

Graph Theory

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Unit 1 – Introduction to Graph Theory

1.1 Historical development and applications of graph theory

Graph theory, born from Euler's Seven Bridges problem, has evolved into a powerful tool for modeling relationships. From Kirchhoff's electrical networks to Erdős' collaborative networks, it's shaped diverse fields and problem-solving approaches.

Today, graph theory drives innovations in computer science, social networks, and biology. Its visual representation of complex systems enables efficient algorithms, decision-making tools, and interdisciplinary research, making it a cornerstone of modern problem-solving.

Historical Development of Graph Theory

Origins of graph theory

- Leonhard Euler (1707-1783) solved Seven Bridges of Königsberg problem 1736 laid foundation for graph theory and topology
- Gustav Kirchhoff (1824-1887) applied graph theory to electrical networks developed Kirchhoff's laws for circuit analysis
- Arthur Cayley (1821-1895) studied trees and their properties contributed to theory of chemical molecules using graphs (benzene rings)
- William Rowan Hamilton (1805-1865) invented Icosian game led to Hamiltonian cycles concept in graph theory
- George Pólya (1887-1985) worked on enumeration of graphs and chemical compounds (isomers)
- Paul Erdős (1913-1996) prolific mathematician made significant contributions introduced Erdős number concept (collaboration distance)

Applications in diverse fields

- Computer Science: network topology and design, data structures and algorithms (binary trees), compiler optimization
- Social Networks: modeling relationships between individuals, analyzing information flow and influence, detecting communities and clusters (Facebook, LinkedIn)
- Transportation: optimizing routes for delivery services (UPS), designing efficient public transit systems, traffic flow analysis
- Biology: modeling protein interactions (protein-protein interaction networks), studying ecological food webs
- Chemistry: representing molecular structures, analyzing chemical reactions (reaction mechanisms)

Significance for problem-solving

- Provides visual and intuitive representation of relationships enables complex system analysis
- Enables efficient algorithms for solving optimization problems: 1. Identify problem as graph 2. Apply appropriate algorithm (Dijkstra's, Kruskal's) 3. Interpret results in context of original problem

- Facilitates analysis of large-scale networks identifying critical nodes and edges studying network resilience and vulnerability
- Allows modeling of complex systems supply chain management communication networks (internet routing)
- Supports decision-making in various fields resource allocation project scheduling (PERT charts)
- Enables development of new technologies recommendation systems search engine algorithms (PageRank)
- Provides framework for interdisciplinary research combining graph theory with other mathematical and scientific disciplines

1.2 Basic definitions and graph types

Graphs are the backbone of network analysis, representing connections between entities. From social networks to transportation systems, graphs model relationships using vertices and edges, with various classifications like directed or undirected.

Special graph types like trees, cycles, and stars have unique properties and applications. Understanding these fundamentals is crucial for solving real-world problems in fields ranging from computer science to sociology, laying the groundwork for advanced graph theory concepts.

Graph Fundamentals

Graph theory terminology basics

- Vertex (plural: vertices) fundamental unit of graph represents point or node (social network user)
- Edge connection between two vertices represented by line or arc (friendship connection)
- Degree of vertex number of edges incident to vertex (Facebook friend count)
- Adjacent vertices two vertices connected by edge (neighboring cities on map)
- Incident edge connected to specific vertex (road connecting two cities)

Directed vs undirected graphs

- Undirected graph edges have no orientation represented by line segments symmetric relationship (Facebook friendships)
- Directed graph (digraph) edges have direction represented by arrows asymmetric relationship (Twitter followers)
- Mixed graph contains both directed and undirected edges (social media platform with followers and mutual friends)

Graph Classifications and Special Types

Classification of graph properties

- Complete graph every pair of vertices connected by edge denoted as K_n (n = number of vertices)
- Bipartite graph vertices divided into two disjoint sets edges connect vertices from different sets (students and classes)

- Planar graph drawn on plane without edge crossings follows Euler's formula: $V - E + F = 2$ (road map)
- Regular graph all vertices have same degree (cube)
- Connected graph path exists between any two vertices (subway system)
- Disconnected graph at least two vertices with no path between them (separate social circles)

Special graph types

- Tree connected graph with no cycles number of edges = number of vertices - 1 (family tree)
- Cycle closed path with no repeated vertices except start/end denoted as C_n (n = number of vertices)
- Wheel formed by connecting single vertex to all vertices of cycle denoted as W_n (n = total number of vertices)
- Path sequence of vertices connected by edges with no repetition (hiking trail)
- Star graph one central vertex connected to all other vertices (airport hub and spoke model)
- Null graph no edges only isolated vertices (strangers in a room)

1.3 Graph representations and visualization

Graphs can be represented in various ways, each with unique advantages. Adjacency matrices, adjacency lists, and incidence matrices offer different approaches to storing graph data, balancing space efficiency and ease of access.

Visualizing graphs helps us understand their structure and properties. Drawing conventions, layout algorithms, and specialized tools enable us to create meaningful visual representations of complex graph structures, aiding analysis and interpretation.

Graph Representation Methods

Graph representation methods

- Adjacency Matrix: Square matrix dimensions equal vertices number, entries indicate edge presence/absence, unweighted graphs use 0 and 1, weighted graphs use edge weights or infinity (Sudoku puzzle graph, Social network connections)
- Adjacency List: Array or list of lists, each vertex has adjacent vertices list, space-efficient for sparse graphs (Web page link structure, Flight connections between airports)
- Incidence Matrix: Rows represent vertices and columns edges, entries indicate vertex-edge incidence (Circuit board connections, Chemical reaction networks)

Conversion of graph formats

- Adjacency Matrix to Adjacency List: Iterate matrix rows, create lists for non-zero entries
- Adjacency List to Adjacency Matrix: Initialize matrix with zeros, set 1 or weight for each edge in list
- Incidence Matrix to Adjacency List: Identify 1s pairs in each column, add corresponding vertices to each other's lists

Graph Visualization

Visualization techniques for graphs

- Drawing Conventions: Vertices as points/circles, edges as lines/curves, directed edges with arrowheads (Social network graphs, Molecular structures)
- Layout Algorithms: Force-directed layout simulates physical forces, circular layout arranges vertices in circle, hierarchical layout organizes in levels (Organizational charts, Family trees)
- Visualization Tools: Graphviz offers command-line graph drawing, Gephi provides interactive visualization, NetworkX with Matplotlib enables customizable plotting (Gene regulatory networks, Computer network topologies)
- Special Graph Types: Planar graph drawings avoid edge crossings, tree layouts emphasize hierarchical structures (Road maps, File system hierarchies)

Analysis of graph visuals

- Identify graph properties: Connectedness reveals overall structure, cycles and paths show navigation possibilities, planarity determines if graph can be drawn without edge crossings (Transportation networks, Electrical circuits)
- Recognize graph structures: Complete graphs have all possible edges, bipartite graphs have two distinct vertex sets, trees and forests lack cycles (Friend networks, Matching problems)
- Analyze graph characteristics: Degree distribution shows connectivity patterns, centrality measures identify important vertices, community structure reveals clusters (Social media influencer networks, Protein interaction networks)
- Compare graph isomorphisms: Vertex mapping preserves connections, edge preservation maintains structure (Chemical compound similarity, Network protocol equivalence)

Unit 2 – Graph Terminology and Basic Properties

2.1 Vertices, edges, and degree

Graphs are made up of vertices and edges, representing entities and their connections. Understanding these components is crucial for analyzing network structures in various fields, from social networks to transportation systems.

Vertex properties, like degree, provide insights into a graph's structure. Degree measures a vertex's connectivity, while concepts like indegree and outdegree in directed graphs reveal the flow of information or resources within a network.

Graph Components and Vertex Properties

Vertices and edges

- Vertices (nodes) form fundamental elements representing entities or points depicted as dots or circles (cities in a transportation network)
- Edges connect vertices representing relationships or links typically shown as lines or arrows (roads connecting cities)
- Graph notation expressed as $G = (V, E)$ where V represents set of vertices and E represents set of edges
- Types of graphs based on edge properties include undirected graphs with no directional edges and directed graphs (digraphs) with specific directional edges (social network connections vs one-way streets)

Degree of a vertex

- Degree defines number of edges incident to a vertex denoted as $\deg(v)$ for a vertex v
- Degree properties range from minimum degree 0 for isolated vertices to maximum degree $|V| - 1$ for vertices connected to all others
- Handshaking lemma states sum of degrees of all vertices equals twice the number of edges:
$$\sum_{v \in V} \deg(v) = 2|E|$$
- Degree sequence lists vertex degrees in non-increasing order
- Regular graphs feature all vertices with same degree (cube graph)

Calculating vertex degrees

- Simple graph examples:
- Path graph end vertices have degree 1 others have degree 2
- Cycle graph all vertices have degree 2
- Complete graph K_n all vertices have degree $n - 1$

- Special graph structures:
- Star graph central vertex has degree $n - 1$ others have degree 1
- Wheel graph central vertex has degree $n - 1$ others have degree 3
- Bipartite graphs may have different degrees for vertices in each partition
- Multigraphs count multiple edges between same pair of vertices
- Loops contribute 2 to degree of vertex

Indegree vs outdegree

- Indegree counts number of edges pointing towards a vertex denoted as $\deg^-(v)$ (incoming emails)
- Outdegree counts number of edges pointing away from a vertex denoted as $\deg^+(v)$ (outgoing emails)
- Total degree in directed graphs sums indegree and outdegree: $\deg(v) = \deg^-(v) + \deg^+(v)$
- Balanced vertices have equal indegree and outdegree
- Source vertices have indegree 0 while sink vertices have outdegree 0 (starting and ending points in a flow network)
- Directed graph properties: sum of all indegrees equals sum of all outdegrees both equaling total number of edges in graph

2.2 Subgraphs and graph operations

Subgraphs are essential building blocks in graph theory, allowing us to analyze smaller parts of larger networks. They're formed by taking subsets of vertices and edges from a bigger graph, giving us tools to study complex structures piece by piece.

Understanding subgraphs opens doors to powerful graph operations like union, intersection, and complement. These operations let us combine, compare, and manipulate graphs in various ways, helping us model real-world systems and solve intricate problems in network analysis.

Understanding Subgraphs

Concept of subgraphs

- Subgraph forms from subset of vertices and edges of another graph
- Types include induced subgraph (contains all edges between its vertices present in original graph) and spanning subgraph (includes all vertices of original graph)
- Subgraphs have fewer or equal vertices and edges compared to original graph inherit properties from parent graph
- Notation $H \subseteq G$ denotes H is a subgraph of G

Identification of subgraphs

- Methods involve vertex deletion and edge deletion

- Process: select subset of vertices from original graph include connecting edges verify resulting structure forms valid subgraph
- Special subgraphs: cliques (complete subgraphs), paths (sequences of connected vertices), cycles (closed paths)
- Subgraph identification crucial in network analysis and pattern recognition (social networks, molecular structures)

Graph operations

- Graph union $G_1 \cup G_2$ combines vertices and edges of two graphs
- Graph intersection $G_1 \cap G_2$ retains only common vertices and edges between two graphs
- Graph complement $\overline{G}$ or G^c creates new graph with edges where original had none
- Edge operations include deletion (removing edges) and contraction (merging vertices)
- Sum of edges in graph and its complement equals total possible edges
- Double complement returns original graph

Creation through graph operations

- Techniques involve combining multiple existing graphs or modifying single graph through operations
- Process: identify desired properties select appropriate operations apply in specific order to achieve result
- Examples: constructing complete bipartite graph from two independent sets creating wheel graph by adding central vertex to cycle graph
- Applications in modeling complex systems (transportation networks) and generating test cases for graph algorithms
- Verify newly created graphs by checking properties like connectivity planarity or regularity

2.3 Walks, paths, and cycles

Graph traversal is all about exploring connections. Walks, paths, and cycles are ways to move through a graph, each with unique rules. These concepts help us understand how information or objects flow through networks.

Open and closed walks, along with length calculations, have real-world applications. From planning delivery routes to analyzing social networks, these ideas shape how we navigate interconnected systems in everyday life.

Fundamental Concepts of Graph Traversal

Walks, paths, and cycles

- Walk traverses graph sequence vertices and edges connecting consecutive vertices may repeat (Manhattan street grid)
- Path special walk no repeated vertices or edges except possibly first/last (GPS route)

- Cycle closed walk starts/ends same vertex no repeats except start/end minimum length 3 simple graphs (Monopoly board)

Open vs closed walks

- Open walk starts ends different vertices any length (train route with multiple stops)
- Closed walk forms loop starts ends same vertex minimum length 1 for self-loops (circular subway line)

Practical Applications

Identifying graph sequences

- Walk verification check consecutive vertex pairs connected by edge allow repetition (social network friend chain)
- Path verification valid walk no repeated vertices/edges except possibly first/last (delivery route)
- Cycle verification valid walk same start/end no repeats except start/end minimum length 3 simple graphs (yearly calendar)

Length of graph structures

- Length calculation count edges traversed weighted graphs sum edge weights (road network with distances)
- Walk length total edge traversals including repeats (tourist sightseeing route)
- Path length edges in path equals vertices minus one (flight itinerary)
- Cycle length edges/vertices in cycle minimum 3 simple graphs (Olympic rings)
- Relation to properties diameter longest shortest path between vertices girth shortest cycle (computer network topology)

2.4 Connectedness and components

Graph connectivity is all about how vertices link up in a network. It's crucial for understanding how information or resources can flow through a system. Connected graphs allow full reachability, while disconnected ones have isolated parts.

Components are the building blocks of graphs. They're like islands of connectivity within the larger structure. Counting and analyzing these components helps us grasp the overall layout and potential of the graph.

Graph Connectivity

Concept of graph connectedness

- Graph connected when path exists between any two vertices enabling full reachability
- Path defined as sequence of vertices with adjacent pairs linked by edges (A-B-C-D)
- Connectedness crucial for many algorithms and applications (network routing, social network analysis)

Connected vs disconnected graphs

- Connected graphs allow reaching every vertex from any other forming single component
- Disconnected graphs contain at least two unreachable vertices resulting in multiple components
- Connectedness determined through graph traversal algorithms like Depth-First Search (DFS) or Breadth-First Search (BFS)

Components in graphs

- Connected component represents maximal connected subgraph with no external connections
- Isolated vertices have no incident edges each forming separate component (standalone nodes)

Number of graph components

- Component counting employs graph traversal algorithms (DFS, BFS)
- Process: initialize count, perform traversal from unvisited vertex, increment count for each new component
- Time complexity $O(V + E)$ for both DFS and BFS approaches

Properties of connected components

- Components form disjoint sets with no shared vertices
- All graph edges contained within components none between different components
- Every vertex in component reachable from all others within same component
- Component graphs represent each component as single vertex useful for analyzing disconnected graph structure

Unit 3 – Graph Representation and Isomorphism

3.1 Adjacency matrices and incidence matrices

Matrices offer powerful ways to represent graphs mathematically. Adjacency matrices use a square format to show connections between vertices, while incidence matrices use a rectangular format to explicitly represent edges.

These matrix representations have different strengths. Adjacency matrices are great for dense graphs and quick edge queries, while incidence matrices work well for sparse graphs and maintaining explicit edge information.

