

Homological Algebra

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Unit 1 – Homological Algebra: Categories

Intro

1.1 Fundamentals of category theory

Category theory provides a powerful framework for understanding mathematical structures. It introduces objects and morphisms as building blocks, with functors and natural transformations connecting different categories. These concepts form the foundation for studying complex algebraic relationships.

Isomorphisms and equivalences play a crucial role in category theory. They allow us to compare objects within a category and entire categories themselves. This abstract approach enables mathematicians to uncover deep connections between seemingly unrelated areas of mathematics.

Categories and Morphisms

Definition and Components of a Category

- Category consists of a collection of objects and morphisms between those objects
- Objects can be thought of as the "points" or "vertices" in the category (groups, vector spaces, topological spaces)
- Morphisms are the "arrows" or "edges" connecting objects in the category (group homomorphisms, linear maps, continuous functions)
- Composition operation assigns to each pair of morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$ a morphism $g \circ f : A \rightarrow C$, called their composite
- Identity morphism for each object A , denoted as $id_A : A \rightarrow A$, satisfies $f \circ id_A = f$ and $id_B \circ f = f$ for any morphism $f : A \rightarrow B$

Properties and Diagrams in Category Theory

- Composition of morphisms is associative: $(h \circ g) \circ f = h \circ (g \circ f)$ for any morphisms $f : A \rightarrow B$, $g : B \rightarrow C$, and $h : C \rightarrow D$
- Commutative diagram is a graphical way to represent the equality of two compositions of morphisms
- In a commutative diagram, if two paths of morphisms starting from the same object and ending at the same object are equal (path independence)
- Commutative diagrams help visualize complex relationships between objects and morphisms in a category (diagram chasing)

Functors and Natural Transformations

Functors as Structure-Preserving Maps

- Functor is a map between categories that preserves the structure of the categories
- Functor $F : C \rightarrow D$ assigns to each object A in C an object $F(A)$ in D , and to each morphism $f : A \rightarrow B$ in C a morphism $F(f) : F(A) \rightarrow F(B)$ in D

- Functors preserve identity morphisms: $F(id_A) = id_{F(A)}$ for every object A in C
- Functors preserve composition of morphisms: $F(g \circ f) = F(g) \circ F(f)$ for any morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ in C (forgetful functor, free functor)

Natural Transformations as Morphisms of Functors

- Natural transformation is a "morphism" between functors, providing a way to compare and relate different functors
- Given two functors $F, G: C \rightarrow D$, a natural transformation $\alpha: F \Rightarrow G$ assigns to each object A in C a morphism $\alpha_A: F(A) \rightarrow G(A)$ in D
- Naturality condition: for any morphism $f: A \rightarrow B$ in C , the following diagram commutes:

$$\alpha_B \circ F(f) = G(f) \circ \alpha_A$$
- Natural transformations capture the idea of "natural" or "canonical" relationships between functors (natural isomorphism, Yoneda lemma)

Isomorphisms and Equivalences

Isomorphisms in Category Theory

- Isomorphism is a morphism $f: A \rightarrow B$ in a category that has an inverse morphism $g: B \rightarrow A$ such that $g \circ f = id_A$ and $f \circ g = id_B$
- Objects A and B are isomorphic if there exists an isomorphism between them, denoted as $A \cong B$
- Isomorphic objects have the same structure and properties within the context of the category (isomorphic groups, isomorphic vector spaces)
- Isomorphisms capture the notion of "sameness" or "equivalence" between objects in a category

Equivalence of Categories

- Equivalence of categories is a stronger notion than isomorphism between individual objects
- Two categories C and D are equivalent if there exist functors $F: C \rightarrow D$ and $G: D \rightarrow C$ such that $G \circ F \cong id_C$ and $F \circ G \cong id_D$ via natural isomorphisms
- Equivalent categories have the same structure and properties, although their objects and morphisms may differ (equivalence between the category of finite-dimensional vector spaces and the category of matrices)
- Equivalence of categories allows for the transfer of results and insights between seemingly different mathematical domains

1.2 Basic algebraic structures and homomorphisms

Algebraic structures like groups, rings, modules, and vector spaces form the foundation of homological algebra. These structures have specific properties and operations that define their behavior and relationships.

