* compactness of unit ball of dual space of Banach space
- Measure theory and integration
- Develops Riesz representation theorem for bounded linear functionals on space of continuous functions
- Constructs Lebesgue integral and studies L^p spaces
- Topological vector spaces
- Key tool in studying locally convex topological vector spaces
- Establishes Alaoglu-Bourbaki theorem, generalizing Banach-Alaoglu Theorem for locally convex spaces
- Approximation theory
- Proves existence of best approximations in certain function spaces
- Studies Chebyshev sets and density of polynomials in various function spaces

Unit 10 – Duality Theory and Reflexive Spaces

10.1 Bidual spaces and natural embeddings

Bidual spaces take us on a journey through layers of linear functionals. We start with a normed space, create its dual of continuous linear functionals, then form the bidual of functionals on functionals. This process reveals deep connections between spaces.

Reflexivity is a key concept in this exploration. A space is reflexive if it's isomorphic to its bidual, meaning they're essentially the same. Non-reflexive spaces, however, have biduals that are strictly larger, showcasing the complexity of functional analysis.

Bidual Spaces

Construction of bidual spaces

- Start with a normed linear space X (Banach spaces, Hilbert spaces)
- Form the dual space X^* consisting of all continuous linear functionals on X
- Continuous linear functionals map elements of X to scalars while preserving linearity and continuity
- Construct the bidual space X^{**} as the dual space of X^*
- Elements of X^{**} are continuous linear functionals on X^* , assigning scalars to functionals
- X^{**} is called the "dual of the dual" or "functionals on functionals"
- Functionals in X^{**} take functionals from X^* as input and output scalar values

Natural embedding and bidual relations

- Define the natural embedding $J : X \rightarrow X^{**}$ as $J(x)(f) = f(x)$ for $x \in X$ and $f \in X^*$
- J maps elements of X to elements of X^{**}
- J is a linear map preserving the norm: $\|J(x)\| = \|x\|$ for all $x \in X$
- J is always injective (one-to-one), embedding X into X^{**}
- Allows viewing X as a subspace of its bidual X^{**}
- Injectivity of J means distinct elements of X map to distinct elements of X^{**}
- Ensures no information is lost when mapping from X to X^{**}

Reflexive Spaces

Isomorphisms in reflexive spaces

- A space X is reflexive if the natural embedding $J : X \rightarrow X^{**}$ is surjective (onto)
- Every element of X^{**} is the image of some element in X under J

- In reflexive spaces, J is an isometric isomorphism between X and X^{**}
- J is bijective (one-to-one and onto) and preserves the norm
- Proof of isometric isomorphism in reflexive spaces: 1. J is surjective by the definition of reflexivity 2. J is injective and norm-preserving for any space 3. Combining surjectivity, injectivity, and norm-preservation, J is an isometric isomorphism
- Reflexive spaces can be identified with their biduals via the isometric isomorphism J
- X and X^{**} are essentially the same space in the reflexive case

Non-reflexive spaces vs biduals

- Examples of non-reflexive spaces: 1. C_0 : space of sequences converging to zero
- Its bidual is ℓ^∞ , the space of bounded sequences 2. $L^1(\mathbb{R})$: space of absolutely integrable functions on the real line
- Its bidual is $L^\infty(\mathbb{R})$, the space of essentially bounded functions
- In non-reflexive spaces, the bidual is strictly larger than the original space
- Natural embedding J is injective but not surjective
- Non-reflexive spaces have elements in the bidual that do not correspond to any element in the original space
- Demonstrates that not all spaces are isomorphic to their biduals
- Non-reflexivity highlights the distinction between a space and its bidual
- Bidual may contain additional elements not present in the original space

10.2 Reflexive spaces and their properties

Reflexive spaces are a crucial concept in functional analysis, linking a space to its dual and bidual. They include finite-dimensional normed spaces, Hilbert spaces, and certain sequence and Lebesgue spaces, but not all Banach spaces are reflexive.

Understanding reflexivity helps analyze properties of direct sums and products of spaces. It's a stronger condition than completeness, offering deeper insights into the structure of normed spaces and their relationships to their dual spaces.

Reflexive Spaces

Examples of reflexive spaces

- Finite-dimensional normed spaces are always reflexive due to the natural isomorphism between a finite-dimensional space and its double dual
- Hilbert spaces, such as the space of square-integrable functions $L^2(\Omega)$ over a measure space Ω , are reflexive because the Riesz representation theorem provides a natural isometric isomorphism between the space and its dual
- The sequence spaces ℓ^p for $1 < p < \infty$ are reflexive, as the dual of ℓ^p is isometrically isomorphic to ℓ^q , where $\frac{1}{p} + \frac{1}{q} = 1$

- The Lebesgue spaces $L^p(\Omega)$ for $1 < p < \infty$ and a measure space Ω are reflexive, with the dual of $L^p(\Omega)$ being isometrically isomorphic to $L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$
- The sequence spaces ℓ^1 and ℓ^∞ are not reflexive, as their duals are ℓ^∞ and $(\ell^1)^{**}$, respectively, which are not isometrically isomorphic to the original spaces
- The Lebesgue spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ for a measure space Ω are not reflexive, as their duals are $L^\infty(\Omega)$ and the space of finitely additive measures on Ω , respectively, which are not isometrically isomorphic to the original spaces

Reflexivity of closed subspaces

- The dual space Y^* can be identified with the quotient space $X^*/Y^\perp$, where $Y^\perp = \{f \in X^* : f(y) = 0 \text{ for all } y \in Y\}$ is the annihilator of Y
- The bidual Y^{**} can be identified with a subspace of X^{**} via the composition of natural embeddings $Y \hookrightarrow X \hookrightarrow X^{**}$, where the first embedding is the inclusion map and the second is the natural embedding J_X
- The restriction of the surjective natural embedding $J_X : X \rightarrow X^{**}$ to Y , denoted by $J_Y : Y \rightarrow Y^{**}$, is also surjective because Y^{**} is a subspace of X^{**} and J_X maps Y onto this subspace
- The surjectivity of J_Y implies that Y is reflexive, as the natural embedding from Y to its bidual is surjective

Properties of Reflexive Spaces

Reflexivity in sums and products

- Direct sum: 1. The dual space $(X \oplus Y)^*$ is isometrically isomorphic to $X^* \oplus Y^*$ by the definition of the dual of a direct sum 2. The bidual space $(X \oplus Y)^{**}$ is isometrically isomorphic to $X^{**} \oplus Y^{**}$ by the definition of the bidual of a direct sum 3. The surjectivity of the natural embeddings $J_X : X \rightarrow X^{**}$ and $J_Y : Y \rightarrow Y^{**}$ implies the surjectivity of their direct sum $J_{X \oplus Y} : X \oplus Y \rightarrow (X \oplus Y)^{**}$ 4. The surjectivity of $J_{X \oplus Y}$ proves that $X \oplus Y$ is reflexive
- Product: 1. The dual space $(\prod_{i \in I} X_i)^*$ is isometrically isomorphic to the direct sum $\bigoplus_{i \in I} X_i^*$ by the definition of the dual of a product space 2. The bidual space $(\prod_{i \in I} X_i)^{**}$ is isometrically isomorphic to the product $\prod_{i \in I} X_i^{**}$ by the definition of the bidual of a product space 3. The surjectivity of each natural embedding $J_{X_i} : X_i \rightarrow X_i^{**}$ implies the surjectivity of their product $J_{\prod X_i} : \prod_{i \in I} X_i \rightarrow (\prod_{i \in I} X_i)^{**}$ 4. The surjectivity of $J_{\prod X_i}$ proves that $\prod_{i \in I} X_i$ is reflexive

Reflexivity vs completeness

- The Banach-Steinhaus theorem states that the dual of a normed space is always a Banach space (complete normed space)
- A reflexive space X is isometrically isomorphic to its bidual X^{**} , which is a Banach space, implying that X itself is a Banach space
- The sequence spaces ℓ^1, ℓ^∞ and the Lebesgue spaces $L^1(\Omega), L^\infty(\Omega)$ for a measure space Ω are examples of Banach spaces that are not reflexive

- Reflexivity is a stronger property than completeness for normed spaces, as it requires the natural embedding from the space to its bidual to be surjective, while completeness only requires Cauchy sequences to converge within the space

10.3 Duality mappings and their applications

Duality mappings connect Banach spaces to their dual spaces, generalizing gradients from Hilbert spaces. They link each point to supporting functionals of the unit ball, offering insights into geometric properties and smoothness of Banach spaces.

These mappings play a crucial role in fixed point theory and nonlinear analysis. They help characterize uniform convexity and smoothness, and are key in studying nonexpansive mappings and accretive operators in Banach spaces.