Graph Representation Matrices

Definition of adjacency matrices

- Square matrix represents graph structure with rows and columns corresponding to vertices
- Entries indicate presence or absence of edges between vertex pairs
- Symmetric matrix for simple undirected graphs with diagonal entries typically 0
- May not be symmetric for directed graphs, entry (i, j) represents edge from vertex i to j
- Matrix entries interpret edge relationships (0: no edge, 1: edge exists, other values: edge weights)

Construction of adjacency matrices

- Undirected graphs fill matrix symmetrically $A_{ij} = A_{ji} = 1$ for existing edges
- Directed graphs set $A_{ij} = 1$ for edges from i to j , A_{ji} may differ
- Weighted graphs replace 1s with edge weights, 0 still indicates no edge
- Self-loops represented by non-zero diagonal entries
- Multiple edges summed in edge counts or weights

Incidence matrices vs adjacency matrices

- Incidence matrix: rectangular, rows for vertices, columns for edges
- Undirected graphs: 1 if vertex incident to edge, 0 otherwise
- Directed graphs: -1 for tail, 1 for head, 0 for non-incident
- Both represent graph structure, incidence matrices show explicit edge info
- Adjacency matrices focus on vertex connections

Conversion between matrix representations

- Adjacency to incidence: create column for each non-zero adjacency entry

- Incidence to adjacency: multiply incidence matrix by transpose $A = MM^T$
- Conversion may lose multiple edge information
- Directed graph conversion requires additional steps

Space complexity of matrix storage

- Adjacency matrix: $O(|V|^2)$ space, efficient for dense graphs
- Incidence matrix: $O(|V||E|)$ space, better for sparse graphs
- Adjacency matrices suit graphs with $|E|$ close to $|V|^2$
- Incidence matrices preferred when $|E|$ much less than $|V|^2$
- Adjacency matrices allow faster edge queries
- Incidence matrices provide explicit edge representation

3.2 Adjacency lists and edge lists

Graph representation techniques are crucial for efficiently storing and manipulating graph structures in computer memory. Adjacency lists, a popular method, offer a space-efficient solution for sparse graphs by storing only connected vertices for each node.

This approach allows for quick access to a vertex's neighbors, making it ideal for many graph algorithms. We'll explore the construction, space complexity, and implementation of adjacency lists, comparing them to other representations like edge lists and matrices.

Graph Representation Techniques

Adjacency lists for graph representation

- Adjacency lists efficiently store graph structure using lists of adjacent vertices for each vertex
- Implemented as array or hash table of linked lists enhances flexibility and performance
- Lists contain only directly connected vertices reducing memory usage for sparse graphs
- Efficient for sparse graphs with few edges compared to total possible connections
- Space-efficient representation especially beneficial for large, sparse networks (social networks)
- Less efficient for dense graphs where most vertices are connected to each other
- Checking edge existence takes $O(d)$ time where d is vertex degree potentially slower for high-degree vertices

Construction of adjacency lists

- Undirected graphs require each edge to appear in two lists maintaining symmetry
- Directed graphs list edges in only one direction following edge orientation
- Weighted graphs store weight with adjacent vertex using pairs or custom objects (vertex, weight)
- Construction process: 1. Initialize empty lists for each vertex 2. Iterate through edges, adding vertices to appropriate lists 3. For undirected graphs, add both vertices for each edge ensuring bidirectional

representation

Space complexity: Lists vs matrices

- Adjacency list space complexity $O(V + E)$ scales with actual graph structure
- Adjacency matrix space complexity $O(V^2)$ regardless of edge count
- Lists more efficient for sparse graphs common in real-world applications (road networks)
- Matrices potentially more efficient for very dense graphs rare in practice
- Lists scale better with increasing graph size for most applications (social networks, web graphs)

Edge lists and adjacency lists

- Edge lists store all edges in single list or array simple and straightforward
- Each element contains source vertex, destination vertex, and weight if applicable
- Can be used to construct adjacency lists by grouping edges by source vertex
- Adjacency lists provide faster access to vertex neighbors beneficial for many algorithms
- Edge lists suitable for algorithms processing all edges sequentially (Kruskal's algorithm)
- Efficient for adding or removing edges without restructuring entire representation

Implementation with graph data structures

- Depth-First Search (DFS) with adjacency lists iterates through neighbor lists recursively or using stack
- Breadth-First Search (BFS) uses queue to explore all neighbors before moving to next level
- Dijkstra's algorithm with adjacency lists uses priority queue to select next vertex, updates distances by iterating neighbor lists
- Kruskal's algorithm with edge lists sorts edges by weight, uses disjoint-set data structure for cycle detection
- Prim's algorithm with adjacency lists uses priority queue for edge selection, updates key values through neighbor iteration

3.3 Graph isomorphism and automorphism

Graph isomorphism and automorphism are key concepts in structural graph analysis. Isomorphism checks if two graphs are structurally identical, while automorphism maps a graph to itself. These ideas help us understand graph symmetry and equivalence.

Determining isomorphism can be tricky, with various strategies like invariant checking and canonical labeling. Automorphisms reveal graph symmetries, with counting techniques providing insights into a graph's structure. These concepts are crucial for comparing and classifying graphs.

Graph Isomorphism

Examples of graph isomorphism

- Graph isomorphism maps vertices between two graphs preserves adjacency relationships

- Isomorphic graphs share same number of vertices, edges, and identical degree sequences
- Cycle graphs with equal vertices exhibit isomorphism (C_4 and rotated C_4)
- Complete graphs K_n isomorphic to any other K_n (K_3 and rotated K_3)
- Path graphs P_n isomorphic when same length (P_3 and flipped P_3)

Complexity of graph isomorphism problem

- Graph isomorphism problem determines if two graphs are structurally identical
- Neither known NP-complete nor in P, belongs to NP (nondeterministic polynomial time)
- Quasi-polynomial time algorithm exists, discovered by László Babai in 2015
- Efficient algorithms available for special graph classes (trees, planar graphs)
- Heuristic approaches employed for general cases in practical applications

Strategies for isomorphism determination

- Invariant checking compares vertex count, edge count, and degree sequences
- Canonical labeling assigns unique vertex labels based on structural properties
- Backtracking algorithms systematically explore possible vertex mappings
- Graph canonization transforms graphs into standard forms for comparison
- Nauty algorithm provides efficient isomorphism testing and canonical labeling

Graph Automorphism

Relationship of automorphism to isomorphism

- Graph automorphism maps a graph to itself preserving adjacency
- Automorphisms subset of isomorphisms, every graph has identity automorphism
- Automorphism group contains all graph automorphisms, closed under composition

Counting graph automorphisms

- Symmetry in graphs leads to multiple automorphisms (rotational, reflectional)
- Orbit-stabilizer theorem and Burnside's lemma aid in automorphism identification
- $|Aut(G)|$ denotes automorphism count, order of automorphism group
- Complete graph K_n has $n!$ automorphisms (all vertex permutations)
- Cycle graph C_n has $2n$ automorphisms (n rotations, n reflections)
- Path graph P_n has 2 automorphisms (identity, reversal)
- Nauty algorithm computes automorphism groups efficiently
- Polynomial-time algorithms exist for trees and planar graphs

Unit 4 – Connectivity and Traversability

4.1 Vertex and edge connectivity

Connectivity in graphs measures how well-connected a structure is. It's crucial for understanding network strength and weak points. Vertex and edge connectivity tell us the minimum elements needed to disconnect a graph.

Calculating connectivity involves algorithms like Ford-Fulkerson and max-flow min-cut. The relationship between connectivity types is expressed as $\kappa(G) \leq \lambda(G) \leq \delta(G)$, with Whitney's theorem providing key insights into graph structure and vulnerability.

Connectivity Concepts

Vertex and edge connectivity

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Vertex connectivity ($\kappa(G)$) represents minimum vertices removed to disconnect graph measures vertex-disjoint paths between any two vertices (complete graphs $\kappa(G) = n - 1$, n = number of vertices)

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Edge connectivity ($\lambda(G)$) represents minimum edges removed to disconnect graph measures edge-disjoint paths between any two vertices (complete graphs $\lambda(G) = n - 1$, n = number of vertices)

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Connectivity relates to graph structure higher connectivity indicates robust well-connected graph lower connectivity suggests vulnerable points (bridges, cut-vertices)

Calculation of graph connectivity

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Vertex connectivity calculation employs Menger's theorem $\kappa(G)$ equals minimum vertex-disjoint paths between non-adjacent vertices uses Ford-Fulkerson algorithm converting graph to flow network finding maximum flow

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Edge connectivity calculation utilizes max-flow min-cut theorem finds minimum cut in graph applies global min-cut algorithms (Karger's, Stoer-Wagner)

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Manual calculation steps 1. Identify potential vertex or edge cuts disconnecting graph 2. Remove vertices or edges systematically to find minimum number required 3. Verify no smaller cut exists

Connectivity vs minimum degree

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Inequality relationship $\kappa(G) \leq \lambda(G) \leq \delta(G)$ where $\kappa(G)$ vertex connectivity $\lambda(G)$ edge connectivity $\delta(G)$ minimum degree of graph

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Implications vertex connectivity always less than or equal to edge connectivity edge connectivity always less than or equal to minimum degree removing edges generally less disruptive than removing vertices

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Special cases include trees $\kappa(G) = \lambda(G) = \delta(G) = 1$ and complete graphs $\kappa(G) = \lambda(G) = \delta(G) = n - 1$

Whitney inequality for connectivity

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Whitney's theorem states for any graph G with at least three vertices $\kappa(G) \leq \lambda(G) \leq \delta(G)$

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Proof outline demonstrates removing $\lambda(G) - 1$ edges cannot disconnect graph shows removing $\lambda(G)$ vertices always disconnects graph concludes $\kappa(G) \leq \lambda(G)$

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Key proof steps use contradiction to prove $\kappa(G) \leq \lambda(G)$ show removing $\delta(G) - 1$ edges from vertex cannot isolate it conclude $\lambda(G) \leq \delta(G)$

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Significance provides bounds for connectivity measures helps estimate graph vulnerability and robustness (network reliability, fault tolerance)

4.2 Cut-vertices and bridges

Cut-vertices and bridges are crucial elements in graph theory. They're the weak points that, when removed, can split a graph into separate pieces. Understanding these concepts helps us analyze network vulnerabilities and design more robust systems.

Identifying cut-vertices and bridges involves clever algorithms like Depth-First Search and Tarjan's method. These techniques are essential for spotting potential bottlenecks in networks, finding biconnected components, and breaking down complex graphs into simpler parts.

Cut-vertices and Bridges in Graph Theory

Define cut-vertices and bridges

- Cut-vertex (articulation point) removal increases connected components in graph crucial for maintaining connectivity
- Bridge (cut-edge) removal increases connected components provides sole path between graph sections

Identify cut-vertices and bridges in a graph

- Methods for identifying cut-vertices utilize Depth-First Search (DFS) algorithm and calculate low-point values
- Techniques for finding bridges employ Tarjan's algorithm and classify edges during DFS traversal

Explain the importance of cut-vertices and bridges in graph connectivity

- Role in network reliability pinpoints single points of failure and identifies bottlenecks (communication networks)
- Applications in graph theory include identifying biconnected components and facilitating graph decomposition

Describe algorithms for finding cut-vertices and bridges

- Tarjan's algorithm achieves linear time complexity $O(V + E)$ utilizes DFS and low-point values
- Hopcroft-Tarjan algorithm extends Tarjan's approach finds biconnected components efficiently

Analyze the impact of removing cut-vertices or bridges on graph properties

- Effects on graph connectivity increase connected components create isolated vertices or subgraphs
- Changes in graph metrics alter diameter average path length clustering coefficient

Discuss the relationship between cut-vertices, bridges, and graph connectivity

- Connectivity measures include vertex connectivity and edge connectivity quantify graph robustness
- Menger's theorem establishes relationship between connectivity and number of disjoint paths
- Block decomposition breaks graph into maximal biconnected subgraphs reveals structural properties

4.3 Network reliability and vulnerability

Network reliability and vulnerability are crucial aspects of graph theory. They measure a network's ability to function under specific conditions and identify weak points. These concepts are essential for understanding how networks maintain connectivity and resist disruptions.

Calculating network reliability involves assessing edge and vertex survival probabilities. Various methods, from exact calculations to approximations, help determine a network's overall reliability. Identifying vulnerable elements and implementing strategies for resilience are key to maintaining robust network performance.

Network Reliability Concepts

Network reliability and vulnerability concepts

- Network reliability measures probability of network functionality under specific conditions maintains connectivity
- Network vulnerability assesses susceptibility to disruptions identifies weak points in structure
- Graph theory represents networks as graphs vertices (nodes or components) connected by edges (links)

Calculation of network reliability

- Edge reliability calculation considers probability of edge survival p and failure $1 - p$ determines likelihood of path existence between all vertex pairs
- Vertex reliability calculation focuses on vertex failures assesses connectivity of remaining vertices
- Methods for calculating network reliability: 1. Exact methods for small networks enumerate all possible states use state space models 2. Approximation methods for large networks employ Monte Carlo simulation bounds and estimates

Identification of vulnerable network elements

- Centrality measures assess importance:
- Degree centrality counts vertex connections
- Betweenness centrality evaluates frequency in shortest paths
- Closeness centrality calculates average distance to all other vertices
- Cut vertices and bridges critical for connectivity:
- Cut vertex removal disconnects graph
- Bridge removal increases connected components
- Network flow analysis applies maximum flow and minimum cut theorems identifies bottlenecks

Strategies for network resilience

- Redundancy adds alternative paths increases edges or vertices
- Load balancing distributes traffic across multiple paths reduces stress on critical components
- Robust network topologies enhance resilience:
- Mesh networks provide high interconnectivity
- Scale-free networks resist random failures
- Backup systems and failover mechanisms implement secondary components enable automatic switching upon failure
- Regular maintenance and monitoring proactively address vulnerabilities update and upgrade components
- Security measures protect against intentional attacks implement access controls and encryption (firewalls, VPNs)

4.4 Graph traversal algorithms (DFS and BFS)

Graph traversal algorithms are essential tools for exploring and analyzing complex networks. DFS and BFS offer distinct approaches to navigating graphs, each with unique strengths and applications in various fields.

These algorithms form the backbone of many real-world systems, from social media platforms to GPS navigation. Understanding their implementations, complexities, and use cases is crucial for solving diverse graph-related problems efficiently.

Graph Traversal Algorithms: DFS and BFS

DFS vs BFS algorithms

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Depth-First Search (DFS)

- Explores branches to their full depth before backtracking
- Utilizes stack or recursion for implementation
- Traverses paths to their ends before exploring alternatives
- Can be implemented recursively or iteratively
- Well-suited for decision trees and puzzle-solving (maze navigation)

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Breadth-First Search (BFS)

- Explores all vertices at current depth before moving deeper
- Employs queue for implementation
- Visits vertices based on distance from starting point
- Always implemented iteratively
- Ideal for finding shortest paths in unweighted graphs (social network connections)

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DFS vs BFS comparison

- Path finding: DFS may not find shortest path, BFS guarantees shortest in unweighted graphs
- Memory usage: DFS efficient for deep graphs, BFS for wide graphs
- Applications: DFS excels in decision trees, BFS in shortest path problems

Implementation of graph traversal

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DFS Implementation

- Recursive approach
 1. Check if current vertex is null or visited (base case)
 2. Mark current vertex as visited and process it
 3. Recursively call DFS on unvisited adjacent vertices
- Iterative approach
 1. Initialize stack with starting vertex
 2. While stack not empty, pop vertex

3. Mark popped vertex as visited and process it
4. Push unvisited adjacent vertices to stack
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BFS Implementation

- Iterative approach only

 1. Initialize queue with starting vertex
 2. While queue not empty, dequeue vertex
 3. Mark dequeued vertex as visited and process it
 4. Enqueue unvisited adjacent vertices

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Graph representation options

- Adjacency list more memory-efficient for sparse graphs
- Adjacency matrix offers faster edge lookup for dense graphs

Complexity analysis of search algorithms

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Time complexity

- Both DFS and BFS: $O(V + E)$ for adjacency list, $O(V^2)$ for adjacency matrix
- V represents vertex count, E represents edge count
- Vertex visitation: $O(V)$
- Edge exploration: $O(E)$
-

Space complexity

- DFS: $O(h)$, h is tree height (worst case $O(V)$)
- BFS: $O(w)$, w is maximum tree width (worst case $O(V)$)
- Adjacency list storage: $O(V + E)$
- Adjacency matrix storage: $O(V^2)$

Applications in graph connectivity

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DFS applications

- Cycle detection in graphs (detecting loops in computer networks)
- Topological sorting of directed acyclic graphs (task scheduling)

- Identifying strongly connected components (analyzing road networks)
- Solving mazes or puzzles (pathfinding in games)
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BFS applications

- Finding shortest paths in unweighted graphs (minimal hops in computer networks)
- Testing graph bipartiteness (two-coloring problems)
- Identifying connected components in undirected graphs (social network clusters)
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Real-world uses of graph traversal

- Web crawling for search engines (Google's indexing system)
- Social network analysis (Facebook's friend suggestion algorithm)
- GPS navigation (finding optimal routes)
- Network broadcast protocols (efficient information dissemination)
- Garbage collection in programming languages (memory management in Java)

Unit 5 – Trees and Forests

5.1 Tree properties and characterizations

Trees are fundamental structures in graph theory, characterized by their connected and acyclic nature. They possess unique properties that make them essential for various applications, from computer science algorithms to network design.