Homomorphisms are crucial tools for studying these structures, preserving their essential properties when mapping between them. Understanding kernels, images, and exact sequences helps analyze the relationships and connections between different algebraic structures.

Algebraic Structures

Groups and Rings

- Group consists of a set G together with a binary operation $*$ satisfying the following axioms:
- Closure: For all $a, b \in G$, $a * b \in G$
- Associativity: For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$
- Identity element: There exists an element $e \in G$ such that for all $a \in G$, $a * e = e * a = a$
- Inverse element: For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$
- Ring extends the concept of a group by adding a second binary operation, usually denoted as addition $+$, and requiring the set to also have the structure of an abelian group under this operation
- A ring $(R, +, *)$ satisfies the following axioms:
- $(R, +)$ is an abelian group with identity element 0
- $(R, *)$ is a monoid with identity element 1
- Distributivity: For all $a, b, c \in R$, $a * (b + c) = (a * b) + (a * c)$ and $(a + b) * c = (a * c) + (b * c)$

Modules and Vector Spaces

- Module generalizes the notion of a vector space by allowing the scalars to be from a ring instead of a field
- A left R -module consists of an abelian group $(M, +)$ and a ring $(R, +, *)$ together with an operation $\cdot : R \times M \rightarrow M$ (called scalar multiplication) such that for all $r, s \in R$ and $x, y \in M$:
- $(r + s) \cdot x = r \cdot x + s \cdot x$
- $(r * s) \cdot x = r \cdot (s \cdot x)$
- $1 \cdot x = x$, where 1 is the multiplicative identity in R
- Vector space is a special case of a module where the ring R is a field $\mathbb{F}$
- A vector space over a field $\mathbb{F}$ is an abelian group $(V, +)$ together with a scalar multiplication operation $\cdot : \mathbb{F} \times V \rightarrow V$ satisfying the following axioms for all $a, b \in \mathbb{F}$ and $u, v \in V$:
- $(a + b) \cdot v = a \cdot v + b \cdot v$
- $a \cdot (u + v) = a \cdot u + a \cdot v$
- $(a * b) \cdot v = a \cdot (b \cdot v)$
- $1 \cdot v = v$, where 1 is the multiplicative identity in $\mathbb{F}$

Homomorphisms and Related Concepts

Homomorphisms and Their Properties

- Homomorphism is a structure-preserving map between two algebraic structures of the same type (e.g., groups, rings, modules, vector spaces)

- A group homomorphism $\varphi : G \rightarrow H$ satisfies $\varphi(a * b) = \varphi(a) * \varphi(b)$ for all $a, b \in G$
- A ring homomorphism $\psi : R \rightarrow S$ satisfies $\psi(a + b) = \psi(a) + \psi(b)$ and $\psi(a * b) = \psi(a) * \psi(b)$ for all $a, b \in R$
- Kernel of a homomorphism $\varphi : G \rightarrow H$ is the set of elements in G that map to the identity element in H , denoted as $\ker(\varphi) = \{g \in G : \varphi(g) = e_H\}$
- The kernel is always a normal subgroup of G
- Image of a homomorphism $\varphi : G \rightarrow H$ is the set of elements in H that are mapped to by some element in G , denoted as $\text{Im}(\varphi) = \{\varphi(g) : g \in G\}$
- The image is always a subgroup of H

Exact Sequences

- Exact sequence is a sequence of homomorphisms between algebraic structures (e.g., groups, rings, modules, vector spaces) such that the image of each homomorphism is equal to the kernel of the next homomorphism
- A short exact sequence is an exact sequence of the form $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$, where 0 denotes the trivial group, ring, module, or vector space
- In a short exact sequence, f is injective (one-to-one), g is surjective (onto), and $\text{Im}(f) = \ker(g)$
- Splitting lemma states that a short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ splits (i.e., $B \cong A \oplus C$) if and only if there exists a homomorphism $h : C \rightarrow B$ such that $g \circ h = \text{id}_C$

1.3 Motivation and historical context of homological algebra

Homological algebra emerged from the study of topology, connecting algebraic structures to geometric spaces. It provides powerful tools for understanding complex mathematical objects, using concepts like cohomology and derived functors to extract meaningful information.