Duality Mappings in Banach Spaces

Definition of duality mapping

- Maps a Banach space X to its dual space X^* associates each point $x \in X$ with the set of supporting functionals of the unit ball at $\frac{x}{\|x\|}$
- Supporting functionals linear functionals that attain their maximum value on the unit ball at a given point (e.g., the hyperplanes tangent to the unit ball)
- Formally defined as $J(x) = \{x^* \in X^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\}$ where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*
- Generalizes the concept of the gradient of the norm in Hilbert spaces to Banach spaces
- In Hilbert spaces, the duality mapping coincides with the Riesz representation (e.g., $J(x) = x$ for all $x \in X$)

Existence in smooth Banach spaces

- Banach space X is smooth if its norm is Gâteaux differentiable at every non-zero point
- Gâteaux differentiability weaker form of differentiability than Fréchet differentiability
- In smooth Banach spaces, the duality mapping J is single-valued and norm-to-weak* continuous
- Single-valued for each $x \in X$, $J(x)$ contains exactly one element (e.g., $J(x) = \{x^*\}$ for some unique $x^* \in X^*$)
- Norm-to-weak continuous if $x_n \rightarrow x$ in norm, then $J(x_n) \rightharpoonup^* J(x)$ (weak convergence in X^*)
- Existence and uniqueness follow from the Hahn-Banach theorem and the Gâteaux differentiability of the norm
- Hahn-Banach theorem guarantees the existence of supporting functionals
- Gâteaux differentiability ensures the uniqueness of the supporting functional at each point

Applications of Duality Mappings

Applications to Banach space geometry

- Characterizes geometric properties of Banach spaces such as uniform convexity and uniform smoothness
- Uniform convexity X is uniformly convex iff J is uniformly monotone, i.e., $\langle J(x) - J(y), x - y \rangle \geq \delta(\|x - y\|)$ for some $\delta : (0, \infty) \rightarrow (0, \infty)$
- Uniform smoothness X is uniformly smooth iff J is uniformly continuous on bounded sets
- Modulus of convexity and modulus of smoothness can be expressed in terms of the duality mapping
- Moduli quantify the degree of uniform convexity and uniform smoothness (L^p spaces for $1 < p < \infty$ are uniformly convex and uniformly smooth)

Use in fixed point theory

- Crucial tool in studying fixed point theorems for nonlinear mappings in Banach spaces
- Fixed point a point $x \in X$ such that $T(x) = x$ for a mapping $T : X \rightarrow X$
- Nonexpansive mappings mappings $T : X \rightarrow X$ satisfying $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in X$
- In uniformly convex Banach spaces, every nonexpansive mapping with a bounded orbit has a fixed point (proof uses duality mapping and modulus of convexity)
- Accretive operators operators $A : X \rightarrow 2^X$ satisfying $\langle J(x - y), u - v \rangle \geq 0$ for all $x, y \in X, u \in A(x)$, and $v \in A(y)$
- Zeros of accretive operators (points $x \in X$ such that $0 \in A(x)$) studied using duality mapping and monotone operator theory

10.4 Characterizations of reflexivity

Reflexivity in Banach spaces is a powerful concept with far-reaching implications. It's characterized by the attainment of norms on dual spaces and the coincidence of weak and weak* compactness. These properties provide valuable tools for analyzing linear functionals and bounded sets.

The Eberlein-Šmulian theorem further connects reflexivity to weak convergence of bounded sequences. This relationship highlights the importance of reflexivity in understanding the behavior of sequences and subsets in Banach spaces, making it a crucial concept in functional analysis.

Characterizations of Reflexivity

James' theorem for reflexivity

- Characterizes reflexivity in terms of the attainment of the norm on the dual space
- Banach space X is reflexive if and only if every continuous linear functional on X attains its norm
- Proof assuming X is reflexive:
- Let $f \in X^*$ be a continuous linear functional on X
- Hahn-Banach theorem guarantees existence of $x^{**} \in X^{**}$ such that $x^{**}(f) = \|f\|$ and $\|x^{**}\| = 1$

- Reflexivity of X implies existence of $x \in X$ such that $x^{**}(g) = g(x)$ for all $g \in X^*$
- In particular, $f(x) = x^{**}(f) = \|f\|$, so f attains its norm at x
- Proof assuming every continuous linear functional on X attains its norm:
- Let $x^{**} \in X^{**}$ be a bounded linear functional on X^*
- Define $f(x) = x^{**}(J(x))$, where $J : X \rightarrow X^{**}$ is the canonical embedding
- $f \in X^*$ and by assumption, there exists $x_0 \in X$ such that $f(x_0) = \|f\|$
- For any $g \in X^*$, $|x^{**}(g) - g(x_0)| = |x^{**}(g) - f(x_0)| \leq \|x^{**}\| \|g - J(x_0)\|$
- Taking the supremum over all g with $\|g\| \leq 1$ yields $\|x^{**} - J(x_0)\| \leq \|x^{**}\| \|J(x_0) - J(x_0)\| = 0$
- Thus, $x^{**} = J(x_0)$, so X is reflexive

Weak compactness and reflexivity

- Weakly compact subset K of Banach space X : every sequence in K has a weakly convergent subsequence
- Weak convergence: sequence (x_n) in X converges weakly to $x \in X$ if $f(x_n) \rightarrow f(x)$ for all $f \in X^*$
- Reflexive spaces characterized by coincidence of weak and weak* compactness
- In reflexive space, bounded set is weakly compact if and only if it is weakly* compact
- Weakly compact sets in Banach space are bounded and closed in norm topology
- In reflexive space, closed unit ball is weakly compact

Eberlein-Šmulian theorem in reflexivity

- Eberlein-Šmulian theorem: for Banach space X , the following are equivalent: 1. X is reflexive 2. Every bounded sequence in X has a weakly convergent subsequence 3. Every bounded subset of X is weakly relatively compact
- Proof $(1 \Rightarrow 2)$: Assume X is reflexive, let (x_n) be bounded sequence in X
- Reflexivity of X implies closed unit ball B_X is weakly compact
- Sequence $(x_n/\|x_n\|)$ contained in B_X , so it has a weakly convergent subsequence
- Corresponding subsequence of (x_n) is also weakly convergent
- Proof $(2 \Rightarrow 3)$: Assume every bounded sequence in X has a weakly convergent subsequence
- Let $A \subseteq X$ be bounded, (x_n) be sequence in A
- By assumption, (x_n) has a weakly convergent subsequence, so A is weakly relatively compact
- Proof $(3 \Rightarrow 1)$: Assume every bounded subset of X is weakly relatively compact
- In particular, closed unit ball B_X is weakly relatively compact
- Since B_X is also weakly closed, it is weakly compact
- Thus, X is reflexive

Reflexivity vs Radon-Nikodým property

- Banach space X has Radon-Nikodým property (RNP) if for every finite measure space (Ω, Σ, μ) and every bounded linear operator $T : L^1(\mu) \rightarrow X$, there exists a Bochner integrable function $g : \Omega \rightarrow X$ such that $T(f) = \int_{\Omega} fg d\mu$ for all $f \in L^1(\mu)$
- Every reflexive Banach space has RNP
- Proof: Let X be reflexive, $T : L^1(\mu) \rightarrow X$ be bounded linear operator
- Define bounded linear functional φ on $L^1(\mu) \otimes X^*$ by $\varphi(f \otimes x^*) = \langle T(f), x^* \rangle$
- Hahn-Banach theorem extends φ to bounded linear functional on $L^1(\mu, X^*)$
- Reflexivity of X implies $L^1(\mu, X^*) \cong L^1(\mu, X)^*$, so there exists $g \in L^1(\mu, X)$ such that $\varphi(F) = \int_{\Omega} F(g) d\mu$ for all $F \in L^1(\mu, X^*)$
- In particular, $\langle T(f), x^* \rangle = \int_{\Omega} f \langle g, x^* \rangle d\mu$ for all $f \in L^1(\mu)$ and $x^* \in X^*$
- Thus, $T(f) = \int_{\Omega} fg d\mu$ for all $f \in L^1(\mu)$, so X has RNP
- There exist non-reflexive Banach spaces with RNP (ℓ^1)
- There exist reflexive Banach spaces without RNP ($L^1[0, 1]$)

Unit 11 – Unbounded and Closed Linear Operators

11.1 Definitions and examples of unbounded operators

Unbounded linear operators are crucial in functional analysis, extending beyond the limitations of bounded operators. They operate on specific domains within Hilbert spaces, allowing for the representation of complex mathematical and physical phenomena.

These operators, like differentiation and multiplication, play key roles in quantum mechanics and partial differential equations. Understanding their properties, such as closedness and spectral characteristics, is essential for solving real-world problems in physics and engineering.

Unbounded Linear Operators

Unbounded linear operators and domains

- Linear operators not necessarily bounded or continuous
- Domain is a proper subspace of the Hilbert space ($L^2([a, b])$)
- Operator $T : D(T) \rightarrow H$ maps from domain $D(T)$ to Hilbert space H
- Satisfies linearity property for $x, y \in D(T)$ and scalars α, β : $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$
- No constant $M > 0$ exists such that $\|Tx\| \leq M\|x\|$ for all $x \in D(T)$, unbounded
- Closed if its graph $G(T) = \{(x, Tx) : x \in D(T)\}$ is a closed subspace of $H \times H$

Examples of unbounded operators

- Differentiation operator on $L^2([a, b])$
- $D(T) = \{f \in L^2([a, b]) : f \text{ absolutely continuous and } f' \in L^2([a, b])\}$
- $(Tf)(x) = f'(x)$ for $f \in D(T)$, $x \in [a, b]$
- Multiplication operator on $L^2(\mathbb{R})$
- $D(T) = \{f \in L^2(\mathbb{R}) : xf(x) \in L^2(\mathbb{R})\}$
- $(Tf)(x) = xf(x)$ for $f \in D(T)$, $x \in \mathbb{R}$
- Laplace operator on $L^2(\mathbb{R}^n)$
- $D(T) = H^2(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \partial^\alpha f \in L^2(\mathbb{R}^n) \text{ for all } |\alpha| \leq 2\}$
-

$$(Tf)(x) = -\Delta f(x) = -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}(x)$$

for $f \in D(T)$, $x \in \mathbb{R}^n$

Bounded vs unbounded operators

- Bounded operators
- Defined on entire Hilbert space
- Continuous, small input changes lead to small output changes
- Finite operator norm $\|T\| = \sup \{\|Tx\| : \|x\| \leq 1\} < \infty$
- Unbounded operators
- Defined on proper subspace (domain) of Hilbert space
- Not necessarily continuous
- No finite operator norm
- Bounded operators have compact spectrum, unbounded can have unbounded spectrum
- Bounded operators always have bounded adjoint, unbounded may have unbounded or non-existent adjoints

Importance in functional analysis

- Quantum mechanics models observables as unbounded self-adjoint operators (position, momentum, Hamiltonian)
- Differential operators in PDEs often unbounded, studying properties helps understand solution behavior
- Unbounded operators with compact resolvents have discrete spectrum, spectral decomposition leads to generalized eigenfunction expansions
- Infinitesimal generators of strongly continuous semigroups often unbounded, applied in evolution equations (heat equation, Schrödinger equation)

11.2 Closed and closable operators

Unbounded operators are a crucial concept in functional analysis, mapping subsets of Banach spaces. Unlike bounded operators, their domains aren't the entire space. Examples include differentiation and multiplication operators on specific function spaces.