Understanding trees is crucial for grasping more complex graph concepts. Their simple yet powerful structure, with unique paths between vertices and specific vertex-edge relationships, forms the basis for many advanced graph algorithms and problem-solving techniques.

Tree Fundamentals

Definition and properties of trees

- Connected acyclic graphs form trees where vertices link via paths without loops
- Key properties characterize trees:
 - Minimally connected structure breaks apart by removing any edge
 - Maximally acyclic configuration forms cycle by adding any edge
- Unique path exists between any two vertices ensures efficient traversal
- Special cases illustrate tree concept:
 - Single vertex represents simplest tree structure
 - Path graph contains exactly two leaf nodes at ends (linear chain)

Unique paths between tree vertices

- Proof by contradiction demonstrates single path between vertices: 1. Assume two distinct paths exist between vertices u and v 2. Combining these paths would create a cycle 3. Cycle presence contradicts acyclic tree property
- Unique path consequences impact tree utility:
 - No redundant connections streamline structure
 - Efficient for algorithms and data structures (binary search trees, decision trees)

Vertex and edge relationships

- Vertex-edge relationship defines tree structure:
 - Number of edges equals number of vertices minus one ($|E| = |V| - 1$)
- Proof uses induction on vertex count: 1. Base case: single vertex tree has zero edges 2. Inductive step: adding new leaf increases both vertices and edges by one
- Applications of this relationship:
 - Quick validation of tree-like structures in networks

- Analysis tool for spanning trees in graphs

Conditions for tree classification

- Equivalent conditions identify trees:
- Connected and acyclic graph structure
- Connected with edge count $|E| = |V| - 1$
- Acyclic with edge count $|E| = |V| - 1$
- Unique path exists between any vertex pair
- Minimally connected (removing any edge disconnects)
- Maximally acyclic (adding any edge creates cycle)
- Verification methods ensure tree properties:
- Depth-First Search (DFS) checks connectivity and cycle absence
- Vertex and edge counting confirms structural relationship
- Path uniqueness testing between vertices
- Importance in graph algorithms:
- Enables efficient tree recognition in complex networks
- Serves as foundation for advanced graph problem-solving (minimum spanning trees, network design)

5.2 Rooted trees and binary trees

Tree structures are fundamental in graph theory, forming hierarchical relationships between nodes. Rooted trees establish a clear structure with a designated root, while binary trees limit nodes to two children. These structures enable efficient data organization and manipulation.

Tree traversal methods like preorder, inorder, and postorder allow systematic exploration of tree nodes. Binary tree properties find applications in various domains, including search algorithms, data compression, and machine learning, showcasing their versatility and importance in computer science.

Tree Structures and Components

Components of rooted trees

- Rooted trees form connected acyclic graphs with designated root node establishing hierarchical structure through directed edges
- Root node sits at top of tree hierarchy without parent node
- Parent nodes connect directly to child nodes moving towards root
- Child nodes connect directly to parent nodes moving away from root
- Leaves (terminal nodes) have no children
- Internal nodes exist between root and leaves with both parent and child nodes
- Siblings share same parent node

- Depth of node measures path length from root
- Height of tree equals maximum depth of any node

Ordered vs unordered rooted trees

- Ordered rooted trees maintain specific left-to-right order for children nodes (file systems)
- Unordered rooted trees disregard sibling order (set representations)
- Isomorphic ordered trees with different sibling arrangements considered distinct
- Isomorphic unordered trees with different sibling arrangements considered equivalent

Binary trees and subtrees

- Binary trees limit nodes to maximum two children designated as left and right
- Full binary trees contain nodes with either 0 or 2 children
- Complete binary trees fill all levels except possibly last from left to right
- Perfect binary trees have all internal nodes with two children and leaves at same level
- Balanced binary trees maintain height difference of at most one between left and right subtrees

Binary tree traversal methods

- Preorder traversal: visit root, traverse left subtree, traverse right subtree (copying trees)
- Inorder traversal: traverse left subtree, visit root, traverse right subtree (sorted order for binary search trees)
- Postorder traversal: traverse left subtree, traverse right subtree, visit root (node deletion)
- Implement traversals recursively or iteratively using stack

Applications of binary tree properties

- Height analysis: $O(\log n)$ for balanced trees, $O(n)$ for skewed trees
- Node count in perfect binary tree: $2^{h+1} - 1$ where h is height
- Array representation for complete binary trees, linked representation for general binary trees
- Binary search trees enable $O(\log n)$ average case search time
- Tree balancing techniques (AVL trees, Red-Black trees) optimize performance
- Expression trees facilitate arithmetic operations
- Huffman coding enables data compression
- Decision trees support machine learning algorithms

5.3 Spanning trees and forests

Spanning trees and forests are crucial concepts in graph theory, connecting all vertices without cycles. They minimize edge count while maintaining connectivity, making them ideal for network backbones and distributed systems. Their properties ensure efficient and resilient structures.

Minimum spanning tree algorithms like Kruskal's and Prim's optimize resource allocation in various applications. These algorithms efficiently find the lowest-weight tree, crucial for designing power grids, road networks, and social networks. Understanding their uniqueness and applications is essential for solving real-world problems.

Spanning Trees and Forests

Properties of spanning trees

- Spanning tree connects all vertices of graph without cycles minimizes edge count to $n - 1$ for n vertices (network backbone)
- Spanning forest extends concept to disconnected graphs creates separate trees for each component (distributed systems)
- Minimally connected removing any edge breaks connectivity maximally acyclic adding edge forms cycle (network resilience)
- All spanning trees of graph have same edge count $n - 1$ provides consistent structure (network planning)
- Applications include efficient network design (computer networks) and hierarchical clustering (data analysis)

Minimum spanning tree algorithms

- Minimum spanning tree (MST) minimizes total edge weight optimizes resource allocation (power grid)
- Kruskal's algorithm: 1. Sort edges by ascending weight 2. Iterate through sorted edges 3. Add edge if no cycle formed 4. Use disjoint set for cycle detection
- Time complexity $O(E \log E)$ or $O(E \log V)$ efficient for sparse graphs (road networks)
- Prim's algorithm: 1. Start with arbitrary vertex 2. Add lowest-weight edge to new vertex 3. Use priority queue for edge selection
- Time complexity $O(E \log V)$ with binary heap $O(E + V \log V)$ with Fibonacci heap suitable for dense graphs (social networks)

Uniqueness of minimum spanning trees

- Proof by contradiction assumes two different MSTs exist
- Find lowest-weight edge in symmetric difference
- Adding edge to one MST creates cycle
- Removing any edge from cycle yields spanning tree
- Resulting tree has lower weight contradicts MST assumption
- Distinct edge weights guarantee unique MST
- Non-distinct weights may result in multiple MSTs (network design flexibility)

Applications of spanning forests

- Network design optimization minimizes cable length (computer networks) optimizes road construction (transportation systems)
- Cluster analysis performs single-linkage clustering identifies communities (social networks)
- Image segmentation partitions images into meaningful regions (computer vision)
- Circuit design minimizes power consumption (integrated circuits)
- Approximation algorithms solve Traveling Salesman Problem (logistics)
- Maze generation creates randomized spanning trees (game development)
- Phylogenetic tree construction reconstructs evolutionary relationships (biology)

Unit 6 – Spanning Trees and Minimum Spanning Trees

6.1 Algorithms for finding spanning trees

Spanning trees are crucial subgraphs that connect all vertices without cycles. They're used in network design and data clustering. With $V-1$ edges, they maintain connectivity while removing any edge disconnects the graph.

DFS and BFS are key algorithms for creating spanning trees. DFS explores deep paths first, using a stack or recursion. BFS visits all neighbors before going deeper, using a queue. Both have $O(V+E)$ time complexity but differ in tree structure and applications.

Spanning Trees and Search Algorithms

Properties of spanning trees

- Spanning tree connects all vertices forming acyclic subgraph of connected undirected graph
- Contains exactly $V - 1$ edges (V = number of vertices) maintaining connectivity without cycles
- Removing any edge disconnects graph while adding edge creates cycle
- Applications include optimizing network design (minimizing cable length) and hierarchical clustering (grouping data points)

DFS for spanning trees

- DFS explores graph depth-first traversing as far as possible before backtracking
- Steps: 1. Start at arbitrary vertex 2. Explore unvisited adjacent vertex 3. Repeat step 2 until dead-end reached 4. Backtrack to last vertex with unexplored neighbors 5. Repeat until all vertices visited
- Implementation uses stack or recursion tracking visited vertices
- Time complexity $O(V + E)$ visiting each vertex and edge once
- Produces "deep" tree with potentially long paths (may not find shortest routes)

BFS for spanning trees

- BFS explores graph breadth-first visiting all neighbors before moving deeper
- Steps: 1. Start at arbitrary vertex 2. Visit all unvisited adjacent vertices 3. Enqueue visited vertices 4. Dequeue vertex and repeat steps 2-3 5. Continue until all vertices visited
- Implementation utilizes queue data structure for vertex processing
- Time complexity $O(V + E)$ examining each vertex and edge once
- Generates "broad" tree with shorter paths (finds shortest routes in unweighted graphs)

Comparison of DFS vs BFS

- Time complexity both $O(V + E)$ traversing all vertices and edges
- Space complexity DFS $O(h)$ (h = tree height) vs BFS $O(w)$ (w = maximum tree width)
- Tree structure DFS produces deep trees while BFS creates broad trees
- Use cases DFS suited for backtracking problems (maze solving) BFS ideal for shortest path finding (social network connections)
- Memory usage DFS generally more efficient for graphs with many long paths
- Implementation DFS easier to implement recursively BFS typically uses iterative approach with queue

6.2 Minimum spanning tree algorithms (Kruskal's and Prim's)

Minimum spanning trees (MSTs) are crucial in graph theory, connecting all vertices with the least total edge weight. They're used in network design, clustering, and solving complex problems efficiently.

Two main algorithms find MSTs: Kruskal's and Prim's. Kruskal's sorts edges and builds the tree, while Prim's grows from a starting point. Both are proven correct using the Cut Property and induction.

Understanding Minimum Spanning Trees

Properties of minimum spanning trees

- Minimum Spanning Tree (MST) connects all vertices in weighted, connected graph minimizes total edge weight without cycles
- Contains exactly $n - 1$ edges where n is number of vertices
- Unique with distinct edge weights may not be unique with equal weights
- Applications include network design optimizes connectivity costs (computer networks, transportation systems)
- Used in clustering algorithms groups similar data points (image segmentation, customer segmentation)
- Aids approximation algorithms for NP-hard problems provides near-optimal solutions (traveling salesman problem)

Algorithms for Finding Minimum Spanning Trees

Kruskal's algorithm for MST

- Steps: 1. Sort edges in non-decreasing weight order 2. Initialize forest with each vertex as separate tree 3. Iterate through sorted edges 4. Add edge to MST if it connects different trees 5. Merge connected trees 6. Continue until $n - 1$ edges added
- Uses disjoint-set data structure for efficient merging and checking (union-find)
- Time Complexity: $O(E \log E)$ or $O(E \log V)$
- E number of edges V number of vertices
- Sorting dominates time complexity

- Efficient for sparse graphs fewer edges to sort and process

Prim's algorithm for MST

- Steps: 1. Start with arbitrary vertex 2. Maintain set of visited vertices 3. Add minimum weight edge connecting visited to unvisited vertex 4. Repeat until all vertices visited
- Uses priority queue for efficient minimum weight edge selection (binary heap, Fibonacci heap)
- Time Complexity:
 - Binary heap: $O((V + E)\log V)$
 - Fibonacci heap: $O(E + V\log V)$
- Efficient for dense graphs processes fewer edges overall

Kruskal's vs Prim's algorithms

- Approach: Kruskal's global considers all edges Prim's local grows single tree
- Efficiency: Kruskal's better for sparse graphs Prim's better for dense graphs
- Implementation: Kruskal's requires edge sorting Prim's needs efficient priority queue
- Memory: Kruskal's stores all edges Prim's stores vertices and adjacent edges
- Parallelization: Kruskal's easier to parallelize Prim's inherently sequential
- Choose based on graph structure and implementation constraints

Correctness proofs of MST algorithms

- Proof strategy uses Cut Property and induction
- Cut Property: Minimum weight edge crossing any cut in graph is in MST
- Kruskal's Proof:
 - Induction on number of edges added
 - Each added edge is safe doesn't create cycle
 - Final tree is spanning and minimum
- Prim's Proof:
 - Induction on number of vertices added to tree
 - Each added edge is minimum weight crossing a cut
 - Final tree is spanning and minimum
- Exchange Argument proves minimality:
 - Non-MST edge can be exchanged with MST edge without increasing total weight
- Both algorithms guaranteed to find correct MST

6.3 Applications of minimum spanning trees

Minimum Spanning Trees (MSTs) are powerful tools for optimizing network design and connectivity. From telecommunications to power grids, MSTs help minimize costs while maximizing efficiency in various real-world applications.

MSTs also play a crucial role in clustering, data compression, and approximation algorithms. They're used in image segmentation, phylogenetic tree construction, and solving complex problems like the Traveling Salesman Problem, showcasing their versatility in graph theory applications.

Real-World Applications of Minimum Spanning Trees

Real-world applications of MSTs

- Network design problems span various sectors minimizing costs and optimizing connectivity
- Telecommunications networks reduce cable length and infrastructure expenses
- Electrical power grids efficiently distribute electricity across regions
- Water supply systems optimize pipe layouts for cost-effective distribution
- Road network planning minimizes construction costs while maximizing accessibility
- Computer network topology designs efficient data transmission paths
- Local Area Networks (LANs) connect devices within a limited area
- Wide Area Networks (WANs) link geographically dispersed locations
- Circuit board design routes electrical connections minimizing wire length and interference
- Transportation systems optimize routes for efficiency and coverage
- Bus routes maximize passenger access while minimizing fuel consumption
- Subway lines connect stations with minimal track length and construction costs
- Cluster analysis groups similar data points or objects
- Image segmentation identifies distinct regions in digital images
- Data clustering organizes large datasets into meaningful groups (customer segmentation)

MSTs for network design

- Kruskal's algorithm constructs MST by iteratively adding edges 1. Sort edges by weight in ascending order 2. Add edges to the tree if they don't create cycles 3. Continue until all vertices are connected
- Prim's algorithm grows MST from a single vertex 1. Start with an arbitrary vertex 2. Add lowest-weight edge connecting tree to unvisited vertex 3. Repeat until all vertices are in the tree
- Applications in network design optimize resource allocation and efficiency
- Minimize total cable length in computer networks reducing material costs
- Optimize pipeline layouts in oil and gas distribution enhancing flow efficiency
- Design efficient transportation routes minimizing fuel consumption and travel time

MSTs in clustering and compression

- Clustering applications group similar data points or objects
- Single-linkage clustering merges clusters based on minimum distance between points
- Hierarchical clustering builds a tree-like structure of nested clusters
- Dendrogram construction visualizes hierarchical relationships in data
- Image segmentation partitions digital images into meaningful regions
- Pixel clustering based on color similarity identifies objects or textures
- Object recognition in computer vision detects and classifies image components
- Data compression reduces file sizes while preserving information
- Huffman coding assigns variable-length codes based on symbol frequency
- Constructing optimal prefix codes minimizes average code length
- Phylogenetic tree construction reveals evolutionary relationships
- Evolutionary biology traces species lineages and common ancestors
- Species classification organizes organisms based on genetic or morphological similarities

MSTs for approximation algorithms

- Traveling Salesman Problem (TSP) approximation finds near-optimal tours
- Double tree algorithm doubles MST edges and shortcuts repeated vertices
- Christofides algorithm combines MST with minimum-weight perfect matching
- Steiner tree problem approximation connects a subset of vertices
- MST-based 2-approximation algorithm provides guaranteed performance bound
- k -MST problem finds minimum weight tree with exactly k vertices
- Used in network design with budget constraints
- Metric TSP leverages MST properties for improved solutions
- Using MST as a lower bound helps evaluate solution quality
- Performance guarantees provide worst-case solution quality assurance
- Approximation ratios for MST-based algorithms quantify solution accuracy (1.5 for Christofides algorithm)

Unit 7 – Eulerian and Hamiltonian Graphs

7.1 Eulerian circuits and trails

Eulerian circuits and trails are fascinating graph structures that traverse every edge exactly once. They're named after Leonhard Euler, who solved the famous Seven Bridges of Königsberg problem in 1736, laying the foundation for graph theory.

To have an Eulerian circuit, all vertices must have even degree and the graph must be connected. For an Eulerian trail, exactly two vertices need odd degree. Algorithms like Hierholzer's and Fleury's help construct these paths efficiently.

Eulerian Circuits and Trails

Definition of Eulerian circuits and trails

- Eulerian trail traverses every edge exactly once allowing repeated vertices starts and ends at different vertices (Seven Bridges of Königsberg)
- Eulerian circuit forms closed path visiting every edge once starts and ends at same vertex (Tour of a museum)
- Concept named after Leonhard Euler originated from Königsberg bridge problem in 1736 laid foundation for graph theory

Conditions for Eulerian paths

- Eulerian circuit requires all vertices have even degree and graph is connected excluding isolated vertices (Complete graph K_4)
- Eulerian trail needs exactly two vertices with odd degree others even and graph connected excluding isolated vertices (Path graph P_4)
- Degree of vertex counts number of edges incident to it determines Eulerian properties
- Connected graph ensures path exists between any two vertices crucial for Eulerian paths

Identification of Eulerian graphs

- Check Eulerian circuit: 1. Verify graph connectivity 2. Count degree of each vertex 3. Ensure all vertices have even degree
- Determine Eulerian trail: 1. Verify graph connectivity 2. Count degree of each vertex 3. Confirm exactly two vertices have odd degree
- Empty graph considered to have Eulerian circuit while single vertex graph has trivial Eulerian circuit

Construction of Eulerian circuits

- Hierholzer's algorithm builds Eulerian circuit: 1. Start at any vertex 2. Follow unused edges until returning to start 3. Repeat for vertices with unused edges 4. Combine circuits
- Fleury's algorithm constructs Eulerian path: 1. Start at odd degree vertex for trail 2. Choose edges that maintain connectivity 3. Remove used edges progressively

- Eulerian trail construction starts at odd degree vertex applies chosen algorithm ends at other odd degree vertex
- Consider time complexity and efficiency when handling large graphs ($O(E)$ for Hierholzer, $O(E^2)$ for Fleury)

7.2 Hamiltonian cycles and paths

Hamiltonian cycles and paths are graph traversals that visit every vertex exactly once. Cycles return to the start, while paths can have different endpoints. These concepts, introduced by William Hamilton, are crucial in graph theory and optimization problems.

Understanding Hamiltonian graphs involves key theorems like Dirac's and Ore's, which provide conditions for their existence. Identifying these structures uses techniques from brute force to heuristic methods, with applications in the famous Traveling Salesman Problem.

Hamiltonian Cycles and Paths Fundamentals

Definition of Hamiltonian cycles and paths

- Hamiltonian path traverses every vertex in a graph exactly once without revisiting allows different start and end vertices (Königsberg Bridge problem)
- Hamiltonian cycle visits every vertex in a graph exactly once returns to starting vertex forms closed loop (Dodecahedron graph)
- William Rowan Hamilton introduced these concepts while studying dodecahedron graph in 1800s led to development of Icosian game

Hamiltonian cycles vs paths

- End points differ Hamiltonian path can have different start and end vertices while cycle must return to start
- Edge count Hamiltonian path uses $n - 1$ edges in n -vertex graph cycle uses n edges
- Closure property Hamiltonian path forms open walk cycle creates closed walk
- Flexibility paths offer more options for traversal cycles more restrictive due to return requirement

Properties of Hamiltonian graphs

- Dirac's Theorem ensures Hamiltonian cycle if minimum vertex degree $\geq n/2$ (Complete graph K_5)
- Ore's Theorem guarantees Hamiltonian cycle if degree sum of non-adjacent vertices $\geq n$ (Petersen graph)
- Complete graphs K_n always contain Hamiltonian cycles for $n \geq 3$ (Triangle, Square, Pentagon)
- Bipartite graphs can have Hamiltonian cycles only with equal-sized partite sets (Hypercube graph)
- All Platonic solids possess Hamiltonian cycles due to symmetry and connectivity (Cube, Tetrahedron)

Techniques for identifying Hamiltonian graphs

- Bondy-Chvátal Theorem uses graph closure by adding edges based on degree sum

- Pósa's Theorem provides sufficient condition for Hamiltonian paths based on vertex degree distribution
- Traveling Salesman Problem closely related to finding optimal Hamiltonian cycles in weighted graphs
- Brute force approach checks all vertex permutations becomes impractical for large graphs
- Backtracking algorithms build and explore potential solutions systematically (Depth-First Search)
- Heuristic methods approximate solutions:
 1. Nearest neighbor algorithm greedily selects closest unvisited vertex
 2. Christofides algorithm guarantees solution within 1.5 times optimal for metric TSP

7.3 Necessary and sufficient conditions for Eulerian and Hamiltonian graphs

Eulerian and Hamiltonian graphs are key concepts in graph theory. Eulerian graphs focus on traversing all edges, while Hamiltonian graphs aim to visit all vertices. These ideas have important applications in network design and optimization.

Eulerian graphs have clear conditions and efficient algorithms. In contrast, Hamiltonian graphs are more complex, with no complete characterization. Understanding these differences helps grasp the varying difficulty levels in graph theory problems.

Eulerian Graphs

Conditions for Eulerian graphs

- Eulerian circuit traverses closed trail visiting every edge exactly once requires connected graph with all vertices having even degree
- Eulerian trail traverses path visiting every edge exactly once needs connected graph with exactly two vertices having odd degree and all others even degree

Proof of Eulerian conditions

- Necessity proof assumes graph has Eulerian circuit implying vertices entered and exited equal number of times resulting in even degree for all vertices
- Sufficiency proof assumes connected graph with all vertices of even degree uses induction on number of edges starting with base case of graph with no edges being Eulerian then removes any cycle and recurses on remaining graph

Vertex degrees in Eulerian circuits

- Sum of vertex degrees in any graph always even
- Eulerian circuit mandates all even degrees
- Eulerian trail permits exactly two odd degree vertices serving as start and end points
- Handshaking lemma states $\sum_{v \in V} \deg(v) = 2|E|$ (total degree equals twice number of edges)

Hamiltonian Graphs

Complexity of Hamiltonian graphs

- Hamiltonian cycle problem classified as NP-complete lacking known polynomial-time algorithm for general graphs
- Contrasts with Eulerian graphs solvable in polynomial time
- No complete characterization of Hamiltonian graphs exists only necessary or sufficient conditions
- Hamiltonian cycle visits every vertex exactly once (Königsberg Bridge Problem)

Sufficient conditions for Hamiltonian graphs

- Dirac's theorem provides sufficient condition for Hamiltonicity stating graph G with $n \geq 3$ vertices and minimum degree $\delta(G) \geq n/2$ is Hamiltonian
- Ore's theorem generalizes Dirac's theorem asserting graph G with $n \geq 3$ vertices and $\deg(u) + \deg(v) \geq n$ for all non-adjacent vertices u and v is Hamiltonian
- Bondy-Chvátal theorem introduces closure of graph adding edges between non-adjacent vertices u, v if $\deg(u) + \deg(v) \geq n$ proving graph is Hamiltonian if and only if its closure is Hamiltonian

Unit 8 – Shortest Paths and Distance in Graphs

8.1 Dijkstra's algorithm for shortest paths

Dijkstra's algorithm finds the shortest path in weighted graphs. It's crucial for optimizing routes in various applications, from navigation systems to network routing. The algorithm efficiently determines the best path by minimizing total weight between start and end nodes.

Understanding Dijkstra's algorithm involves grasping its steps, time complexity, and implementation. It uses a priority queue to improve efficiency, making it suitable for sparse graphs. However, it can't handle negative edge weights or cycles.

Understanding Dijkstra's Algorithm

Shortest paths in weighted graphs

- Path between vertices minimizes total weight sum connects start to end node
- Weight numerical value assigned to edges represents cost, distance, or time (fuel consumption, travel time)
- Optimal route determination crucial for efficient resource allocation and decision-making
- Applications span navigation systems, network routing, logistics optimization, social network analysis
- Fundamental in graph theory solves real-world problems across diverse domains
- Types include single-source (one starting point) and all-pairs (between every pair of vertices)

Steps of Dijkstra's algorithm

1. Initialize: - Set source vertex distance to 0, others to infinity - Create unvisited vertices set
2. Main loop: - Select minimum distance vertex from unvisited set - Mark selected vertex as visited - Update neighboring vertices' distances
3. Relaxation: - Apply formula $d[v] = \min(d[v], d[u] + w(u, v))$ - $d[v]$: current distance to v, $d[u]$: distance to u, $w(u, v)$: edge weight
4. Terminate: - Algorithm concludes when all vertices visited
5. Reconstruct path: - Use predecessor information to backtrack from destination to source

Time complexity of Dijkstra's algorithm

- Naive implementation runs in $O(V^2)$ time (V: number of vertices)
- Priority queue improves to $O((V + E)\log V)$ (E: number of edges)
- Space complexity remains $O(V)$ for storing distances and visited status
- Efficient for sparse graphs fewer edges than V^2
- Less suitable for dense graphs many edges approaching V^2

- Cannot handle negative edge weights may lead to incorrect results
- Fails to find optimal solution in graphs with negative cycles

Implementation of Dijkstra's algorithm

- Represent graph using adjacency list or matrix for efficient access
- Utilize priority queue (min-heap) for vertex selection minimizes extraction time
- Store distances in array or hash table for quick updates and lookups
- Pseudocode outlines initialization, main loop, and path reconstruction
- Adapt implementation for directed/undirected and weighted/unweighted graphs
- Incorporate error handling for invalid inputs (disconnected vertices, self-loops)
- Optimize for large graphs using advanced data structures (Fibonacci heap)
- Verify correctness with test cases known shortest paths
- Consider variations bidirectional search, A* algorithm for informed search

8.2 Bellman-Ford algorithm for negative edge weights

Bellman-Ford algorithm tackles shortest path problems with negative edge weights, a scenario where Dijkstra's algorithm falls short. It can handle graphs with negative weights and detect negative cycles, making it useful for financial modeling and currency exchange analysis.

The algorithm works by relaxing all edges repeatedly, ensuring correct shortest paths even with negative weights. While slower than Dijkstra's, its versatility in handling negative weights and cycle detection makes it valuable for complex graph problems.

Understanding Bellman-Ford Algorithm

Limitations of Dijkstra's algorithm

- Dijkstra's algorithm fails with negative edge weights assumes all edge weights are non-negative and greedy approach doesn't work with negative weights
- Negative edge weights lead to incorrect shortest paths calculations
- Dijkstra's algorithm cannot detect negative cycles potentially causing infinite loops (currency exchange arbitrage)
- Fails in graphs with negative cycles renders shortest paths undefined

Bellman-Ford algorithm for negative weights

- Solves single-source shortest path problem works with negative edge weights
- Algorithm steps: 1. Initialize distances to all vertices as infinity 2. Set distance to source vertex as 0 3. Relax all edges $|V| - 1$ times 4. For each edge (u, v) with weight w , update $d[v] = \min(d[v], d[u] + w)$
- Correctly computes shortest paths even with negative weights (stock market analysis)
- Detects negative cycles through additional iteration to check for further improvements

Time complexity of Bellman-Ford

- Overall time complexity $O(|V||E|)$ where $|V|$ is number of vertices and $|E|$ is number of edges
- Main loop runs $|V| - 1$ times processes all edges in each iteration
- Space complexity $O(|V|)$ for distance array and predecessor array
- Slower but more versatile than Dijkstra's algorithm handles negative weights and detects cycles

Negative cycle detection with Bellman-Ford

- Run main loop for additional iteration (total $|V|$ times) if any distance updates occur in last iteration negative cycle exists
- Helps identify arbitrage opportunities in financial modeling (currency exchange)
- Extract negative cycle by backtracking from updated vertex in last iteration follow predecessor links until vertex revisited
- Applications include currency exchange rate analysis resource allocation problems detecting inconsistencies in systems of difference constraints

8.3 Floyd-Warshall algorithm for all-pairs shortest paths

The Floyd-Warshall algorithm tackles the all-pairs shortest path problem in graphs. It calculates the shortest distances between every pair of vertices, providing a comprehensive view of the network's structure and connectivity.

Unlike single-source algorithms, Floyd-Warshall uses dynamic programming to iteratively refine distance estimates. It considers all possible intermediate vertices, making it efficient for dense graphs and capable of handling negative edge weights.

Understanding the Floyd-Warshall Algorithm

All-pairs shortest path problem

- Computes shortest paths between every vertex pair in weighted graphs encompasses directed and undirected structures
- Problem holds critical importance in network routing optimizes transportation systems and enhances communication network efficiency
- Contrasts with single-source algorithms by calculating distances between all vertex pairs provides more comprehensive network analysis (Dijkstra's algorithm)

Floyd-Warshall algorithm fundamentals

- Employs dynamic programming approach iteratively refines distance estimates between vertices
- Utilizes intermediate vertices concept examines paths through all possible intermediate points to find shortest routes
- Algorithm process: 1. Initialize distance matrix with direct edge weights 2. For each vertex k as intermediate update distances: $d_{ij} = \min(d_{ij}, d_{ik} + d_{kj})$ 3. Repeat step 2 for all vertices as intermediates

- Incorporates path reconstruction uses predecessor matrix to store and retrieve actual shortest paths

Complexity of Floyd-Warshall algorithm

- Time complexity: $O(V^3)$ where V represents number of vertices involves three nested loops each iterating V times
- Space complexity: $O(V^2)$ for distance matrix additional $O(V^2)$ if storing predecessor matrix
- Demonstrates higher efficiency than running Dijkstra's algorithm V times particularly for dense graphs

Implementation of Floyd-Warshall algorithm

- Pseudocode structure employs three nested loops (k, i, j) updates distance matrix in-place during execution
- Input represented as adjacency matrix initialized with edge weights and infinity for non-edges
- Outputs final distance matrix after all iterations optionally includes predecessor matrix for path reconstruction
- Handles negative edge weights effectively detects presence of negative cycles in graph
- Practical considerations:
- Utilizes infinity value for unreachable vertex pairs
- Addresses numerical precision issues for large graphs or weights
- Implements efficient data structures to optimize memory usage and computation speed

8.4 Graph distance and diameter

Graph distances measure how far apart things are in networks. They help us understand connections, find central points, and see how spread out a network is. These concepts are crucial for analyzing everything from social media to transportation systems.

Calculating distances in graphs involves smart algorithms like Breadth-First Search and Dijkstra's. These methods help us find the shortest paths, central nodes, and overall network span. They're key to solving real-world problems in logistics, social networks, and urban planning.