Closed operators are characterized by the Closed Graph Theorem. They're essential in studying partial differential equations and quantum mechanics. Closable operators, a related concept, have graphs that can be extended to closed operators.

Closed and Closable Operators

Definition of unbounded operators

- An unbounded operator T maps a subset $D(T)$ of a Banach space X to another Banach space Y
- The domain $D(T)$ is a linear subspace of X but not necessarily the entire space (unlike bounded operators)
- The graph of T is the set of ordered pairs (x, Tx) where $x \in D(T)$ ($G(T) = \{(x, Tx) : x \in D(T)\}$)

- Examples of unbounded operators include differentiation operator on $C^1([a, b])$ and multiplication operator on $L^2(\mathbb{R})$

Characterization of closed operators

- Closed Graph Theorem states an unbounded operator $T : D(T) \subset X \rightarrow Y$ between Banach spaces is closed iff for any sequence $(x_n) \subset D(T)$ with $x_n \rightarrow x$ and $Tx_n \rightarrow y$, we have $x \in D(T)$ and $Tx = y$
- Proof: 1. ($\Rightarrow$) Assume T is closed. Let $(x_n) \subset D(T)$ with $x_n \rightarrow x$ and $Tx_n \rightarrow y$. Since $G(T)$ is closed and $(x_n, Tx_n) \in G(T)$ for all n , $(x, y) \in G(T)$, so $x \in D(T)$ and $Tx = y$. 2. ($\Leftarrow$) Assume the condition holds. Let (x_n, Tx_n) be a sequence in $G(T)$ converging to (x, y) . Then $x_n \rightarrow x$ and $Tx_n \rightarrow y$. By the condition, $x \in D(T)$ and $Tx = y$, so $(x, y) \in G(T)$. Thus, $G(T)$ is closed, and T is closed
- Examples of closed operators include the Laplace operator on $H^2(\mathbb{R}^n)$ and the multiplication operator on $L^2(\mathbb{R})$ with a bounded measurable function

Identification of closed vs closable operators

- To check if an unbounded operator T is closed, consider a sequence $(x_n) \subset D(T)$ with $x_n \rightarrow x$ and $Tx_n \rightarrow y$. If $x \in D(T)$ and $Tx = y$ for all such sequences, then T is closed
- To check if T is closable, consider the closure of its graph $\overline{G(T)}$. If $\overline{G(T)}$ is the graph of an operator, then T is closable, and the associated operator is the closure of T
- Equivalently, T is closable iff for any sequence $(x_n) \subset D(T)$ with $x_n \rightarrow 0$ and $Tx_n \rightarrow y$, we have $y = 0$
- Examples of closable operators include the differentiation operator on $C([a, b])$ and the multiplication operator on $L^2(\mathbb{R})$ with an unbounded measurable function

Construction of operator closures

- The closure of a closable operator $T : D(T) \subset X \rightarrow Y$, denoted $\bar{T}$, is defined by:
- $D(\bar{T}) = \{x \in X : \exists (x_n) \subset D(T) \text{ such that } x_n \rightarrow x \text{ and } Tx_n \text{ converges in } Y\}$
- For $x \in D(\bar{T})$, $\bar{T}x = \lim_{n \rightarrow \infty} Tx_n$, where (x_n) is a sequence as in the definition of $D(\bar{T})$
- Properties of the closure:
- $\bar{T}$ is a closed extension of T
- $\bar{T}$ is the smallest closed extension of T , i.e., if S is any other closed extension of T , then $\bar{T} \subset S$
- The graph of $\bar{T}$ is the closure of the graph of T , i.e., $G(\bar{T}) = \overline{G(T)}$
- Examples of closures include the closure of the differentiation operator on $C([a, b])$ is the weak derivative operator on $W^{1,p}([a, b])$ for $1 \leq p < \infty$

11.3 Adjoints of unbounded operators

Unbounded operators in Hilbert spaces are crucial in functional analysis. They extend beyond bounded operators, allowing for a wider range of applications in physics and mathematics. Understanding their adjoints is key to grasping their behavior and properties.

Adjoints of unbounded operators help analyze operator properties like symmetry and self-adjointness. These concepts are fundamental in quantum mechanics and differential equations, where unbounded operators often represent physical observables or differential operators.

Adjoint of Unbounded Operators

Definition of unbounded operator adjoint

- Considers a densely defined unbounded operator $A : D(A) \subset H \rightarrow H$ on a Hilbert space H
- Defines the adjoint A^* of A as follows:
- Domain $D(A^*)$ consists of all $y \in H$ for which there exists $z \in H$ satisfying $\langle Ax, y \rangle = \langle x, z \rangle$ for all $x \in D(A)$
- For each $y \in D(A^*)$, A^*y is the unique element $z \in H$ fulfilling the condition $\langle Ax, y \rangle = \langle x, z \rangle$ for all $x \in D(A)$

Properties of unbounded operator adjoints

- Proves that if A is densely defined, then its adjoint A^* is a closed operator
- Considers a sequence $(y_n) \subset D(A^*)$ converging to y with $A^*y_n \rightarrow z$
- Shows that for any $x \in D(A)$, $\langle x, z \rangle = \lim_{n \rightarrow \infty} \langle x, A^*y_n \rangle = \lim_{n \rightarrow \infty} \langle Ax, y_n \rangle = \langle Ax, y \rangle$
- Concludes that $y \in D(A^*)$ and $A^*y = z$, establishing the closedness of A^*
- Demonstrates that if A is densely defined and $A \subset B$, then $B^* \subset A^*$
- Takes $y \in D(B^*)$ and shows that for any $x \in D(A) \subset D(B)$, $\langle Ax, y \rangle = \langle Bx, y \rangle = \langle x, B^*y \rangle$
- Deduces that $y \in D(A^*)$ and $A^*y = B^*y$, confirming $B^* \subset A^*$
- Establishes that if A is densely defined, then $(A^*)^* = \overline{A}$, the closure of A
- Proves $A \subset (A^*)^*$ by showing that for $x \in D(A)$ and $y \in D(A^*)$, $\langle (A^*)^*x, y \rangle = \langle x, A^*y \rangle = \langle Ax, y \rangle$
- Utilizes the closedness of $(A^*)^*$ as the adjoint of the densely defined operator A^*
- Argues that $\overline{A} \subset (A^*)^*$ since $A \subset (A^*)^*$ and $(A^*)^*$ is closed, and the reverse inclusion follows from the previous property

Computation of common unbounded adjoints

- Considers the multiplication operator $M_f : D(M_f) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ defined by $M_f\varphi = f\varphi$, where $f \in L^\infty(\Omega)$ and $D(M_f) = \{\varphi \in L^2(\Omega) : f\varphi \in L^2(\Omega)\}$
- Computes the adjoint as $M_f^* = M_{\bar{f}}$ with $D(M_f^*) = D(M_f)$
- Examines the differential operator $A : D(A) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ defined by $A\varphi = -i\varphi'$ with $D(A) = \{\varphi \in H^1(0, 1) : \varphi(0) = \varphi(1) = 0\}$
- Determines the adjoint as $A^* = i\frac{d}{dx}$ with $D(A^*) = H^1(0, 1)$

Unbounded operators vs their adjoints

- Shows that if A is symmetric (i.e., $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in D(A)$), then $A \subset A^*$
- Demonstrates that for any $x, y \in D(A)$, $\langle Ax, y \rangle = \langle x, Ay \rangle$ implies $y \in D(A^*)$ and $A^*y = Ay$
- Proves that if A is self-adjoint (i.e., $A = A^*$), then A is closed

- Argues that since $A = A^*$ and A^* is always closed, A must be closed
- Establishes that if A is positive (i.e., $\langle Ax, x \rangle \geq 0$ for all $x \in D(A)$), then A^* is also positive
- Considers $y \in D(A^*)$ and a sequence $(x_n) \subset D(A)$ such that $x_n \rightarrow y$ and $Ax_n \rightarrow A^*y$
- Shows that $\langle A^*y, y \rangle = \lim_{n \rightarrow \infty} \langle Ax_n, x_n \rangle \geq 0$, confirming the positivity of A^*

11.4 Spectral theory for unbounded self-adjoint operators

Unbounded self-adjoint operators are crucial in functional analysis, extending beyond bounded operators. They're defined on dense subspaces of Hilbert spaces and have unique spectral properties, making them essential for understanding complex mathematical structures.

The spectral theorem for these operators provides a powerful tool for analysis. It allows for operator decomposition, domain characterization, and diagonalization, enabling applications in quantum mechanics, differential equations, and operator algebras.