Graph Distance Fundamentals

Concepts of graph metrics

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Graph distance measures shortest path between vertices counting edges traversed $d(u, v)$ (social networks, transportation routes)

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Eccentricity $e(v)$ determines maximum distance from vertex to any other vertex $e(v) = \max_{u \in V} d(v, u)$ (network diameter, worst-case scenarios)

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Radius $rad(G)$ identifies minimum eccentricity among all vertices $rad(G) = \min_{v \in V} e(v)$ (central nodes, optimal locations)

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Diameter $diam(G)$ represents maximum eccentricity across all vertices $diam(G) = \max_{v \in V} e(v)$ (network span, communication delays)

Calculation of vertex distances

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Breadth-First Search explores vertices in layers $O(V + E)$ time complexity (social network connections, web crawling)

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Dijkstra's algorithm uses priority queue for weighted graphs $O((V + E) \log V)$ time complexity (GPS navigation, network routing)

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Floyd-Warshall algorithm computes all-pairs shortest paths $O(V^3)$ time complexity (flight connections, supply chain optimization)

Advanced Graph Distance Concepts

Eccentricity and radius determination

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Eccentricity calculation runs shortest path algorithm from vertex to all others finding maximum distance (network extremities, resource distribution)

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Radius calculation computes eccentricity for all vertices identifying minimum value (central locations, emergency response centers)

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Center of graph comprises vertices with eccentricity equal to radius (optimal facility placement, distribution hubs)

Diameter computation and implications

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Diameter calculation methods: 1. Brute force computes distances between all vertex pairs 2. Efficient approach uses eccentricity of each vertex

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Diameter implications measure worst-case distance indicating overall connectivity (network performance, information dissemination)

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Diameter relationships:

- Bounded by radius: $\text{Diameter} \leq 2 * \text{radius}$
- In trees equals longest path edge count (hierarchical structures, decision trees)

Applications of graph distances

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Network analysis identifies critical nodes optimizes resource distribution (supply chains, communication networks)

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Social network applications determine degrees of separation analyze information spread (viral marketing, influence propagation)

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Transportation planning optimizes routes minimizes travel times (urban planning, logistics optimization)

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Computer networks minimize data transmission latency design efficient topologies (internet infrastructure, distributed systems)

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Facility location problems determine optimal service placement minimize maximum distance (warehouse locations, public service accessibility)

Unit 9 – Network Flows and Cuts

9.1 Maximum flow problem and Ford-Fulkerson algorithm

Network flow theory is all about maximizing flow through a system. The maximum flow problem finds the most stuff we can push from a start point to an end point, following rules about how much each connection can handle.

The Ford-Fulkerson algorithm solves this by repeatedly finding paths to send more flow until we can't anymore. It's used in real-world situations like water systems and power grids to figure out the most efficient way to move resources around.

Understanding the Maximum Flow Problem

Components of maximum flow problem

- Maximum flow problem optimizes network flow theory finds maximum amount of flow through network
- Network components include source (s) starts flow, sink (t) ends flow, edges connect nodes, nodes serve as vertices
- Capacity $c(u, v)$ limits maximum flow on edge (u, v)
- Flow $f(u, v)$ represents amount passing through edge (u, v)
- Flow constraints ensure capacity constraint $0 \leq f(u, v) \leq c(u, v)$ and flow conservation (incoming equals outgoing except source and sink)
- Residual network shows remaining available capacity on edges (transportation networks, communication systems)

Ford-Fulkerson Algorithm and Its Application

Steps of Ford-Fulkerson algorithm

1. Initialize flow to zero for all edges
 2. Find augmenting path from source to sink in residual network
 3. Determine bottleneck capacity along augmenting path
 4. Update flow along augmenting path
 5. Update residual network
 6. Repeat steps 2-5 until no augmenting path exists
- Augmenting path connects source to sink in residual network with available capacity
 - Bottleneck capacity represents minimum residual capacity along augmenting path
 - Algorithm terminates when no more augmenting paths exist in residual network (water supply systems, electrical grids)

Application of Ford-Fulkerson algorithm

1. Draw initial network with capacities
 2. Create residual network
 3. Find augmenting path using depth-first search or breadth-first search
 4. Identify bottleneck capacity
 5. Update flow and residual network
 6. Repeat until no augmenting path exists
- Applies to simple networks with few nodes and edges, complex networks with multiple intermediate nodes, networks with parallel edges, networks with bidirectional flows
 - Special cases include networks with multiple sources or sinks, networks with edge lower bounds (transportation routing, supply chain optimization)

Time complexity of Ford-Fulkerson algorithm

- Worst-case time complexity $O(E \cdot |f_{max}|)$, E represents number of edges, $|f_{max}|$ denotes maximum flow value
- Time complexity depends on number of augmenting paths and choice of augmenting path selection method
- Edmonds-Karp algorithm uses breadth-first search, time complexity $O(VE^2)$, V represents number of vertices
- Dinic's algorithm improves with blocking flows, time complexity $O(V^2E)$
- Practical considerations include performance on sparse vs dense networks and impact of network structure on iterations (computer networks, social network analysis)

9.2 Min-cut max-flow theorem

The min-cut max-flow theorem is a powerful tool in graph theory, linking network flow to cut capacity. It states that the maximum flow from source to sink equals the minimum capacity of any cut separating them, revealing bottlenecks and optimizing resource allocation.

This theorem has wide-ranging applications, from transportation and communication networks to image segmentation and social network analysis. By modeling scenarios as flow networks, we can solve complex problems in scheduling, resource allocation, and even sports team elimination.

Understanding the Min-Cut Max-Flow Theorem

Min-cut max-flow theorem fundamentals

- Min-Cut Max-Flow Theorem states maximum flow from source to sink equals minimum capacity of any cut separating source from sink in any network
- Duality between flow and cut problems reveals upper bound on maximum flow and helps identify bottlenecks in network flow
- Flow represents amount moving through network (water in pipes)

- Cut partitions vertices into two disjoint subsets (North and South hemispheres)
- Capacity limits maximum amount flowing through an edge (pipe diameter)
- Theorem applies to transportation networks (highway systems), communication systems (internet traffic), and resource allocation (supply chains)

Proof of min-cut max-flow theorem

- Proof begins with maximum flow in network and constructs minimum cut using residual graph
- Define set S containing source and all vertices reachable in residual graph
- S and its complement form a cut with no augmenting paths in residual graph
- Flow across cut equals its capacity due to flow conservation and capacity constraints
- Constructed cut has capacity equal to maximum flow, no cut can have smaller capacity
- Proof utilizes flow conservation (input equals output), capacity constraints ($\text{flow} \leq \text{capacity}$), and residual graph structure (backward edges)

Minimum cuts and maximum flow

- Minimum cuts separate source and sink with smallest total capacity among all cuts
- Correspond to saturated edges in maximum flow
- Ford-Fulkerson, Edmonds-Karp, and Push-relabel algorithms find minimum cuts
- Minimum cut capacity equals maximum flow value, saturated edges in max flow form min cut
- Multiple minimum cuts may exist in a network, all with same capacity
- Identify critical edges or vulnerabilities in network analysis (bridge closures, power grid weak points)

Applications of min-cut max-flow theorem

- Model scenario as flow network, identify source, sink, and edge capacities
- Apply max flow algorithm (Ford-Fulkerson) and interpret results in context
- Transportation optimizes traffic flow (rush hour routing) and airline scheduling (flight connections)
- Computer networks allocate bandwidth (internet service providers) and assess network reliability (data center connections)
- Bipartite matching solves job assignments (employees to tasks) and resource allocation (machines to projects)
- Advanced techniques include multi-commodity flow problems (multiple types of goods), minimum cost flow (cheapest transportation), and maximum flow over time (dynamic networks)
- Real-world considerations involve handling uncertainty in capacities (weather effects), dealing with dynamic network changes (road construction), and incorporating additional constraints (budget limitations)

Applications and Extensions of Min-Cut Max-Flow

Applications of min-cut max-flow theorem

- Image segmentation represents image as graph, uses min-cut to separate foreground and background (object recognition)
- Social network analysis identifies community structures and measures network resilience (friend groups, information spread)
- Project scheduling models tasks and dependencies as network, uses max flow to determine critical path (construction projects)
- Sports team elimination determines if team can still win league by modeling games as edges (playoff scenarios)
- Distributed computing balances load across servers and minimizes communication bottlenecks (cloud computing)
- Evacuation planning models building or city as network to determine maximum evacuation rate (emergency response)
- Approximation algorithms use min-cut max-flow as subroutine to solve NP-hard problems approximately (traveling salesman problem)

9.3 Applications of network flows

Network flow problems are all about optimizing resources and connections. From transportation and job assignments to scheduling and social networks, these problems pop up everywhere. They help us make the most of what we've got.

To tackle these problems, we set up a network with nodes and edges, add constraints, and define our goals. Then we use algorithms to find the best solution. The results help us spot bottlenecks, make smart decisions, and improve our systems.

Network Flow Problem Modeling

Network flow problem identification

- Transportation problems involve optimizing supply chain management streamlines product distribution and traffic flow optimization reduces urban congestion (rush hour traffic)
- Resource allocation tackles job assignment matches employees to tasks and production planning maximizes factory output
- Scheduling addresses project management tracks task dependencies and airline crew scheduling ensures efficient staffing (flight routes)
- Other applications include image segmentation in computer vision enhances object recognition and bipartite matching in social networks improves friend recommendations

Network flow problem formulation

- Identify network components with nodes representing locations or entities (cities, warehouses) and edges representing connections or flows (roads, pipelines)

- Define capacity constraints specifying maximum flow on each edge (bandwidth limits) and node capacity limitations (storage capacities)
- Determine flow conservation rules ensuring inflow equals outflow for intermediate nodes maintains balance
- Specify objective function to maximize total flow increases system throughput or minimize cost of flow reduces operational expenses
- Handle additional constraints like time windows for scheduling problems (delivery deadlines) and budget limitations for resource allocation (project funding)
- Choose appropriate network flow model such as maximum flow problem optimizes throughput or minimum cost flow problem minimizes expenses

Network Flow Algorithm Application and Interpretation

Application of network flow algorithms

- Ford-Fulkerson algorithm uses augmenting path method to find maximum flow iteratively through residual graph concept
- Edmonds-Karp algorithm employs breadth-first search for finding augmenting paths improves time complexity
- Push-relabel algorithm performs local operations push and relabel to achieve maximum flow efficiently
- Dinic's algorithm utilizes blocking flow concept to find maximum flow in stages
- Implement algorithms by choosing appropriate data structures (adjacency lists) and handling edge cases and termination conditions
- Solve variations of network flow problems including minimum cut problem identifies network vulnerabilities and circulation problem balances flow in closed systems

Interpretation of network flow results

- Analyze optimal flow distribution by identifying bottlenecks in the network (congested roads) and determining unused capacity (underutilized resources)
- Translate flow values to real-world quantities such as number of units transported (shipping containers) or amount of resources allocated (work hours)
- Evaluate solution quality by comparing to problem constraints and objectives and assessing practical feasibility
- Perform sensitivity analysis to understand impact of changing edge capacities (road expansions) and effect of adding or removing nodes (new distribution centers)
- Make data-driven decisions to optimize resource utilization improves efficiency and identify areas for infrastructure improvement guides investments
- Communicate results effectively by visualizing flow patterns (heat maps) and summarizing key findings for stakeholders
- Iterate and refine the model by incorporating feedback from domain experts and adjusting constraints based on real-world limitations enhances accuracy

Unit 10 – Planar Graphs and Graph Coloring

10.1 Planar graph properties and Euler's formula

Planar graphs are a fascinating subset of graphs that can be drawn without edge crossings. They have unique properties, like Euler's formula and maximum edge limits, that set them apart from other graph types.

Understanding planar graphs is crucial for real-world applications. From circuit design to map coloring, these concepts help solve complex problems. Kuratowski's theorem provides a powerful tool for determining graph planarity.

Planar Graphs and Their Properties

Properties of planar graphs

- Planar graphs drawn on plane without edge crossings edges intersect only at vertices
- Every planar graph has planar embedding subgraphs also planar
- Maximum edges in planar graph with n vertices: $3n - 6$ (for $n \geq 3$)
- Every planar graph 4-colorable (Four Color Theorem)
- Dual graphs constructed by placing vertex in each face of original planar graph edges correspond to adjacent faces

Faces in planar graphs

- Faces regions bounded by edges in planar embedding include outer (infinite) face
- Each face bounded by cycle of edges degree of face number of surrounding edges
- Connected planar graph faces simply connected disconnected graphs may have faces with holes

Euler's Formula and Graph Planarity

Applications of Euler's formula

- Euler's formula for planar graphs: $V - E + F = 2$ (V vertices, E edges, F faces)
- Determine number of faces in planar graph
- Calculate maximum edges in planar graph
- Prove non-planarity of certain graphs (K_5 and $K_{3,3}$)
- Variations for graphs on other surfaces (torus) and multiple connected components

Kuratowski's theorem for planarity

- Graph planar if and only if no subgraph homeomorphic to K_5 or $K_{3,3}$

- Homeomorphism graphs obtained by inserting or removing degree 2 vertices
- Identify K_5 or $K_{3,3}$ subdivisions in graph to prove non-planarity
- Wagner's theorem equivalent formulation using graph minors instead of homeomorphism

10.2 Vertex coloring and chromatic number

Vertex coloring assigns colors to graph vertices, ensuring adjacent ones differ. It's crucial for solving real-world problems like map coloring and scheduling. The chromatic number represents the minimum colors needed for proper coloring.

Determining the chromatic number involves algorithms and theorems. Applications of vertex coloring are diverse, from solving Sudoku puzzles to creating exam timetables. The Four Color Theorem for planar graphs is a significant result in this field.

Vertex Coloring Fundamentals

Concept of vertex coloring

- Vertex coloring assigns colors to graph vertices ensuring adjacent vertices have different colors
- Proper vertex coloring uses minimum colors needed while maintaining color difference between adjacent vertices
- Vertex coloring solves real-world problems (map coloring, scheduling, frequency assignment in radio networks, register allocation in compiler optimization)

Determining chromatic number

- Chromatic number represents minimum colors required for proper vertex coloring
- Greedy coloring algorithm and backtracking algorithm help find chromatic number
- Upper bound: Brooks' theorem states $\chi(G) \leq \Delta(G) + 1$
- Lower bound: Clique number $\chi(G) \geq \omega(G)$
- Special cases include bipartite graphs (chromatic number 2), complete graphs (chromatic number n), cycle graphs (chromatic number 2 for even cycles, 3 for odd cycles)

Advanced Concepts and Applications

Planar graphs vs vertex coloring

- Four Color Theorem proves every planar graph is 4-colorable historically significant complex proof
- Face coloring in planar graphs dually relates to vertex coloring
- Outerplanar graphs possess 3-colorable property
- Graph thickness correlates with chromatic number

Applications of coloring algorithms

- Graph coloring heuristics employ largest degree first and saturation degree ordering techniques
- Vertex coloring algorithms solve Sudoku puzzles, create exam timetables, and schedule sports events

- Optimization uses integer programming formulation and local search algorithms
- Graph coloring is NP-complete requiring approximation algorithms for large graphs

10.3 Edge coloring and chromatic index

Edge coloring assigns colors to graph edges, ensuring no adjacent edges share colors. The chromatic index represents the minimum colors needed, always at least equal to the maximum graph degree.

Vizing's Theorem sets an upper bound.

Edge coloring relates to matching, as each color class creates a matching. It has practical applications in scheduling and resource allocation. While NP-complete for general graphs, efficient algorithms exist for specific graph classes.

Edge Coloring Fundamentals

Edge coloring properties

- Edge coloring assigns colors to graph edges ensuring each edge receives one color and no adjacent edges share colors
- Chromatic index $\chi'(G)$ represents minimum colors required always greater than or equal to maximum graph degree
- Proper edge coloring prevents adjacent edges from having same color while strong edge coloring prohibits same color for edges within distance 2 (path graph, cycle graph)

Chromatic index of graphs

- Chromatic index determines minimum colors for proper edge coloring with lower bound $\Delta(G)$ (maximum graph degree)
- Vizing's Theorem states $\chi'(G) \leq \Delta(G) + 1$ classifying graphs into Class 1 $\chi'(G) = \Delta(G)$ or Class 2 $\chi'(G) = \Delta(G) + 1$
- Specific graph types have known chromatic indices bipartite graphs $\chi'(G) = \Delta(G)$ complete graphs $\chi'(K_n) = n - 1$ (even n) or $\chi'(K_n) = n$ (odd n) cycles $\chi'(C_n) = 2$ (even n) or $\chi'(C_n) = 3$ (odd n)

Applications and Advanced Concepts

Edge coloring vs matching

- Matching forms set of edges without common vertices maximum matching represents largest possible set
- Edge coloring relates to matching as each color class creates a matching minimum edge coloring partitions edges into fewest matchings
- König's Theorem equates chromatic index to maximum matching size in bipartite graphs
- Edmonds' Matching Algorithm efficiently finds maximum matchings in general graphs helps determine chromatic index for some cases

Applications in scheduling and allocation

- Edge coloring solves scheduling problems timetabling (classes to time slots) sports tournament scheduling (team matches)
- Resource allocation uses edge coloring for frequency assignment (wireless networks) traffic light synchronization (transportation)
- Algorithms include greedy coloring Misra-Gries algorithm heuristic approaches (local search genetic algorithms for large-scale problems)
- Edge coloring complexity NP-complete for general graphs polynomial-time algorithms exist for specific graph classes (bipartite graphs)

10.4 Map coloring and the Four Color Theorem

Map coloring assigns colors to regions, ensuring adjacent areas differ while minimizing color count. This problem, originating in 1852, sparked interest in graph theory and led to new proof techniques, extending beyond cartography to various applications.

The Four Color Theorem states any planar map can be colored using at most four colors. It involves planar graphs, dual graphs, and chromatic numbers. This theorem, proved using computer assistance, sparked debates on mathematical proofs' nature.

Map Coloring and the Four Color Theorem

Map coloring problem significance

- Map coloring problem assigns colors to regions on a map ensuring adjacent regions have different colors while minimizing the number of colors used
- Originated in 1852 when Francis Guthrie observed color patterns while coloring counties of England
- Sparked interest in graph theory led to development of new proof techniques
- Extended beyond cartography to solve scheduling problems, allocate resources, and assign frequencies in telecommunications (cellular networks)

Four Color Theorem fundamentals

- Four Color Theorem states any planar map can be colored using at most four colors
- Involves key concepts of planar graphs (can be drawn on a plane without edge crossings), dual graphs (derived from original graph), and chromatic number (minimum number of colors needed)
- Sets upper bound on chromatic number for planar graphs establishes relationship between map coloring and graph coloring
- First major theorem proved using computer assistance sparked debates on nature of mathematical proofs

Applications of Four Color Theorem

- Convert map to planar graph by identifying vertices and edges then apply coloring algorithm
- Greedy coloring algorithm assigns colors sequentially choosing first available color for each region

- Kempe chain method swaps colors along connected regions to resolve conflicts
- Consider efficiency of coloring algorithms and handle special cases (enclaves, exclaves)

Proof techniques for Four Color Theorem

- Initial proof attempts included Kempe's flawed proof in 1879 and Heawood's correction leading to five-color theorem
- Reducibility concept breaks down problem into smaller manageable cases
- Discharging method redistributes "charge" among graph elements to maintain balance
- Appel and Haken's 1976 computer-assisted proof used computer to check large number of cases
- Sparked controversy over reliability of computer-assisted proofs led to simplified proofs by Robertson, Sanders, Seymour, and Thomas in 1997
- Ongoing research seeks purely mathematical proof and explores generalizations to other surfaces and higher dimensions

Unit 11 – Matchings and Coverings

11.1 Matchings in bipartite graphs

Bipartite graphs split vertices into two sets, with edges connecting between them. They lack odd-length cycles and are two-colorable. Matchings in these graphs pair elements uniquely, with perfect matchings covering all vertices and maximum matchings optimizing allocation.

Applications of bipartite matchings are widespread. They solve assignment problems, optimize network flow, and improve scheduling efficiency. Algorithms like Ford-Fulkerson and Hopcroft-Karp find maximum matchings, while theorems guide their implementation and analysis.

Bipartite Graphs and Matchings

Properties of bipartite graphs

- Bipartite graph splits vertices into two disjoint sets with edges connecting vertices between sets (students-classes)
- No odd-length cycles exist in bipartite graphs prevents closed loops within same set
- Two-colorable property allows vertex coloring using only two colors (red-blue)
- König's theorem states maximum matching size equals minimum vertex cover size in bipartite graphs
- Complete bipartite graphs $K_{m,n}$ connect every vertex in one set to every vertex in other set (job applicants-available positions)
- Trees always bipartite due to lack of cycles

Matchings in bipartite graphs

- Matching selects edges without common vertices pairs elements uniquely (employees-tasks)
- Perfect matching covers all vertices ensures one-to-one correspondence (students-dorm rooms)
- Maximum matching finds largest possible set of paired vertices optimizes allocation (organ donors-recipients)
- Alternating path alternates between matched and unmatched edges explores potential improvements
- Augmenting path starts and ends with unmatched vertices increases matching size when found

Maximum matching size

- Ford-Fulkerson algorithm finds maximum matching by treating graph as flow network
- Hopcroft-Karp algorithm improves time complexity for sparse graphs $O(\sqrt{VE})$
- Hall's Marriage Theorem provides necessary and sufficient condition for perfect matching existence
- Berge's Theorem states matching is maximum if no augmenting path exists guides algorithm termination

Real-world applications of matchings

- Assignment problems match workers to jobs or students to projects optimize resource allocation
- Network flow problems use matchings to determine maximum flow in transportation systems (roads-vehicles)
- Scheduling applications assign time slots to tasks minimize conflicts and maximize efficiency
- Resource allocation matches limited resources to consumers fairly distributes goods or services (donors-charities)

11.2 Maximum matchings and augmenting paths

Augmenting paths are key to finding maximum matchings in bipartite graphs. These paths alternate between matched and unmatched edges, revealing opportunities to increase matching size and optimize resource allocation.

Algorithms like Hungarian and Hopcroft-Karp use augmenting paths to find maximum matchings efficiently. Their correctness is backed by Berge's theorem, while time complexity varies based on graph size and density.

Maximum Matchings and Augmenting Paths

Concept of augmenting paths

- Bipartite graphs divide vertices into two disjoint sets with edges only connecting vertices between sets (students and courses)
- Matchings pair vertices without sharing endpoints maximizes resource allocation (job assignments)
- Augmenting paths alternate between matched and unmatched edges starting and ending with unmatched vertices
- Odd-length paths reveal opportunities to increase matching size
- Existence of augmenting paths indicates non-maximum matching allows for optimization
- Exploiting augmenting paths increases matching size improves overall pairing

Application of augmenting path algorithms

- Hungarian algorithm iteratively finds augmenting paths updates matching accordingly
- Hopcroft-Karp algorithm locates multiple shortest augmenting paths in each phase improves efficiency
- Efficient path finding utilizes specialized data structures (adjacency lists, priority queues)
- Tracking visited vertices and edges prevents redundant searches optimizes performance
- Incremental matching updates maintain partial solutions throughout execution

Correctness of augmenting path algorithms

- Berge's theorem states maximum matching has no augmenting paths provides theoretical foundation
- Proof demonstrates augmenting path existence implies non-maximality establishes algorithm correctness

- Contradiction proof shows maximum matching cannot have augmenting paths reinforces theorem
- Algorithms maintain invariants ensure consistent progress towards optimal solution
- Termination occurs when no more augmenting paths exist guarantees maximum matching

Time complexity of augmenting algorithms

- Hungarian algorithm runs in $O(|V||E|)$ time scales with graph size and density
- Hopcroft-Karp achieves $O(\sqrt{|V|}|E|)$ complexity improves performance on sparse graphs
- Graph density significantly impacts runtime denser graphs require more computation
- Initial matching quality affects convergence speed better starting points reduce iterations
- Space requirements depend on graph representation methods (adjacency matrices, edge lists)
- Additional memory needed for path finding and matching storage impacts overall efficiency

11.3 Vertex and edge covers

Vertex and edge covers are crucial concepts in graph theory. Vertex covers include vertices that touch every edge, while edge covers use edges to connect all vertices. These ideas help solve real-world problems like network security and facility location.

Finding minimum covers is a key challenge. For vertex covers, it's NP-hard in general but solvable for some graphs. Edge covers are easier, using maximum matching algorithms. These concepts link to other graph theory ideas and have wide-ranging applications.

Vertex and Edge Covers in Graphs

Vertex and edge covers

- Vertex cover encompasses vertices including at least one endpoint of every edge covers all edges in graph (domino arrangement)
- Edge cover comprises edges covering all vertices every vertex incident to at least one edge in cover (spider web structure)
- Vertex covers emphasize vertices while edge covers prioritize edges vertex covers include one endpoint per edge edge covers connect to every vertex

Minimum cover determination

- Minimum vertex cover smallest set of vertices covering all edges NP-hard problem for general graphs polynomial-time algorithms for specific classes (bipartite graphs)
- Minimum edge cover smallest set of edges covering all vertices found in polynomial time using maximum matching algorithms
- Finding minimum covers utilizes:
 1. Greedy algorithms
 2. Approximation algorithms
 3. Integer linear programming formulations

- Minimum vertex cover complement yields maximum independent set (chessboard analogy)

Maximum matchings vs vertex covers

- Maximum matching largest set of edges without common endpoints (dance partners)
- König's theorem bipartite graphs maximum matching size equals minimum vertex cover size
- Gallai's theorem sum of minimum vertex cover and maximum matching sizes equals vertex count
- Maximum matching provides lower bound for minimum vertex cover size
- Algorithms leverage relationship to approximate minimum vertex covers construct covers from matchings in bipartite graphs

Applications of cover concepts

- Network security vertex covers determine minimum monitored nodes (security cameras)
- Facility location edge covers find optimal service placement covering all locations (pizza delivery)
- Task scheduling vertex covers assign minimum resources to cover all tasks (project management)
- Traffic control edge covers place traffic lights at minimum intersections (city planning)
- Optimization techniques employ:
 1. Linear programming relaxations
 2. Branch and bound algorithms
 3. Local search heuristics
- Real-world applications include sensor placement in IoT networks camera placement for surveillance systems protein-protein interaction networks in computational biology

Unit 12 – Independent Sets and Cliques

12.1 Independent sets and maximum independent sets

Independent sets in graphs are groups of vertices with no edges between them. They're crucial for modeling conflict-free resource allocation in networks and scheduling. Understanding independent sets helps solve problems in social networks and molecular biology.

Identifying independent sets involves checking for edge absence between vertices. Various algorithms exist for finding maximum independent sets, from brute force to greedy approaches and approximation methods. The problem's complexity often necessitates trade-offs between solution quality and runtime.

Independent Sets in Graphs

Definition of independent sets

- Independent set represents vertices in graph without edges connecting them forming stable set or anti-clique
- Complements vertex cover used to model conflict-free resource allocation (computer networks, scheduling)
- Characterizes non-adjacent vertices enabling efficient problem-solving in various domains (social networks, molecular biology)

Identification of independent sets

- Verify by checking absence of edges between all vertex pairs in set ensures independence
- Efficient method iterates through set vertices examining adjacency lists or matrices confirming no neighbors present
- Time complexity varies $O(n^2)$ for dense graphs with adjacency matrix $O(n + m)$ for sparse graphs using adjacency list (n vertices m edges)

Maximum independent set algorithms

- Brute force generates all vertex subsets checks independence tracks largest set found exhaustive but inefficient
- Greedy approaches: 1. Minimum degree iteratively selects lowest degree vertex removes it and neighbors 2. Maximum degree iteratively removes highest degree vertex and its neighbors
- Branch and bound uses upper/lower bounds to prune search space recursively explores promising branches
- Dynamic programming optimal for specific graph classes (trees series-parallel graphs) enables efficient solutions
- Approximation algorithms provide guaranteed ratios for general graphs where exact solutions infeasible

Complexity of independent set algorithms

- Brute force exhibits $O(2^n)$ complexity exponential growth with graph size

- Greedy algorithms run in $O(n^2)$ for dense $O(n + m)$ for sparse graphs polynomial time but may not find optimal solution
- Branch and bound worst-case exponential average-case often outperforms brute force
- Dynamic programming on trees achieves $O(n)$ complexity linear time for optimal tree structure solutions
- Approximation algorithms offer polynomial time complexity trading solution quality for runtime
- Maximum independent set problem NP-hard lacking known polynomial-time exact algorithm for general graphs justifies approximation and heuristic approaches

12.2 Cliques and maximum cliques

Cliques in graph theory are subsets of vertices where every pair is connected by an edge. They're like tight-knit groups in social networks. Understanding cliques is crucial for analyzing network structures and solving complex problems in various fields.

Finding maximum cliques is computationally challenging. Algorithms range from brute force to sophisticated techniques like Bron-Kerbosch. The problem's NP-completeness means we often use approximations or heuristics for large graphs. Cliques also relate to independent sets and vertex covers.

Clique Fundamentals

Definition and examples of cliques

- Clique definition represents subset of vertices in undirected graph where every two distinct vertices are adjacent (connected by edge)
- Properties of cliques include complete subgraph and fully connected subset of vertices
- Examples of cliques encompass triangle (3-clique), square with both diagonals (4-clique), pentagram (5-clique)
- Real-world applications involve social networks (groups of mutual friends) and biological networks (protein interactions)

Identification of graph cliques

- Steps to verify a clique involve selecting subset of vertices, checking if every pair of vertices is connected by edge, confirming clique status if all pairs are connected
- Edge count method requires counting number of edges in subgraph, comparing with expected number of edges for clique of size n : $\frac{n(n-1)}{2}$
- Adjacency matrix approach utilizes submatrix for selected vertices, checking if all off-diagonal elements are 1 (indicating edges)

Maximum Clique Algorithms and Analysis

Algorithms for maximum clique

- Brute force approach generates all possible subsets of vertices, checks each subset for clique property, keeps track of largest clique found

- Bron-Kerbosch algorithm employs recursive backtracking using three sets: R (current clique), P (prospective additions), X (excluded vertices), prunes search space using pivot selection
- Greedy algorithms repeatedly select vertices with highest degree, remove non-adjacent vertices after each selection
- Heuristic approaches include simulated annealing, genetic algorithms, tabu search

Time complexity of clique algorithms

- Brute force complexity: $O(2^n * n^2)$ (generating all subsets and checking each subset)
- Bron-Kerbosch algorithm: Worst-case $O(3^{n/3})$, often performs better in practice
- Greedy algorithm complexity: $O(n^2)$ for degree calculations and vertex selection, not guaranteed to find maximum clique
- NP-completeness of maximum clique problem means no known polynomial-time algorithm for optimal solution, approximation algorithms and heuristics often used in practice

Cliques vs independent sets

- Complementary nature: Maximum clique in graph G equals maximum independent set in complement graph G'
- Problem transformation allows algorithms for one problem to be adapted for the other, solving one problem implicitly solves the other
- Applications include graph coloring (relation to both cliques and independent sets) and resource allocation (cliques for conflict, independent sets for compatibility)
- Computational complexity: Both problems are NP-complete, approximation algorithms apply to both problems
- Vertex cover connection: Complement of maximal independent set is minimal vertex cover, relates clique problem to vertex cover problem

12.3 Relationships between independent sets, cliques, and vertex covers

Graph theory explores fascinating relationships between structures like independent sets and vertex covers. These concepts are crucial for understanding network connectivity and optimizing graph-based problems.

Cliques and independent sets in graph complements reveal hidden connections. By converting between problems, we unlock powerful problem-solving techniques applicable to real-world scenarios like social networks and resource allocation.