Spectral Theory for Unbounded Self-Adjoint Operators

Self-adjoint unbounded operators

- Unbounded operators are linear operators $T : D(T) \subset H \rightarrow H$ where the domain $D(T)$ is a dense subspace of the Hilbert space H
- Not necessarily bounded or continuous, unlike bounded operators defined on the entire Hilbert space
- Self-adjoint unbounded operators satisfy $T = T^*$, where T^* is the adjoint of T
- Domain of T equals the domain of its adjoint: $D(T) = D(T^*)$, ensuring the operator is well-defined
- Inner product equality: $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in D(T)$, expressing symmetry
- Spectrum of a self-adjoint unbounded operator consists of:
 - Point spectrum: eigenvalues λ satisfying $Tx = \lambda x$ for some $x \in D(T)$
 - Continuous spectrum: $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not surjective, but its range is dense in H
 - Residual spectrum is empty for self-adjoint operators, unlike non-self-adjoint operators

Spectral theorem for unbounded operators

- The spectral theorem for unbounded self-adjoint operators guarantees the existence of a unique spectral measure E on the Borel σ -algebra of $\mathbb{R}$
- Operator decomposition: $T = \int_{\mathbb{R}} \lambda dE(\lambda)$, expressing T as an integral with respect to the spectral measure
- Domain characterization: $D(T) = \left\{ x \in H : \int_{\mathbb{R}} |\lambda|^2 d\langle E(\lambda)x, x \rangle < \infty \right\}$, describing the domain of T using the spectral measure
- Interpretation of the spectral theorem:
 - The spectral measure E decomposes the Hilbert space H into a direct integral of Hilbert spaces H_λ corresponding to the spectral values λ
 - The operator T acts as multiplication by λ on each H_λ , simplifying its action

- The spectral theorem provides a way to diagonalize unbounded self-adjoint operators, generalizing the concept of eigendecomposition

Diagonalization of unbounded operators

- Diagonalization using the spectral theorem involves defining a unitary operator $U : H \rightarrow L^2(\mathbb{R}, \mu)$: 1. $(Ux)(\lambda) = \frac{d\langle E(\lambda)x, x \rangle}{d\mu(\lambda)}$, mapping elements of H to functions in $L^2(\mathbb{R}, \mu)$ 2. μ is the scalar-valued spectral measure given by $\mu(B) = \langle E(B)x, x \rangle$ for Borel sets $B \subset \mathbb{R}$
- The operator T is unitarily equivalent to the multiplication operator M_λ on $L^2(\mathbb{R}, \mu)$:
- $UTU^{-1} = M_\lambda$, where $(M_\lambda f)(\lambda) = \lambda f(\lambda)$, demonstrating the diagonalization of T
- Functional calculus allows the construction of new operators from the spectral measure of T :
- For a bounded Borel function $f : \mathbb{R} \rightarrow \mathbb{C}$, define the operator $f(T)$ by $f(T) = \int_{\mathbb{R}} f(\lambda) dE(\lambda)$
- Functional calculus extends the notion of applying functions to operators, enabling the study of operator properties

Applications of unbounded operator theory

- Quantum mechanics heavily relies on self-adjoint operators to represent observables
- The spectral theorem allows the decomposition of the Hilbert space into eigenspaces corresponding to possible measurement outcomes
- Expectation values and probabilities can be calculated using the spectral measure, providing a probabilistic interpretation
- Differential operators, such as the Laplace operator or the Schrödinger operator, are often unbounded self-adjoint operators
- The spectral theorem can be used to analyze the spectrum and eigenfunctions of these operators
- Spectral properties of differential operators have applications in physics (quantum mechanics, wave equations), engineering (vibration analysis), and partial differential equations (existence and uniqueness of solutions)
- Functional calculus and operator algebras benefit from the spectral theorem
- The functional calculus allows the construction of new operators from a given self-adjoint operator, enriching the operator algebra
- The spectral theorem provides a powerful tool for analyzing the structure of operator algebras generated by self-adjoint operators, such as von Neumann algebras and C^* -algebras

Unit 12 – Differential Equations & Quantum Mechanics

12.1 Sturm-Liouville theory and eigenvalue problems

Sturm-Liouville theory tackles eigenvalue problems for second-order linear differential equations. It's crucial for solving boundary value problems and understanding the behavior of differential operators.

The theory reveals key properties of eigenvalues and eigenfunctions, including their orthogonality and completeness. These insights allow us to represent functions as series expansions, a powerful tool in mathematical analysis.

Sturm-Liouville Theory

Definition of Sturm-Liouville problems

- Eigenvalue problems for second-order linear differential equations of the form $-(p(x)y')' + q(x)y = \lambda w(x)y$
- Boundary conditions $y(a) = y(b) = 0$ specify the values of the solution at the endpoints of the interval $[a, b]$
- Functions $p(x)$, $q(x)$, and $w(x)$ are real-valued and continuous on $[a, b]$ with $p(x) > 0$ and $w(x) > 0$ (ensures the problem is well-defined and has unique solutions)
- Properties of Sturm-Liouville problems include:
 - Real eigenvalues λ_n that can be ordered as an increasing sequence $\lambda_1 < \lambda_2 < \lambda_3 < \dots$
 - Eigenfunctions $y_n(x)$ corresponding to different eigenvalues are orthogonal with respect to the weight function $w(x)$, satisfying $\int_a^b w(x)y_n(x)y_m(x)dx = 0$ for $n \neq m$
 - Eigenfunctions form a complete orthonormal basis for the space of square-integrable functions on $[a, b]$ (allows for the representation of arbitrary functions as series expansions)

Eigenvalue problems in Sturm-Liouville theory

- Solving a Sturm-Liouville problem involves:
 1. Writing the differential equation in Sturm-Liouville form $-(p(x)y')' + q(x)y = \lambda w(x)y$
 2. Applying boundary conditions $y(a) = y(b) = 0$
 3. Determining the eigenvalues λ_n by solving the characteristic equation obtained from the differential equation and boundary conditions
 4. Finding the corresponding eigenfunction $y_n(x)$ for each eigenvalue λ_n
- Example: Sturm-Liouville problem $-y'' = \lambda y$ with $y(0) = y(\pi) = 0$ has eigenvalues $\lambda_n = n^2$ for $n = 1, 2, 3, \dots$ and corresponding eigenfunctions $y_n(x) = \sin(nx)$ (sine functions)

Application to boundary value problems

- Sturm-Liouville theory used to solve non-homogeneous boundary value problems of the form $-(p(x)y')' + q(x)y = f(x)$ with boundary conditions $y(a) = \alpha$ and $y(b) = \beta$
- Solution expressed as a series expansion using the eigenfunctions of the corresponding Sturm-Liouville problem:

$$y(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

- Coefficients c_n determined by the forcing function $f(x)$ and the boundary conditions α and β (allows for the incorporation of non-homogeneous terms and boundary values)

Spectrum and Eigenfunctions of Operators

Spectrum and eigenfunctions of operators

- Sturm-Liouville operator L defined as $Ly = -(p(x)y')' + q(x)y$
- Spectrum of L is the set of all eigenvalues λ_n , which is discrete and consists of an infinite sequence of real numbers (characterizes the operator's behavior)
- Eigenfunctions $y_n(x)$ have the following properties:
 - Infinitely differentiable on the open interval (a, b) (smooth functions)
 - Satisfy the boundary conditions $y_n(a) = y_n(b) = 0$
 - Have exactly $n - 1$ zeros in the open interval (a, b) (oscillatory behavior)
- Eigenfunctions are complete, meaning any square-integrable function $f(x)$ on $[a, b]$ can be represented as a series expansion

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

- Coefficients c_n given by $c_n = \int_a^b w(x)f(x)y_n(x)dx$ (inner product with the weight function $w(x)$)

12.2 Sobolev spaces and weak solutions of PDEs

Sobolev spaces are powerful tools in functional analysis, extending differentiability to function spaces. They're crucial for solving partial differential equations, providing a framework for weak solutions when classical derivatives don't exist.

These spaces, denoted as $W^{k,p}(\Omega)$, contain functions with weak derivatives up to order k in $L^p(\Omega)$. They're equipped with norms that make them Banach spaces, and for $p=2$, they become Hilbert spaces, ideal for many applications.

Sobolev Spaces

Definition of Sobolev spaces

- Generalize the notion of differentiability to function spaces denoted as $W^{k,p}(\Omega)$
- k represents the order of weak derivatives (highest order of derivatives considered)
- p represents the integrability of the function and its weak derivatives (Lebesgue space L^p)
- Ω is the domain of the functions (open subset of $\mathbb{R}^n$)
- Contain functions with weak derivatives up to order k that belong to $L^p(\Omega)$
- Equipped with the Sobolev norm $\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}$, making them Banach spaces

- For $p = 2$, Sobolev spaces are Hilbert spaces denoted as $H^k(\Omega)$
- Possess important properties such as density (smooth functions are dense), embedding (continuous embeddings between Sobolev spaces), and trace theorems (restrictions of functions to the boundary)

Concept of weak derivatives

- Extend the notion of classical derivatives to less regular functions
- A function $u \in L^1_{loc}(\Omega)$ has a weak derivative $v \in L^1_{loc}(\Omega)$ if $\int_{\Omega} u \frac{\partial \varphi}{\partial x_i} dx = - \int_{\Omega} v \varphi dx$ for all test functions $\varphi \in C_c^\infty(\Omega)$
- Test functions are smooth and compactly supported
- Weak derivatives allow for the study of solutions to PDEs that may not have classical derivatives

Applications in differential equations

- Provide a suitable framework for solving partial differential equations (PDEs)
- Weak formulation of a PDE is posed in a Sobolev space setting
- Solution space and test function space are chosen based on the regularity requirements of the PDE
- Examples of PDEs solved using Sobolev spaces:
- Poisson equation: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$
- Weak formulation: Find $u \in H_0^1(\Omega)$ such that $\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx$ for all $v \in H_0^1(\Omega)$
- Elliptic equations:

$$- \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + c(x)u = f(x)$$

in Ω , $u = 0$ on $\partial\Omega$

- Weak formulation: Find $u \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + c(x)uv dx = \int_{\Omega} f v dx$$

for all $v \in H_0^1(\Omega)$

Weak solutions via Sobolev spaces

- Solutions to PDEs in the weak sense, obtained by multiplying the PDE by a test function and integrating by parts
- A function $u \in V$ (a suitable Sobolev space) is a weak solution to a PDE if it satisfies the weak formulation $a(u, v) = \langle f, v \rangle$ for all test functions $v \in V$
- Existence and uniqueness of weak solutions can be proven using functional analysis tools:
- Lax-Milgram theorem: Guarantees existence and uniqueness for coercive ($a(u, u) \geq \alpha \|u\|_V^2$) and bounded ($|a(u, v)| \leq M \|u\|_V \|v\|_V$) bilinear forms
- Riesz representation theorem: Establishes the existence of a unique solution $u \in V$ to $\langle f, v \rangle = (u, v)_V$ for all $v \in V$, where $f \in V'$ is a bounded linear functional

- These theorems are applied to the weak formulation of the PDE to prove the existence and uniqueness of weak solutions in the appropriate Sobolev space setting

12.3 Operator methods in quantum mechanics

Quantum mechanics uses linear operators on Hilbert spaces to describe physical systems. State vectors represent physical states, while observables are represented by self-adjoint operators. Commutation relations between operators determine the compatibility of observables.

The spectral theorem is crucial for understanding quantum operators. It states that self-adjoint operators can be decomposed into projection operators, allowing for calculation of functions of operators. This theorem is essential for analyzing observables and their measurement outcomes.

Mathematical Formulation and Operator Methods in Quantum Mechanics

Mathematical formulation of quantum mechanics

- Quantum mechanics formulated using linear operators acting on a Hilbert space (complex vector space with inner product)
- State vectors represent physical states are elements of the Hilbert space (wavefunctions)
- Observables represented by self-adjoint (Hermitian) operators (position, momentum, energy)
- Commutation relations between operators play a crucial role determine compatibility of observables
- Canonical commutation relations: $[\hat{x}, \hat{p}] = i\hbar$ ($\hat{x}$ position operator, $\hat{p}$ momentum operator)
- Operators that commute have simultaneous eigenstates can be measured simultaneously
- Expectation values of observables calculated using the inner product gives average value of observable
- $\langle A \rangle = \langle \psi | \hat{A} | \psi \rangle$ ($|\psi\rangle$ state vector, $\hat{A}$ observable operator)
- Time evolution governed by the time-dependent Schrödinger equation describes dynamics of quantum system
- $i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$ ($\hbar$ reduced Planck's constant, t time)
- The Hamiltonian operator $\hat{H}$ determines the time evolution of the system encodes energy and interactions

Spectral theorem for self-adjoint operators

- Spectral theorem states any self-adjoint operator can be decomposed into a sum of projection operators
- $\hat{A} = \sum_n a_n |\phi_n\rangle \langle \phi_n|$ (a_n eigenvalues, $|\phi_n\rangle$ corresponding eigenstates)
- Eigenvalues real, eigenstates orthonormal and complete
- Spectral decomposition allows for calculation of functions of operators enables algebraic manipulation
- $f(\hat{A}) = \sum_n f(a_n) |\phi_n\rangle \langle \phi_n|$ (f arbitrary function)

- Spectral theorem essential for understanding properties of observables relates spectrum to measurement outcomes
- Eigenvalues represent possible measurement outcomes discrete or continuous
- Eigenstates form a complete orthonormal basis for the Hilbert space span the space of states

Solving time-independent Schrödinger equation

- Time-independent Schrödinger equation an eigenvalue problem determines stationary states and energy levels
- $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$ (E_n energy eigenvalues, $|\psi_n\rangle$ corresponding eigenstates)
- Stationary states have well-defined energy do not evolve in time
- Solving eigenvalue problem involves finding eigenvalues and eigenstates of the Hamiltonian
- Analytical solutions possible for simple systems (harmonic oscillator, hydrogen atom)
- Numerical methods used for more complex systems (variational methods, perturbation theory)
- Eigenstates of the Hamiltonian form a complete orthonormal basis allows expansion of arbitrary states
- Any state can be expanded in terms of the energy eigenstates superposition principle
- $|\psi\rangle = \sum_n c_n |\psi_n\rangle$, where $c_n = \langle \psi_n | \psi \rangle$ (expansion coefficients)

Spectrum analysis of quantum operators

- Spectrum of an operator set of its eigenvalues characterizes the possible measurement outcomes
- Discrete spectrum: isolated eigenvalues (bound states in potential wells)
- Continuous spectrum: continuous range of eigenvalues (scattering states, free particles)
- Eigenfunctions corresponding to discrete eigenvalues square-integrable normalizable
- Represent bound states typically localized in space (atomic orbitals)
- Eigenfunctions corresponding to continuous eigenvalues not square-integrable non-normalizable
- Represent scattering states typically delocalized (plane waves)
- Properties of spectrum and eigenfunctions provide insights into physical behavior of the system
- Energy levels, transition probabilities, selection rules determined by spectrum and symmetries
- Symmetries of the Hamiltonian reflected in degeneracy of the spectrum (rotational, translational invariance)

Spectral Theory and Applications in Quantum Mechanics

12.4 Spectral analysis of Schrödinger operators

Schrödinger operators are key to quantum mechanics, describing energy and dynamics of quantum systems. They're defined as linear differential operators with specific properties, including self-adjointness and an unbounded domain.

The spectrum of Schrödinger operators reveals crucial information about quantum systems. It includes point, continuous, and residual spectra, each corresponding to different physical states. Understanding these spectra is vital for analyzing quantum behavior.

Schrödinger Operators and Their Spectrum

Definition of Schrödinger operators

- Linear differential operators used in quantum mechanics describe the energy and dynamics of a quantum system
- Defined as $H = -\frac{\hbar^2}{2m}\nabla^2 + V(x)$, where:
 - $\hbar$ reduced Planck's constant
 - m mass of the particle
 - ∇^2 Laplacian operator
 - $V(x)$ potential energy function
- Properties:
 - Self-adjoint $\langle Hf, g \rangle = \langle f, Hg \rangle$ for all f, g in the domain of H
 - Unbounded domain of H is a proper subset of the Hilbert space
 - Closed graph of H is a closed subset of the product Hilbert space

Spectrum of Schrödinger operators

- Set of all $\lambda \in \mathbb{C}$ for which $H - \lambda I$ is not invertible
- Types of spectra:
 - Point spectrum (discrete spectrum) eigenvalues of H
 - Continuous spectrum λ for which $H - \lambda I$ is not bounded below
 - Residual spectrum λ for which $H - \lambda I$ is injective but not surjective
- Related to the physical properties of the quantum system
- Eigenvalues correspond to bound states (hydrogen atom)
- Continuous spectrum corresponds to scattering states (free particle)

Spectral theory for Schrödinger operators

- Spectral theorem self-adjoint operator has a unique spectral decomposition
- $H = \int_{\sigma(H)} \lambda dE(\lambda)$, where $E(\lambda)$ is the spectral measure
- Resolvent operator-valued function $R(z) = (H - zI)^{-1}$ for $z \in \mathbb{C} \setminus \sigma(H)$
- Analyzes the spectrum and provides information about the eigenfunctions
- Spectral measures projection-valued measures associated with self-adjoint operators
- Construct the functional calculus and study the spectrum

Applications and Interpretation

Spectral methods in Schrödinger equation

- Time-independent Schrödinger equation $H\psi = E\psi$
- ψ wavefunction
- E energy eigenvalue
- Spectral methods expand the wavefunction in terms of the eigenfunctions of H
- $\psi(x) = \sum_n c_n \phi_n(x)$
- $\phi_n(x)$ eigenfunctions
- c_n expansion coefficients
- Eigenvalues and eigenfunctions determined by solving the eigenvalue problem
- Variational methods (Rayleigh-Ritz method) approximate the eigenvalues and eigenfunctions

Physical interpretation of Schrödinger spectra

- Spectrum represents the possible energy values of the quantum system
- Discrete eigenvalues correspond to bound states with well-defined energies (electron in an atom)
- Continuous spectrum represents scattering states or unbound states (free electron)
- Eigenfunctions provide information about the spatial distribution of the particle
- Probability of finding the particle at a given position is proportional to $|\psi(x)|^2$
- Eigenfunctions with different eigenvalues are orthogonal, representing distinct quantum states
- Spectral decomposition allows for the calculation of expectation values and time evolution
- Expectation values $\langle A \rangle = \langle \psi | A | \psi \rangle = \int_{\sigma(H)} a(\lambda) d\langle \psi | E(\lambda) | \psi \rangle$
- $a(\lambda)$ observable associated with the operator A
- Time evolution $\psi(x, t) = e^{-iHt/\hbar} \psi(x, 0)$
- $\psi(x, 0)$ initial wavefunction

Unit 13 – Optimization in Banach Spaces

13.1 Convex analysis in Banach spaces

Convex sets and functions play a crucial role in Banach spaces. They're the building blocks for many important theorems and applications in functional analysis. Understanding these concepts is key to grasping more advanced topics.