Relationships in Graph Theory

Relationship of independent sets and vertex covers

- Independent sets
- Set of vertices in graph where no two vertices adjacent forms independent structure

- No edges between any pair of vertices in set maintains independence
- Maximal independent set cannot add more vertices without violating independence
-

Maximum independent set largest possible in graph (maximum cardinality)

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Vertex covers

- Set of vertices covers all edges in graph ensures complete edge coverage
- Every edge incident to at least one vertex in cover guarantees full coverage
- Minimal vertex cover removal of any vertex leaves edge uncovered
-

Minimum vertex cover smallest possible for graph (minimum cardinality)

-

Relationship between independent sets and vertex covers

- Complement property vertex cover complement forms independent set
- Independent set complement forms vertex cover reciprocal relationship
- Sum of sizes $|I| + |V| = n$, I independent set, V vertex cover, n total vertices
- Optimization duality maximum independent set equivalent to minimum vertex cover

Cliques vs independent sets in complements

- Cliques
- Subset of vertices where every pair adjacent forms complete substructure
- Forms complete subgraph all vertices interconnected
- Maximal clique cannot add more vertices without violating completeness
-

Maximum clique largest possible in graph (maximum size)

-

Complement of a graph

- Graph with same vertices as original edges exist where absent in original and vice versa
-

Notation $\overline{G}$ represents complement of graph G

-

Relationship between cliques and independent sets in complement

- Duality principle clique in G forms independent set in $\overline{G}$

- Inverse relationship independent set in G forms clique in $\overline{G}$
- Size preservation maximum clique size in G equals maximum independent set size in $\overline{G}$

Conversion between graph problems

- Problem equivalence
- Maximum independent set (MIS) $\equiv$ Minimum vertex cover (MVC) $\equiv$ Maximum clique (MC)
-

Computationally equivalent problems (NP-hard complexity)

-

Conversion techniques 1. Find MIS, complement yields MVC 2. Find MVC, complement yields MIS 3. Find MIS in G , corresponding vertices form MC in $\overline{G}$ 4. Find MC in G , corresponding vertices form MIS in $\overline{G}$

-

Implications of conversions

- Solving one problem provides solutions to others enables problem-solving flexibility
- Algorithms for one problem adaptable to others promotes algorithmic versatility
- Hardness results apply across all three problems establishes computational complexity bounds

Application of graph problem relationships

- Problem-solving strategies
- Identify most suitable representation (independent set, clique, or vertex cover)
- Transform problem if necessary using known relationships
-

Apply appropriate algorithms or techniques for chosen representation

-

Applications in graph theory

- Network analysis identifies influential nodes or clusters (social networks)
- Resource allocation assigns non-conflicting resources (scheduling)
-

Pattern recognition finds recurring structures in data (bioinformatics)

-

Algorithmic approaches

- Greedy algorithms for approximations provide fast suboptimal solutions
- Branch and bound for exact solutions guarantees optimality
-

Heuristics for large-scale problems balance efficiency and solution quality

-

Real-world examples

- Social network analysis finds groups of mutually unconnected individuals (community detection)
- Scheduling allocates non-overlapping time slots (course timetabling)
- Molecular biology predicts protein structure using maximum clique (structural bioinformatics)

Unit 13 – Ramsey Theory and Extremal Graph Theory

13.1 Ramsey's theorem and Ramsey numbers

Ramsey's Theorem is a powerful tool in graph theory. It guarantees the existence of ordered substructures in large graphs, given specific conditions. This theorem has wide-ranging implications, from social network analysis to solving party problems.

Calculating Ramsey numbers is a complex task that reveals patterns in graphs. While some values are known, finding others remains challenging. Proof methods include constructive proofs, exhaustive searches, and probabilistic approaches, often utilizing the pigeonhole principle.

Ramsey's Theorem and Its Applications

Ramsey's theorem in graph theory

- Ramsey's Theorem guarantees existence of ordered substructures in large graphs for given number of colors c and integers $n_1, n_2, \dots, n_c$
- Defines $R(n_1, n_2, \dots, n_c)$ as minimum vertices needed in complete graph to ensure monochromatic subgraph of order n_i for some color i
- Significance extends to extremal graph theory and connects to number theory and combinatorics
- Practical applications include analyzing social networks and solving party problems (6 people guarantee 3 mutual friends or 3 mutual strangers)

Calculation of Ramsey numbers

- Ramsey number $R(m, n)$ represents smallest number of vertices needed for clique of size m or independent set of size n
- Known values: $R(3, 3) = 6$, $R(4, 4) = 18$, $R(5, 5)$ between 43 and 48
- Indicate complexity of finding patterns in graphs
- Calculation methods involve: 1. Constructive proofs for lower bounds 2. Exhaustive search for small cases 3. Probabilistic methods for upper bound estimation

Proof of Ramsey numbers

- Utilizes pigeonhole principle: n items in m containers, $n > m$, at least one container has multiple items
- Proof for $R(3, 3) \leq 6$: 1. Consider complete graph on 6 vertices, edges colored red or blue 2. Each vertex has 5 incident edges 3. Pigeonhole principle ensures at least 3 edges of same color 4. Forms red or blue triangle
- General strategy uses induction on vertices and applies pigeonhole principle for monochromatic subgraphs

Applications of Ramsey's theorem

- Graph coloring determines minimum colors to avoid monochromatic subgraphs
- Clique finding establishes lower bounds on graph size for guaranteed cliques
- Problem-solving approach identifies relevant Ramsey numbers, applies theorem for substructures, guides solution search
- Real-world uses include network analysis (social network clusters), algorithm complexity studies, and information theory (data compression)

13.2 Turán's theorem and extremal graphs

Turán's theorem is a key result in extremal graph theory. It establishes the maximum number of edges a graph can have without containing certain complete subgraphs, linking edge density to structural properties.

This theorem forms the basis for many other results in the field. It introduces important concepts like clique number, edge density, and Turán graphs, which are crucial for understanding graph structure and properties.

Turán's Theorem and Extremal Graph Theory

Turán's theorem in graph theory

- Turán's theorem fundamental result in extremal graph theory provides upper bound on edge count in graphs without complete subgraphs
- Extremal graph theory studies maximum or minimum graphs satisfying certain properties focuses on relationships between graph parameters and structural properties
- Turán's theorem establishes connection between edge density and presence of complete subgraphs serves as foundation for many other results in extremal graph theory
- Key concepts include clique number (size of largest complete subgraph), edge density (ratio of edges to maximum possible edges), and Turán graph (extremal graph achieving maximum edges without containing specific complete subgraph)

Maximum edges without subgraphs

- Turán's theorem states for graph G with n vertices and no K_{r+1} subgraph, maximum number of edges is $e(G) \leq \frac{r-1}{2r} n^2$
- Applies to triangle-free graphs ($r = 2$): $e(G) \leq \frac{n^2}{4}$ and K_4 -free graphs ($r = 3$): $e(G) \leq \frac{n^2}{3}$
- Erdős-Stone-Simonovits theorem generalizes Turán's result to forbidden subgraphs other than complete graphs
- Techniques for determining maximum edges include induction on vertex count, averaging arguments, and extremal graph constructions

Properties of Turán graphs

- Turán graph complete r -partite graph with n vertices divided as equally as possible

- Construction process: 1. Divide n vertices into r parts 2. Connect all vertices between different parts 3. Ensure size difference between parts is at most 1
- Properties include maximality (contains maximum edges without K_{r+1} subgraph), symmetry (vertex-transitive), and degree distribution (vertices have degree $\frac{r-1}{r}n$ or $\frac{r-1}{r}n - 1$)
- Examples include $T(n,2)$ (complete bipartite graph with parts of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$) and $T(n,3)$ (complete tripartite graph with parts differing by at most 1 in size)

Applications of Turán's theorem

- Problem-solving strategies involve identifying forbidden subgraph, applying Turán's theorem for edge upper bound, and constructing extremal graph to show bound tightness
- Extensions include generalized Turán problems (forbidding multiple subgraphs), Turán-type problems for hypergraphs, and spectral Turán theorems (using eigenvalues to bound subgraph occurrences)
- Applications to other areas include Ramsey theory (connections between Turán's theorem and Ramsey numbers) and probabilistic method (using Turán's theorem to prove existence of certain graphs)
- Advanced techniques involve stability results (characterizing graphs close to extremal), supersaturation (counting forbidden subgraphs in dense graphs), and flag algebras (computer-assisted proofs in extremal graph theory)

13.3 Applications of Ramsey theory

Ramsey theory explores guaranteed substructures in large mathematical objects. It's like finding patterns in chaos, with applications ranging from social networks to computer algorithms. This powerful tool helps us understand complex systems and predict behaviors.

In graph theory, Ramsey theory shines by finding monochromatic subgraphs in large, colored graphs. It's connected to other areas like the Erdős-Szekeres theorem, showing how seemingly different concepts share underlying principles in mathematics.

Ramsey Theory and Its Applications

Applications of Ramsey theory

- Combinatorics applications extend graph coloring problems by finding monochromatic substructures in large graphs (complete graphs)
- Extremal graph theory uses Ramsey theory to determine minimum graph sizes for specific subgraphs
- Computer science applications improve algorithm design and analysis through guaranteed substructure existence
- Complexity theory leverages Ramsey theory for lower bound proofs in computational problems
- Social network analysis employs Ramsey theory for clique detection identifying tightly connected groups within large networks
- Community structure identification uses Ramsey-type arguments to find cohesive subgroups
- Information theory applies Ramsey theory to construct error-correcting codes with specific properties (Reed-Solomon codes)

- Game theory utilizes Ramsey theory for strategy development in multi-player games predicting opponent behavior patterns

Ramsey theory vs Erdős-Szekeres theorem

- Erdős-Szekeres theorem guarantees existence of monotone subsequences in large sequences (increasing or decreasing)
- Relation to Ramsey theory stems from both ensuring substructures in large structures (cliques in graphs, monotone subsequences)
- Proof techniques employ pigeonhole principle assigning elements to categories
- Diagonal argument used in both to construct desired substructures
- Generalizations extend to higher-dimensional versions considering multidimensional arrays
- Geometric Ramsey theory applies concepts to geometric objects (points, lines, planes)

Advanced Concepts and Future Directions

Ramsey theory in probabilistic combinatorics

- Probabilistic method introduces using probability to prove existence of structures without explicit construction
- Erdős pioneered probabilistic Ramsey theory developing random graph models (Erdős-Rényi model)
- Applications in extremal combinatorics prove lower bounds on Ramsey numbers using probabilistic arguments
- Lovász Local Lemma extends probabilistic methods to events with limited dependencies
- Szemerédi's Regularity Lemma connects to Ramsey theory and probabilistic methods by partitioning large graphs into quasi-random parts

Limitations in Ramsey theory

- Computational complexity poses challenges in calculating exact Ramsey numbers beyond small cases ($R(5, 5)$ unknown)
- Asymptotic behavior of Ramsey numbers exhibits known bounds and gaps with room for improvement
- Structural Ramsey theory explores sparse analogues of classical results adapting to real-world network structures
- Infinite Ramsey theory presents challenges and open questions in countably infinite sets
- Applications to other areas potentially impact computer science and physics through pattern recognition
- Ramsey theory on hypergraphs investigates higher-dimensional generalizations with numerous open problems

Unit 14 – Random Graphs and Probabilistic Methods

14.1 Erdős-Rényi random graph model

Random graphs are mathematical models that simulate complex networks. The Erdős-Rényi model, introduced in 1959, is a foundational approach for generating these graphs with specific properties.

This model comes in two variants: $G(n,p)$ and $G(n,M)$. Both create graphs with n vertices, but differ in how edges are assigned. Understanding these models is crucial for analyzing real-world networks in various fields.

Erdős-Rényi Random Graph Model Fundamentals

Erdős-Rényi random graph model

- Erdős-Rényi model generates random graphs with specific properties
- Named after mathematicians Paul Erdős and Alfréd Rényi who introduced it in 1959
- Two main variants $G(n,p)$ and $G(n,M)$ differ in how edges are assigned
- $G(n,p)$ model uses n vertices and probability p for edge formation
- $G(n,M)$ model uses n vertices and fixed number M of edges
- Widely used in network science to study complex systems (social networks, biological networks)

Generation of random graphs

- $G(n,p)$ model generation process creates graphs with varying edge counts: 1. Begin with n isolated vertices 2. Examine each vertex pair 3. Add edge between pair with probability p 4. Repeat for all $\binom{n}{2}$ possible edges
- $G(n,M)$ model generation process ensures exact edge count: 1. Start with n isolated vertices 2. Randomly choose M pairs of vertices 3. Connect each selected pair with an edge
- $G(n,p)$ uses probabilistic approach while $G(n,M)$ guarantees fixed edge count
- Both models produce graphs with uniform distribution within their constraints

Probability of graph structures

- Probability of specific graph G with m edges in $G(n,p)$ model calculated as $P(G) = p^m(1-p)^{\binom{n}{2}-m}$
- Expected number of edges in $G(n,p)$ model given by $E(|E|) = \binom{n}{2}p$
- Connectivity probability increases with n and p , sharp threshold at $p = \frac{\ln n}{n}$
- Giant component emerges when $np > 1$, crucial for network connectivity
- k -clique formation probability: $\binom{n}{k}p^{\binom{k}{2}}$, important for community detection
- Threshold phenomena occur in both models as n approaches infinity

G(n,p) vs G(n,M) model properties

- Both produce n-vertex random graphs with uniform distribution
- G(n,p) allows variable edge count, G(n,M) fixes edge count at M
- G(n,M) equivalent to G(n,p) conditioned on exactly M edges
- Similar asymptotic behavior as n approaches infinity
- G(n,p) often easier for theoretical analysis
- G(n,M) useful when specific edge count required (sparse graph analysis)
- Computational trade-offs: G(n,p) faster for large n, G(n,M) more controlled

14.2 Properties of random graphs

Random graphs are fascinating models that reveal how network structures emerge from randomness. They exhibit surprising properties like sudden connectivity and the emergence of giant components as edge probabilities increase.

These graphs showcase intriguing thresholds where properties rapidly change. As edges are added, we see the birth of triangles, cycles, and cliques. The degree distribution follows predictable patterns, offering insights into vertex connectivity in random networks.

Random Graph Properties and Evolution

Connectivity in random graphs

- Connectivity concept in random graphs reflects likelihood of all vertices being reachable
- Connected graph contains path between any two vertices
- Connectivity probability increases with edge probability
- Giant component emerges suddenly at critical threshold
- Threshold occurs around $p = 1/n$ for $G(n, p)$ model
- Giant component size grows rapidly above threshold
- Graph diameter typically small in random graphs
- Diameter measures maximum distance between any two vertices
- Expected diameter often logarithmic in number of vertices
- Random graphs exhibit high robustness
- Maintain connectivity despite random vertex or edge removals
- More resilient than regular lattices, less than scale-free networks

Subgraphs in random graphs

- Subgraph containment likelihood depends on edge probability
- Subgraph subset of graph vertices and edges

- Induced subgraph retains all edges between selected vertices
- Triangle appearance threshold around $p = n^{-1}$
- Edge probability required for expected one triangle
- Number of triangles grows cubically with p above threshold
- Complete subgraphs (cliques) emerge as p increases
- Larger cliques require higher edge probabilities
- Maximal clique size typically logarithmic in n
- Cycles of various lengths appear progressively
- Shorter cycles emerge at lower edge probabilities
- Expected number of k -cycles $\approx (np)^k/2k$ for fixed k

Threshold Functions and Degree Distribution

Threshold functions for graph properties

- Threshold functions mark rapid transitions in graph properties
- Critical points where properties become highly probable
- Sharp thresholds have abrupt transitions, coarse more gradual
- Graph connectivity threshold approximately $p = \ln(n)/n$
- Probability approaches 1 for $p > (1 + \epsilon)\ln(n)/n$
- Almost certainly disconnected for $p < (1 - \epsilon)\ln(n)/n$
- Hamiltonian cycle threshold around $p = \ln(n)/n$
- Coincides with minimum degree ≥ 2 threshold
- Probability approaches 1 above this threshold
- Perfect matching threshold approximately $p = \ln(n)/n$
- Ensures high probability of pairing all vertices
- Closely related to minimum degree ≥ 1 threshold
- Chromatic number evolves with increasing edge probability
- Transitions from 2-colorable to requiring $\Theta(n/\log n)$ colors
- Asymptotically approaches $n/2 \log n$ for dense graphs

Degree distribution of vertices

- Degree distribution characterizes vertex connectivity
- Vertex degree counts number of incident edges
- Probability mass function gives likelihood of each degree

- Expected degree $\mathbb{E}[d] = p(n-1)$ in $G(n, p)$ model
- Variance $\text{Var}(d) = p(1-p)(n-1)$
- Binomial distribution for finite n
- Asymptotic degree distribution approximates Poisson
- $\Pr(d = k) \approx e^{-\lambda} \lambda^k / k!$ where $\lambda = np$
- Accurate for large n and constant expected degree
- Degree correlations typically absent in Erdős-Rényi graphs
- Zero assortativity expected due to random edge formation
- Contrasts with many real-world networks (social, biological)

Evolution of random graphs

- Random graph evolution models edge addition process
- $G(n, p)$ increases p , $G(n, m)$ adds edges sequentially
- Both models equivalent for large n when $p = m/\binom{n}{2}$
- Phase transitions mark sudden structural changes
- Critical thresholds for connectivity, giant component, etc
- Often occur around $p \sim \ln(n)/n$ or $p \sim 1/n$
- Connected components evolve predictably
- Initially isolated vertices, then small components form
- Double jump: many small components to giant component to fully connected
- Giant component formation critical point at $p = 1/n$
- Below threshold, all components logarithmic in size
- Above threshold, unique component of size $\Theta(n)$ emerges
- Convergence to complete graph as p approaches 1
- Rate of convergence $\Theta(n^2(1-p))$ expected missing edges
- Almost surely complete when $p = 1 - o(1/n^2)$

14.3 Probabilistic method in graph theory

The probabilistic method is a powerful tool in graph theory, using probability to prove the existence of graphs with specific properties. It simplifies complex proofs and provides results for graphs that are difficult to construct explicitly.

Key techniques include the first moment method, second moment method, and Lovász Local Lemma. These approaches analyze expected values, variances, and dependencies to demonstrate the existence of graphs with desired characteristics like large girth, specific colorings, or avoiding certain subgraphs.

Fundamentals of the Probabilistic Method

Principles of probabilistic method

- Core concept uses probability to prove graph property existence without explicit construction
- Key steps define probability space of graphs and show desired properties occur with positive probability
- Advantages simplify proofs for complex properties and provide existence results for hard-to-construct graphs
- Common distributions include uniform distribution on graphs and Erdős-Rényi random graph model ($G(n,p)$)

Specific Techniques and Applications

First moment method applications

- Based on linearity of expectation $E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$
- Steps define random variable X , calculate $E[X]$, use $E[X]$ to prove existence
- Proves existence of graphs with large girth and chromatic number (Erdős' construction)
- Demonstrates graphs with specific edge density properties (Turán's theorem)

Second moment method analysis

- Uses expectation and variance $\text{Var}(X) = E[X^2] - (E[X])^2$
- Key components Chebyshev's inequality bounds probability of deviation from mean
- Steps define X , calculate $E[X]$ and $\text{Var}(X)$, apply $P(|X - E[X]| \geq t) \leq \frac{\text{Var}(X)}{t^2}$
- Analyzes threshold functions in random graphs (connectivity, Hamilton cycles)
- Proves concentration of graph parameters around means (chromatic number, independence number)

Lovász Local Lemma for graphs

- Proves existence of objects satisfying multiple constraints in dependency graph
- Key components include dependency graph and probability bounds for individual events
- General form if events have limited dependence and small individual probabilities, avoiding all events has positive probability
- Proves existence of graphs with specific colorings (acyclic edge coloring)
- Demonstrates graphs avoiding certain subgraphs (Ramsey-type results)
- Variants include symmetric version and algorithmic versions for finding desired structures

Unit 15 – Graph Theory in Computer Science & Networks

15.1 Graph algorithms in computer science

Graph traversal and search algorithms are essential tools for exploring and analyzing complex networks. These techniques, including Breadth-First Search and Depth-First Search, allow us to navigate graphs efficiently, uncovering hidden patterns and connections.

Minimum spanning trees, shortest path algorithms, and complexity analysis round out our toolkit for tackling graph-related challenges. From optimizing network designs to finding the quickest routes, these algorithms power many real-world applications we encounter daily.

Graph Traversal and Search Algorithms

Graph traversal algorithms

- Breadth-First Search (BFS) uses queue-based implementation for level-order traversal finds shortest path in unweighted graphs
- Web crawling systematically explores web pages
- Social network analysis discovers connections and communities
- Finding connected components identifies isolated subgraphs
- Depth-First Search (DFS) employs stack-based or recursive implementation explores deeper paths before backtracking
- Topological sorting orders tasks based on dependencies
- Cycle detection identifies loops in directed graphs
- Maze solving finds path from start to end
- Time and space complexity analysis reveals efficiency
- BFS: $O(V + E)$ time visits all vertices and edges, $O(V)$ space stores queue
- DFS: $O(V + E)$ time explores all paths, $O(V)$ space in worst case (stack depth)

Minimum spanning trees

- Minimum Spanning Tree (MST) connects all vertices with minimum total edge weight forms acyclic subgraph
- Kruskal's Algorithm uses greedy approach with Union-Find data structure
- Time complexity: $O(E \log E)$ or $O(E \log V)$ sorts edges and processes them
- Prim's Algorithm employs priority queue implementation
- Time complexity: $O(E \log V)$ selects minimum weight edges
- Applications of MSTs optimize network design

- Telecommunications network planning minimizes cable length
- Transportation route optimization reduces total distance
- Cluster analysis groups similar data points
- Image segmentation identifies object boundaries

Shortest path algorithms

- Dijkstra's Algorithm finds single-source shortest path for non-negative edge weights uses greedy approach with priority queue
- Time complexity: $O((V + E)\log V)$ with binary heap
- GPS navigation systems calculate optimal routes
- Network routing protocols determine efficient data paths
- Bellman-Ford Algorithm handles graphs with negative edge weights employs dynamic programming approach
- Time complexity: $O(VE)$ iterates through all edges $V-1$ times
- Routing in computer networks handles varying link costs
- Arbitrage detection in financial markets identifies profit opportunities
- Floyd-Warshall Algorithm solves all-pairs shortest path problem uses dynamic programming approach
- Time complexity: $O(V^3)$ considers all vertex triplets

Complexity of graph algorithms

- Time complexity analysis evaluates algorithm efficiency
- Best, average, and worst-case scenarios consider input variations
- Big O notation expresses upper bound of growth rate
- Space complexity analysis measures memory usage
- Data structures and algorithms consume varying amounts of memory
- Factors affecting algorithm efficiency impact performance
- Graph density (sparse vs. dense graphs) influences traversal time
- Graph representation (adjacency matrix vs. adjacency list) affects memory usage and access time
- Comparison of algorithm efficiencies guides selection
- Trade-offs between time and space complexity balance resources
- Scalability for large graphs determines practical applicability
- Optimization techniques improve performance
- Fibonacci heap for Dijkstra's algorithm: $O(E + V\log V)$ reduces time complexity
- Johnson's algorithm for sparse graphs: $O(V^2\log V + VE)$ efficiently handles large, sparse networks

- NP-complete graph problems present computational challenges
- Traveling Salesman Problem optimizes route through multiple cities
- Graph Coloring assigns colors to vertices without adjacent same colors
- Hamiltonian Cycle finds path visiting each vertex exactly once

15.2 Social network analysis

Graph theory in social network analysis provides powerful tools for understanding complex social structures. Nodes represent individuals, while edges model relationships, allowing us to quantify importance, identify communities, and analyze network resilience.

This approach enables us to study information diffusion, predict future connections, and make recommendations. By applying graph theory concepts, we can uncover hidden patterns and dynamics within social networks, offering valuable insights for various applications.

Graph Theory in Social Network Analysis

Graph concepts for social networks

- Graph representation of social networks models individuals as nodes and relationships as edges facilitates analysis of complex social structures
- Centrality measures quantify node importance in networks
- Degree centrality counts direct connections reveals popular individuals
- Betweenness centrality measures node's role in information flow identifies key intermediaries
- Closeness centrality calculates average shortest path length indicates efficient information spreaders
- Eigenvector centrality assesses node importance based on neighbors' importance identifies influential nodes
- Connectivity concepts analyze network structure and resilience
- Connected components identify distinct subgroups in network
- Bridges and articulation points highlight critical connections and nodes
- k -connectivity measures network robustness to node/edge removals
- Small-world networks exhibit high clustering and low average path length (Facebook, LinkedIn)
- Scale-free networks display power-law degree distribution few highly connected hubs (Twitter, citation networks)

Influential nodes and communities

- Community detection algorithms identify cohesive subgroups in networks
- Girvan-Newman algorithm iteratively removes high betweenness edges
- Louvain method optimizes modularity through hierarchical clustering
- Clique percolation finds overlapping communities based on k -cliques

- Influence maximization selects seed nodes for maximum information spread
- Independent Cascade model simulates probabilistic influence propagation
- Linear Threshold model uses node thresholds for activation
- PageRank algorithm ranks node importance based on network structure
- k -core decomposition identifies densely connected subgraphs
- Modularity optimization finds community structure by maximizing inter-community edges
- Hierarchical clustering groups nodes based on similarity measures
- Spectral clustering uses eigenvectors of graph Laplacian for community detection

Graph theory in information diffusion

- Diffusion models simulate spread of information or diseases in networks
- Susceptible-Infected-Recovered (SIR) model tracks progression through states
- Susceptible-Infected-Susceptible (SIS) model allows reinfection
- Cascade models predict information spread
- Threshold models require minimum fraction of active neighbors
- Probabilistic models use infection probabilities for each edge
- Network effects on diffusion shape information flow
- Homophily tendency of similar nodes to connect influences spread patterns
- Structural equivalence nodes with similar network positions behave similarly
- Influence propagation studies how information spreads from seed nodes
- Seed selection strategies optimize initial spreaders
- Time-dependent diffusion models temporal aspects of spread
- Opinion dynamics models evolution of beliefs in social networks
- Voter models simulate binary opinion changes
- DeGroot model represents opinion as continuous values
- Epidemic thresholds determine conditions for widespread diffusion
- Rumor spreading models simulate gossip propagation in networks

Link prediction and recommendations

- Link prediction methods estimate likelihood of future connections
- Common neighbors count shared connections between nodes
- Jaccard coefficient normalizes common neighbors by total neighbors
- Adamic-Adar index weighs common neighbors by their degree

- Preferential attachment predicts based on node degrees
- Resource allocation index considers paths through common neighbors
- Matrix factorization techniques decompose adjacency matrix for predictions
- Random walk-based approaches explore network structure
- SimRank measures similarity based on random walks
- Personalized PageRank adapts PageRank for specific nodes
- Supervised learning for link prediction extracts features from graph structure
- Collaborative filtering recommends items based on user similarities
- User-based compares user preferences
- Item-based analyzes item similarities
- Graph embedding techniques map nodes to vector space
- Node2Vec uses biased random walks for embeddings
- DeepWalk applies skip-gram model to random walks
- Time-aware link prediction incorporates temporal information
- Cold start problem addresses recommendations for new users/items
- Evaluation metrics assess prediction performance
- Area Under the ROC Curve (AUC) measures overall prediction quality
- Precision at k evaluates top- k predictions
- Mean Average Precision (MAP) combines precision and ranking

15.3 Transportation and communication networks

Transportation networks are the arteries of our modern world. From roads to railways, these systems can be modeled as weighted graphs, with edges representing routes and weights indicating distances or costs.

Network flow problems optimize resource allocation in transportation. Algorithms like Ford-Fulkerson and push-relabel help solve maximum flow issues, enhancing everything from traffic routing to supply chain management and cargo shipping efficiency.

Transportation Networks

Transportation networks as weighted graphs

- Weighted graphs model transportation systems edges represent routes (roads, railways, flight paths) weights indicate distances, costs, or travel times
- Graph properties crucial for transportation analysis connectivity measures network accessibility shortest paths optimize routes minimum spanning trees minimize infrastructure costs
- Modeling techniques visualize and analyze networks node-link diagrams show network structure adjacency matrices represent connections incidence matrices relate nodes to edges

- Analysis methods optimize transportation networks Dijkstra's algorithm finds shortest paths Kruskal's or Prim's algorithm constructs minimum spanning trees centrality measures (degree, betweenness, closeness) identify important nodes

Network flow for logistics optimization

- Network flow problems optimize resource allocation maximum flow determines capacity (highways, pipelines) minimum cost flow minimizes transportation expenses multi-commodity flow handles multiple goods simultaneously
- Ford-Fulkerson algorithm solves max flow problems residual networks track remaining capacity augmenting paths increase flow iteratively
- Push-relabel algorithm improves efficiency preflow allows excess at nodes height function guides flow towards sink
- Applications enhance transportation systems traffic routing alleviates congestion supply chain optimization streamlines distribution cargo shipping maximizes freight capacity

Communication Networks

Graph theory in communication networks

- Network topologies define connection patterns bus (Ethernet) star (Wi-Fi) ring (token ring) mesh (internet backbone) tree (hierarchical networks)
- Graph-theoretic concepts guide network design diameter measures worst-case delay average path length indicates efficiency clustering coefficient quantifies local connectivity
- Routing algorithms direct data flow breadth-first search explores level by level depth-first search probes deeply Bellman-Ford algorithm handles negative weights
- Network protocols ensure proper functioning spanning tree protocol prevents loops link-state routing shares network topology information

Robustness of networks through graph measures

- Network resilience metrics quantify fault tolerance node connectivity measures node removal impact edge connectivity assesses link failure effects algebraic connectivity relates to overall connectedness
- Vulnerability analysis identifies weak points critical nodes and edges significantly impact performance articulation points disconnect network when removed bridges are sole connections between subgraphs
- Robustness measures evaluate network strength average node degree indicates connection density clustering coefficient measures local redundancy assortativity reveals connection patterns between high/low degree nodes
- Failure scenarios test network resilience random failures simulate accidental outages targeted attacks focus on critical components
- Percolation theory analyzes large-scale connectivity giant component represents largest connected subgraph percolation threshold marks critical point for network fragmentation
- Cascading failures model interdependent systems load redistribution occurs after initial failures failure propagation models simulate domino effects

15.4 Biological networks and systems biology

Biological networks in graph theory offer powerful tools for understanding complex systems in biology. From protein interactions to metabolic pathways, these networks use nodes and edges to represent relationships between biological entities.

Graph measures and algorithms reveal key properties of biological networks. Degree distribution, clustering coefficients, and centrality measures help identify important nodes and modules. These insights drive drug research and disease understanding, aiding in target identification and drug repurposing.

Biological Networks in Graph Theory

Biological systems as graphs

- Protein-Protein Interaction (PPI) Networks model physical protein interactions with proteins as nodes and interactions as edges (yeast two-hybrid system)
- Metabolic Networks depict biochemical reactions connecting metabolites as nodes through reaction edges (glycolysis pathway)
- Gene Regulatory Networks illustrate gene expression control with genes as nodes and regulatory interactions as edges (lac operon)
- Signal Transduction Networks represent cellular signaling with molecules as nodes and signal transmission as edges (MAPK cascade)
- Graph types in biological networks:
 - Undirected graphs for PPI networks show mutual interactions
 - Directed graphs for metabolic and gene regulatory networks indicate reaction or regulation direction
 - Weighted graphs quantify interaction strengths (affinity measurements)

Graph measures for biological networks

- Degree distribution often follows power-law in biological networks indicating scale-free properties (hub proteins)
- Clustering coefficient measures local connectivity revealing functional modules (protein complexes)
- Path length calculates average shortest path between nodes relating to network efficiency (small-world property)
- Centrality measures:
 - Degree centrality counts connections identifying highly interactive nodes
 - Betweenness centrality assesses information flow importance (bottleneck proteins)
 - Eigenvector centrality evaluates node influence considering neighbor importance
- Network motifs are recurring patterns with functional significance (feed-forward loops)
- Modularity identifies densely connected subgraphs corresponding to functional units (metabolic pathways)

Graph Algorithms and Applications

Graph algorithms in biological systems

- Clustering algorithms identify protein complexes in PPI networks:
 1. Markov Clustering (MCL) simulates random walks
 2. Affinity Propagation finds exemplars and assigns cluster members
- Path finding algorithms locate shortest paths in metabolic networks:
 1. Dijkstra's algorithm for single-source shortest paths
 2. Floyd-Warshall algorithm for all-pairs shortest paths
- Network alignment algorithms detect conserved pathways across species (orthologous proteins)
- Community detection algorithms reveal functional modules:
 1. Louvain method optimizes modularity
 2. Infomap minimizes description length of random walks
- Topological analysis identifies hubs and bottlenecks crucial for network robustness (essential genes)

Graph approaches for drug research

- Network-based drug target identification uses centrality analysis to find key proteins (kinase inhibitors)
- Disease network analysis constructs disease-gene networks to identify disease modules (cancer pathways)
- Drug-target interaction networks use bipartite graphs to connect drugs and targets predicting interactions (polypharmacology)
- Network-based drug repurposing analyzes similarity between drug and disease networks for new indications (thalidomide for leprosy)
- Perturbation analysis simulates drug intervention effects on network structure (synthetic lethality)
- Multi-omics data integration combines genomic, proteomic, and metabolomic data in layered networks for holistic disease and drug effect views (personalized medicine)