The Hahn-Banach theorem is a powerful tool for extending linear functionals and separating convex sets. It has far-reaching consequences, including the existence of non-trivial dual spaces and the characterization of reflexive Banach spaces.

Convex Sets and Functions in Banach Spaces

Convex sets in Banach spaces

- Convex set contains all points on the line segment connecting any two points within the set
- Examples of convex sets in Banach spaces include closed balls $B[x_0, r] = \{x \in X : \|x - x_0\| \leq r\}$, hyperplanes $\{x \in X : f(x) = \alpha\}$ where f is a continuous linear functional and $\alpha \in \mathbb{R}$, and halfspaces $\{x \in X : f(x) \leq \alpha\}$ or $\{x \in X : f(x) \geq \alpha\}$

Convex functions in Banach spaces

- Convex function satisfies the inequality $f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$ for any $x, y \in X$ and $t \in [0, 1]$, meaning the line segment connecting any two points on the graph lies above or on the graph
- Examples of convex functions in Banach spaces include norms $f(x) = \|x\|$, continuous linear functionals $f(x) = \langle x^*, x \rangle$ where $x^* \in X^*$ (dual space), and affine functions $f(x) = \langle x^*, x \rangle + \alpha$ where $x^* \in X^*$ and $\alpha \in \mathbb{R}$

Hahn-Banach Theorem and Its Consequences

Hahn-Banach theorem applications

- Hahn-Banach theorem (analytic form) extends any continuous linear functional defined on a subspace to the whole space without increasing its norm
- Hahn-Banach theorem (geometric form) separates any closed convex set and a point outside it by a hyperplane
- Consequences of the Hahn-Banach theorem: 1. Normed spaces have enough continuous linear functionals to separate points 2. The dual space X^* of a Banach space X is non-trivial ($X^* \neq \{0\}$) 3. The bidual space X^{**} is isometrically isomorphic to X for reflexive Banach spaces

Subdifferentials in Banach Spaces

Subdifferentials in Banach spaces

- Subdifferential of a convex function f at $x \in X$ is the set $\partial f(x) = \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x \rangle, \forall y \in X\}$, generalizing the concept of the gradient for non-differentiable convex functions

- Properties of subdifferentials:
- If f is differentiable at x , then $\partial f(x) = \{\nabla f(x)\}$
- If f is continuous at x , then $\partial f(x)$ is a non-empty, convex, and weak* compact subset of X^*
- The subdifferential is a monotone operator, satisfying $\langle x^* - y^*, x - y \rangle \geq 0$ for any $x, y \in X$, $x^* \in \partial f(x)$, and $y^* \in \partial f(y)$
- Applications of subdifferentials include optimality conditions (x is a minimizer of f if and only if $0 \in \partial f(x)$) and characterizing proximal operators $\text{prox}_f(x) = \text{argmin}_{y \in X} \{f(y) + \frac{1}{2}\|y - x\|^2\}$

Convex Conjugate Functions

Properties of convex conjugates

- Convex conjugate (or Fenchel conjugate) of a convex function f is defined as $f^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}$ and is always convex, even if the original function is not
- Fenchel-Young inequality: $f(x) + f^*(x^*) \geq \langle x^*, x \rangle$ for all $x \in X$ and $x^* \in X^*$
- Double conjugate: $(f^*)^* = f$ if and only if f is convex, lower semicontinuous, and proper
- Subdifferential characterization: $x^* \in \partial f(x)$ if and only if $f(x) + f^*(x^*) = \langle x^*, x \rangle$

Examples of convex conjugates

- Indicator function

$$\delta_C(x) = \begin{cases} 0, & x \in C \\ +\infty, & x \notin C \end{cases}$$

where C is a closed convex set has conjugate equal to the support function $\delta_C^*(x^*) = \sup_{x \in C} \langle x^*, x \rangle$

- Norm $f(x) = \|x\|$ has conjugate equal to the indicator function of the unit ball in the dual space, $f^*(x^*) = \delta_{B[0,1]}(x^*)$
- Quadratic function $f(x) = \frac{1}{2}\|x\|^2$ has conjugate that is also a quadratic function, $f^*(x^*) = \frac{1}{2}\|x^*\|^2$

13.2 Variational principles and extremum problems

Variational principles are all about finding the best solution in a complex space. They're like searching for the highest peak or lowest valley in a mathematical landscape, using special tools to navigate the terrain.

These principles help us solve real-world problems by turning them into mathematical puzzles. We'll learn how to find solutions, prove they exist, and figure out if they're the only ones. It's like being a math detective!

Variational Principles

Variational problems in Banach spaces

- Involve finding an element in a Banach space (complete normed vector space) that minimizes or maximizes a given functional (real-valued function on the space)
- Require defining a functional $J : X \rightarrow \mathbb{R}$ on a Banach space X (examples: L^p spaces, Sobolev spaces)

- Goal is to find $u \in X$ such that $J(u) \leq J(v)$ for all $v \in X$ (minimization problem) or $J(u) \geq J(v)$ for all $v \in X$ (maximization problem)

- Solving variational problems involves: 1. Determining the appropriate Banach space X and functional J based on the problem's context and constraints 2. Establishing the existence of a solution using techniques such as the direct method of the calculus of variations or the Ekeland variational principle 3. Characterizing the solution using necessary and sufficient conditions, such as the Euler-Lagrange equation or the Karush-Kuhn-Tucker conditions

Direct method of calculus

- A powerful tool for establishing the existence of solutions to variational problems in Banach spaces
- Requires the functional J to be lower semicontinuous (does not "jump up" in the limit) and coercive (grows to infinity as the norm of the argument tends to infinity)
- Lower semicontinuity: A functional $J : X \rightarrow \mathbb{R}$ is lower semicontinuous if for any sequence $\{u_n\}$ in X converging to u , we have $J(u) \leq \liminf_{n \rightarrow \infty} J(u_n)$
- Coercivity: A functional $J : X \rightarrow \mathbb{R}$ is coercive if $J(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$, guaranteeing that the functional attains its minimum value within the space
- Applying the direct method involves: 1. Verifying lower semicontinuity and coercivity of the functional J 2. Choosing a minimizing sequence $\{u_n\}$ such that $J(u_n) \rightarrow \inf_{v \in X} J(v)$ 3. Showing that the sequence $\{u_n\}$ has a subsequence that converges to a limit $u \in X$ (using compactness arguments or weak convergence) 4. Proving that u is a minimizer of J using the lower semicontinuity property

Ekeland variational principle

- A powerful result in nonlinear analysis with numerous applications in optimization and variational problems
- States that for a lower semicontinuous functional J bounded from below on a complete metric space X , there exists a "perturbed" minimizer u_ε for any $\varepsilon > 0$
- Perturbed minimizer: For a given $\varepsilon > 0$, a point $u_\varepsilon \in X$ is called an ε -minimizer of J if $J(u_\varepsilon) \leq J(v) + \varepsilon d(u_\varepsilon, v)$ for all $v \in X$, approximately minimizing the functional J with an error controlled by ε
- Applying the Ekeland variational principle involves: 1. Verifying the lower semicontinuity of the functional J and the completeness of the metric space X 2. Choosing an arbitrary $\varepsilon > 0$ and applying the principle to obtain an ε -minimizer u_ε 3. Studying the properties of the perturbed minimizer u_ε and its relationship to the original variational problem 4. Passing to the limit as $\varepsilon \rightarrow 0$ to derive information about the solution to the original problem

Solutions to extremum problems

- Existence of solutions often relies on compactness arguments (Rellich-Kondrachov theorem for Sobolev spaces) or the direct method of the calculus of variations (lower semicontinuity and coercivity of the functional)
- Uniqueness of solutions typically depends on the convexity properties of the functional (strictly convex functional has at most one minimizer)
- Analyzing existence and uniqueness involves: 1. Identifying the appropriate function space and functional for the given extremum problem 2. Establishing the existence of solutions using compactness

arguments or the direct method 3. Investigating the convexity properties of the functional to determine the uniqueness of solutions 4. Considering additional regularity assumptions or constraints that may lead to uniqueness if it cannot be guaranteed

13.3 Calculus of variations and Euler-Lagrange equations

The calculus of variations is a powerful mathematical tool for finding optimal functions. It's used to solve problems like finding the shortest path between two points or the shape of a hanging chain.

At its core, the calculus of variations uses the Euler-Lagrange equation to find stationary points of functionals. This leads to solutions for various optimization problems in physics, engineering, and economics.

Calculus of Variations

Euler-Lagrange equations derivation

- Provide necessary conditions for a function to be a stationary point of a functional (mapping from vector space to real numbers)
- Consider functional $J[y] = \int_{x_1}^{x_2} F(x, y(x), y'(x)) dx$ and assume $y(x)$ is a stationary point
- Introduce small variation $\varepsilon \eta(x)$ around $y(x)$ and expand $J[y + \varepsilon \eta]$ using Taylor series
- Set first-order term to zero, integrate by parts, and apply fundamental lemma of calculus of variations
- Resulting Euler-Lagrange equation: $\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0$
- Express balance between change in F with respect to y and change in F with respect to y'
- Solutions make functional $J[y]$ stationary (minimum, maximum, or saddle points)

Variational problem solutions

- Identify functional $J[y] = \int_{x_1}^{x_2} F(x, y(x), y'(x)) dx$ to be minimized or maximized
- Write Euler-Lagrange equation: $\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0$
- Solve differential equation for $y(x)$ subject to boundary conditions
- Verify solution satisfies sufficient conditions for optimality (if applicable)
- Examples:
 - Geodesics: Find shortest curve connecting two points on a surface
 - Brachistochrone: Find curve of fastest descent between two points under gravity
 - Isoperimetric problems: Maximize or minimize quantity subject to perimeter or area constraint

Advanced Topics in Variational Calculus

Legendre transform in variations

- Convert between Lagrangian (generalized coordinates and velocities) and Hamiltonian (generalized coordinates and momenta) formulations

- Legendre transform definition: Given $f(x)$, transform $g(p) = \sup_x (px - f(x))$
- Relates Lagrangian $L(q, \dot{q}, t)$ and Hamiltonian $H(q, p, t)$
- Define generalized momentum $p = \frac{\partial L}{\partial \dot{q}}$
- Hamiltonian given by Legendre transform of Lagrangian: $H(q, p, t) = p\dot{q} - L(q, \dot{q}, t)$
- Hamilton's equations of motion: $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$
- Advantages: Symmetric treatment of coordinates and momenta, rich mathematical structure (symplectic geometry, Poisson brackets)

Optimality conditions for variations

- Ensure solution to Euler-Lagrange equations is minimum or maximum of functional
- Second variation $\delta^2 J[y]$: Second-order term in Taylor expansion of $J[y + \varepsilon \eta]$
- For minimum (maximum), must be non-negative (non-positive) for all admissible variations $\eta(x)$
- Legendre condition: Necessary condition for second variation to be non-negative (non-positive)
- For minimum: $\frac{\partial^2 F}{\partial y'^2} \geq 0$
- For maximum: $\frac{\partial^2 F}{\partial y'^2} \leq 0$
- Jacobi's condition: Strengthened Legendre condition requiring absence of conjugate points along solution curve
- Conjugate points: Points where solution to Jacobi equation (linearized Euler-Lagrange equation) vanishes
- Presence indicates solution is not strict minimum or maximum
- Weierstrass-Erdmann corner conditions: Apply when solution curve has corners (discontinuities in first derivative)
- At corner: $F - y' \frac{\partial F}{\partial y'}$ and $\frac{\partial F}{\partial y'}$ must be continuous

13.4 Applications to optimal control theory

Optimal control theory tackles finding control functions that optimize performance indices in dynamic systems. It's a powerful tool for solving complex problems in engineering, economics, and other fields where decisions must be made over time.

The Pontryagin maximum principle and Hamilton-Jacobi-Bellman equation are key methods in optimal control. They provide necessary and sufficient conditions for optimality, helping us find the best control strategies for various systems and objectives.

Optimal Control Theory

Formulation of optimal control problems

- Optimal control problems involve finding a control function that optimizes a performance index

- Performance index is a functional that depends on the state and control variables (cost function, objective function)
- Variational problems involve finding a function that optimizes a functional
- Functional is a mapping from a space of functions to real numbers (energy functional, action functional)
- To formulate an optimal control problem as a variational problem:
- Define the state variables (position, velocity) and control variables (force, acceleration)
- Specify the system dynamics that describe how the state variables evolve over time (equations of motion, state equations)
- Define the performance index to be optimized, which is a functional of the state and control variables (minimum time, minimum energy)
- Specify the initial and final conditions (initial state, target state), as well as any constraints on state and control variables (bounds on control inputs, state constraints)

Application of Pontryagin maximum principle

- Pontryagin maximum principle provides necessary conditions for optimality in optimal control problems
- Steps to apply the Pontryagin maximum principle: 1. Define the Hamiltonian function:

$$H(x, u, \lambda, t) = L(x, u, t) + \lambda^T f(x, u, t)$$
- $L(x, u, t)$ is the integrand of the performance index (Lagrangian, running cost)
- $f(x, u, t)$ represents the system dynamics (state equations, equations of motion)
- λ is the costate vector, also known as adjoint variables or Lagrange multipliers 2. Derive the optimal control by maximizing the Hamiltonian with respect to the control variable u (first-order necessary condition) 3. Obtain the state and costate equations:
- State equation: $\dot{x} = \frac{\partial H}{\partial \lambda}$ describes the evolution of the state variables
- Costate equation: $\dot{\lambda} = -\frac{\partial H}{\partial x}$ describes the evolution of the costate variables 4. Solve the state and costate equations with the optimal control, initial conditions, and transversality conditions (boundary conditions, final state constraints)

Dynamic Programming and the Hamilton-Jacobi-Bellman Equation

Hamilton-Jacobi-Bellman equation in practice

- The Hamilton-Jacobi-Bellman (HJB) equation is a partial differential equation that provides a sufficient condition for optimality in optimal control problems
- The HJB equation is based on the principle of optimality and the concept of the value function
- Value function $V(x, t)$ represents the optimal cost-to-go from state x at time t (optimal value, optimal cost)
- The HJB equation is given by:
- $$-\frac{\partial V}{\partial t} = \min_u \{L(x, u, t) + \frac{\partial V}{\partial x}^T f(x, u, t)\}$$

- $L(x, u, t)$ is the running cost, which is the integrand of the performance index (Lagrangian, instantaneous cost)
- $f(x, u, t)$ represents the system dynamics (state equations, equations of motion)
- Solving the HJB equation yields the optimal value function and the optimal control policy (feedback control law, state feedback)

Optimal control vs dynamic programming

- Dynamic programming is a solution approach for optimal control problems that breaks down the problem into smaller subproblems (principle of optimality, Bellman equation)
- The principle of optimality states that an optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decisions
- The value function in dynamic programming satisfies the HJB equation
- The HJB equation can be derived using the principle of optimality and the dynamic programming approach (Bellman equation, dynamic programming recursion)
- The optimal control can be obtained from the value function by minimizing the right-hand side of the HJB equation with respect to the control variable (argmin, optimal policy)
- Solving the HJB equation using dynamic programming involves discretizing the state space and time, and then solving the resulting discrete-time optimal control problem using backward induction (value iteration, policy iteration)

Unit 14 – Recent Developments in Functional Analysis

14.1 Nonlinear functional analysis and fixed point theorems

Nonlinear operators are functions between normed spaces that don't follow linear rules. They're tricky because their properties can vary widely. Some are continuous, some aren't. Some are bounded, others aren't. It's a mixed bag!

Fixed point theorems are super useful for dealing with nonlinear operators. The Banach Fixed Point Theorem is a big deal here. It helps us find solutions to nonlinear equations and prove they exist and are unique. It's a powerful tool in our math toolkit.

Nonlinear Operators and Fixed Point Theorems

Nonlinear operators and properties

- Nonlinear operators are functions between normed spaces that do not satisfy the linearity properties of additivity ($F(x + y) \neq F(x) + F(y)$) and homogeneity ($F(\alpha x) \neq \alpha F(x)$)
- Examples include $F(x) = x^2$, $F(x) = \sin(x)$, and $F(x) = e^x$ on $\mathbb{R}$
- Properties of nonlinear operators vary and may not hold for all nonlinear operators
- Continuity: Some nonlinear operators are continuous ($F(x) = x^2$) while others are not ($F(x) = \lfloor x \rfloor$)
- Differentiability: Some nonlinear operators are differentiable ($F(x) = x^3$) while others are not ($F(x) = \lfloor x \rfloor$)
- Boundedness: Some nonlinear operators are bounded ($F(x) = \sin(x)$) while others are not ($F(x) = e^x$)
- Compactness: Some nonlinear operators are compact ($F(x) = \frac{x}{1+|x|}$) while others are not ($F(x) = x^2$)

Banach Fixed Point Theorem

- The Banach Fixed Point Theorem, also known as the Contraction Mapping Theorem, states that a contraction mapping on a complete metric space has a unique fixed point
- A contraction mapping $T : X \rightarrow X$ satisfies $d(T(x), T(y)) \leq k \cdot d(x, y)$ for all $x, y \in X$, where $0 \leq k < 1$
- The unique fixed point x^* can be found by iterating $x_{n+1} = T(x_n)$ starting from any initial point $x_0 \in X$
- Proof outline of the Banach Fixed Point Theorem: 1. Show that the sequence $\{x_n\}$ is Cauchy using the contraction property 2. Use the completeness of X to conclude that $\{x_n\}$ converges to some $x^* \in X$ 3. Prove that x^* is a fixed point of T using the continuity of T (implied by the contraction property) 4. Show that the fixed point is unique by assuming the existence of another fixed point and deriving a contradiction
- Applications of the Banach Fixed Point Theorem include solving nonlinear equations ($x = T(x)$), proving the existence and uniqueness of solutions to differential and integral equations, and

establishing the convergence of iterative methods

Solutions of nonlinear equations

- Existence of solutions to nonlinear equations can be established by applying fixed point theorems (Banach Fixed Point Theorem, Brouwer Fixed Point Theorem, Schauder Fixed Point Theorem) to the equation $x = T(x)$
- If the conditions of the chosen fixed point theorem are satisfied, the existence of a solution is guaranteed
- Uniqueness of solutions can be proved using the properties of the nonlinear operator
- Contraction property (Banach Fixed Point Theorem) ensures the uniqueness of the fixed point (solution)
- Strict contraction property ($d(T(x), T(y)) < d(x, y)$ for all $x \neq y$) also guarantees uniqueness

Applications in differential equations

- Differential equations can be converted into equivalent integral equations using an integral operator
- Apply a suitable fixed point theorem to the integral operator to establish the existence and uniqueness of the solution
- Example: For the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, define the integral operator $T(y)(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds$ and apply a fixed point theorem
- Integral equations can be solved directly by applying a fixed point theorem to the integral operator
- Example: For the Fredholm integral equation $y(t) = f(t) + \int_a^b K(t, s, y(s)) ds$, define the integral operator $T(y)(t) = f(t) + \int_a^b K(t, s, y(s)) ds$ and apply a fixed point theorem

14.2 Distributions and generalized functions

Distributions are powerful mathematical tools that extend the concept of functions. They allow us to work with objects like the Dirac delta, which aren't traditional functions but are crucial in physics and engineering.

Operations on distributions, like derivatives and convolutions, open up new ways to solve complex problems. These techniques are especially useful in differential equations, helping us tackle issues that stumped classical methods.

Distributions and Their Properties

Properties of distributions

- Distributions are continuous linear functionals that map test functions (smooth, compactly supported functions) to real or complex numbers
- A distribution T is continuous if for every sequence of test functions ϕ_n converging to ϕ , $T(\phi_n)$ converges to $T(\phi)$
- The derivative of a distribution T is defined by $\langle T', \phi \rangle = -\langle T, \phi' \rangle$ for every test function ϕ , with higher-order derivatives defined iteratively

- A sequence of distributions T_n converges to a distribution T if for every test function ϕ , $\langle T_n, \phi \rangle$ converges to $\langle T, \phi \rangle$

Distributions vs classical functions

- Locally integrable functions f define distributions T_f by $\langle T_f, \phi \rangle = \int f(x)\phi(x)dx$
- Regular distributions are defined by locally integrable functions
- Singular distributions are not defined by locally integrable functions (Dirac delta function and its derivatives)

Operations on Distributions and Applications

Operations on distributions

- The derivative of a distribution is defined by duality: $\langle T', \phi \rangle = -\langle T, \phi' \rangle$, which is a continuous operation on the space of distributions
- The convolution of two distributions S and T is defined by $\langle S * T, \phi \rangle = \langle S(x), \langle T(y), \phi(x + y) \rangle \rangle$, which is associative and commutative
- The Fourier transform of a distribution T is defined by $\langle \mathcal{F}(T), \phi \rangle = \langle T, \mathcal{F}(\phi) \rangle$, which is a continuous, linear, and bijective operation on the space of tempered distributions

Applications in differential equations

- Distributions can be used to find fundamental solutions to linear partial differential equations, which solve the PDE with a Dirac delta function as the source term
- Green's functions are fundamental solutions to linear PDEs with specific boundary conditions and can be used to solve inhomogeneous boundary value problems
- Distributions provide a rigorous framework for dealing with discontinuities and singularities in physical systems, such as modeling point masses and charge distributions

14.3 Operator algebras and C*-algebras

Operator algebras and C*-algebras are powerful tools in functional analysis. They extend the concept of algebras to spaces of bounded linear operators, providing a rich framework for studying quantum mechanics and noncommutative geometry.

The Gelfand-Naimark Theorem is a cornerstone, showing that every C*-algebra can be realized as an algebra of operators on a Hilbert space. This connection bridges abstract algebra and concrete operator theory, opening doors to various applications in physics and mathematics.

Operator Algebras and C*-Algebras

Operator and C*-algebras

- Operator algebras are subalgebras of the algebra of bounded linear operators on a Hilbert space that are closed under the operator norm topology
- The algebra of all bounded linear operators on a Hilbert space and the algebra of all compact operators on a Hilbert space are examples of operator algebras

- C-algebras are operator algebras that are closed under the involution operation (adjoint) and satisfy the C-identity: $\|A^* A\| = \|A\|^2$
- Examples include the algebra of all bounded linear operators on a Hilbert space, the algebra of continuous functions on a compact Hausdorff space with pointwise addition and multiplication and the supremum norm, and the group C*-algebra of a locally compact group

Gelfand-Naimark Theorem

- The Gelfand-Naimark Theorem states that every C-algebra is *-isomorphic to a C*-algebra of bounded operators on some Hilbert space
- The proof involves constructing a Hilbert space using the positive linear functionals on the C-algebra, defining a representation of the C-algebra on this Hilbert space using the GNS construction, and showing that this representation is faithful and preserves the involution and norm
- The theorem implies that every abstract C-algebra can be concretely realized as an operator algebra and that C-algebras provide a natural framework for studying noncommutative topology and geometry

Spectral properties in C*-algebras

- The spectrum $\sigma(a)$ of an element a in a C*-algebra A is the set of all $\lambda \in \mathbb{C}$ such that $a - \lambda 1$ is not invertible in A
- The spectrum is always a non-empty compact subset of $\mathbb{C}$ and the spectral radius of a equals the radius of the smallest disk containing $\sigma(a)$
- The spectral theorem for normal elements states that for a normal element a in a C*-algebra, there exists a unique projection-valued measure E on the Borel sets of $\sigma(a)$ such that $a = \int_{\sigma(a)} \lambda dE(\lambda)$
- This implies that normal elements can be diagonalized in a suitable sense and allows for the construction of new elements from normal elements using continuous functions via the functional calculus

Applications of C*-algebras

- In quantum mechanics, observables are represented by self-adjoint elements in a C-algebra, while states are represented by positive linear functionals of norm 1 on the C-algebra
- The Gelfand-Naimark Theorem ensures that these abstract observables can be realized as concrete operators on a Hilbert space
- C-algebras provide a natural framework for studying quantum statistical mechanics, with the KMS condition characterizing equilibrium states in terms of a condition on the C-algebra of observables
- In noncommutative geometry, C*-algebras are used to generalize the concepts of topology and geometry to noncommutative spaces
- The Connes-Kasparov conjecture relates the K-theory of C*-algebras to the geometry of the underlying noncommutative space

14.4 Wavelets and frames in Hilbert spaces

Wavelets and frames are powerful tools for analyzing signals and functions in Hilbert spaces. They provide flexible ways to decompose and represent complex data, offering unique insights into signal characteristics at different scales and positions.

These techniques have revolutionized signal processing, image compression, and numerical analysis. By exploiting the sparsity and multi-resolution nature of wavelet representations, we can efficiently solve problems in various fields of mathematics and engineering.

Wavelets and Frames in Hilbert Spaces

Wavelets and frames in Hilbert spaces

- Wavelets are functions $\psi \in L^2(\mathbb{R})$ used to decompose and analyze signals or functions
- Obtained by translating and dilating a mother wavelet ψ
- Translation shifts the wavelet: $\psi_{a,b}(x) = \psi(x - b)$
- Dilation scales the wavelet: $\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right)$, where $a, b \in \mathbb{R}, a \neq 0$
- Must satisfy admissibility condition: $\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty$, where $\hat{\psi}$ is the Fourier transform of ψ
- Ensures wavelet can be used for signal reconstruction
- Frames are a sequence $\{f_n\}_{n=1}^{\infty}$ in a Hilbert space H that provide stable, redundant representations
- Exist constants $A, B > 0$ such that for all $f \in H$:

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2$$

- A and B are frame bounds that quantify stability and redundancy
- Tight frames have equal frame bounds $A = B$
- Parseval frames are tight frames with $A = B = 1$, behave like orthonormal bases

Construction of wavelets and frames

- Haar wavelet is a simple example of an orthonormal wavelet basis for $L^2(\mathbb{R})$
- Mother wavelet:

$$\psi(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

- Haar wavelets: $\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$, where $j, k \in \mathbb{Z}$
- j controls scale (dilation) and k controls position (translation)
- Gabor frames are constructed from a window function $g \in L^2(\mathbb{R})$ and lattice parameters $a, b > 0$
- Gabor atoms: $g_{m,n}(x) = e^{2\pi i b n x} g(x - am)$, where $m, n \in \mathbb{Z}$
- m and n control time and frequency shifts, respectively
- The set $\{g_{m,n}\}_{m,n \in \mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$ if $ab < 1$
- Oversampling in time-frequency plane ensures completeness and stability

Convergence of wavelet expansions

- For $f \in L^2(\mathbb{R})$, the wavelet expansion is given by $f = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}$
- Wavelet coefficients $\langle f, \psi_{j,k} \rangle$ capture signal information at different scales and positions
- Convergence in $L^2(\mathbb{R})$ norm: $\lim_{N \rightarrow \infty} \left\| \sum_{j,k \in \mathbb{Z}, |j|, |k| \leq N} \langle f, \psi_{j,k} \rangle \psi_{j,k} - f \right\| = 0$
- Partial sums converge to the original signal as more terms are included
- Wavelet coefficients are stable, bounded by the L^2 norm of the signal: $|\langle f, \psi_{j,k} \rangle| \leq C \|f\|$
- Frame expansions converge in Hilbert space norm:

$$\lim_{N \rightarrow \infty} \left\| \sum_{n=1}^N \langle f, S^{-1} f_n \rangle f_n - f \right\| = 0$$

- S is the frame operator, S^{-1} is its inverse
- Frame coefficients $\langle f, S^{-1} f_n \rangle$ are stable, bounded by the Hilbert space norm of f

Applications of wavelets and frames

- Signal processing uses wavelet transforms for: 1. Denoising: Thresholding wavelet coefficients to remove noise 2. Compression: Exploiting sparsity of wavelet coefficients to reduce data size 3. Feature extraction: Identifying important signal characteristics at different scales
- Image compression algorithms (JPEG2000) use wavelets to: 1. Decompose image into wavelet coefficients 2. Threshold and quantize coefficients to achieve high compression ratios 3. Reconstruct image with minimal perceptual loss
- Numerical analysis employs wavelets and frames for:
 - Adaptive mesh refinement in finite element methods
 - Wavelets identify regions requiring higher resolution
 - Efficient representation and computation of operators and functions
 - Wavelet bases can diagonalize certain operators
 - Discretization and solution of partial differential equations
- Frames provide stable, flexible discretizations of function spaces