Cohomology theories assign algebraic objects to topological spaces, capturing essential properties. Derived functors extend this approach, measuring how functors deviate from exactness. These tools have wide-ranging applications in mathematics, from algebraic geometry to representation theory.

Cohomology and Topological Invariants

Cohomology Theories and Eilenberg-Steenrod Axioms

- Cohomology theories assign algebraic objects (abelian groups or vector spaces) to topological spaces in a functorial way
- Cohomology theories capture important topological information about spaces
- Eilenberg-Steenrod axioms provide a set of properties that characterize a cohomology theory
- Homotopy invariance: Homotopy equivalent spaces have isomorphic cohomology groups
- Exactness: A short exact sequence of spaces induces a long exact sequence of cohomology groups
- Excision: Cohomology of a space can be computed using an excised subspace and its complement
- Dimension: The cohomology of a point is trivial except in dimension 0, where it is the coefficient group

- Examples of cohomology theories satisfying the Eilenberg-Steenrod axioms include singular cohomology and de Rham cohomology

Topological Invariants and Algebraic Topology Connection

- Topological invariants are properties of topological spaces that remain unchanged under homeomorphisms or homotopy equivalences
- Cohomology groups serve as topological invariants, providing a way to distinguish non-homeomorphic or non-homotopy equivalent spaces
- Betti numbers: The ranks of the cohomology groups are topological invariants (the first Betti number counts the number of connected components, the second counts the number of holes, etc.)
- Cup product structure: The cup product gives the cohomology ring additional structure, which is a finer invariant than just the cohomology groups
- Cohomology plays a central role in algebraic topology, which studies topological spaces using algebraic tools
- Algebraic topology uses functors to translate topological problems into algebraic ones, which are often easier to solve
- Results obtained algebraically can then be interpreted back in the topological setting, providing insights into the original problem

Derived Functors in Homological Algebra

Derived Functors and Their Importance

- Derived functors are a way to extend functors between abelian categories (such as modules over a ring) to take into account additional homological information
- Derived functors measure the failure of a functor to be exact, i.e., to preserve exact sequences
- For a left exact functor F , its right derived functors $R^i F$ measure the deviation from being right exact
- For a right exact functor G , its left derived functors $L_i G$ measure the deviation from being left exact
- Derived functors are important tools in homological algebra and have applications in various areas of mathematics, including algebraic geometry and representation theory

Ext and Tor Functors

- Ext and Tor are two fundamental examples of derived functors in homological algebra
- For R -modules A and B , $Ext_R^i(A, B)$ is the i -th right derived functor of the $Hom_R(-, B)$ functor applied to A
- $Ext_R^i(A, B)$ measures the failure of the functor $Hom_R(-, B)$ to be exact when applied to A
- $Ext_R^1(A, B)$ classifies extensions of B by A , i.e., short exact sequences of the form $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$
- For R -modules A and B , $Tor_i^R(A, B)$ is the i -th left derived functor of the tensor product functor $- \otimes_R B$ applied to A
- $Tor_i^R(A, B)$ measures the failure of the functor $- \otimes_R B$ to be exact when applied to A

- $Tor_0^R(A, B)$ is isomorphic to the tensor product $A \otimes_R B$

1.4 Category-theoretic approach to homological algebra

Homological algebra blends category theory and algebra, offering powerful tools for studying algebraic structures. It introduces abelian categories, which generalize abelian groups and provide a framework for working with kernels, cokernels, and exact sequences.

This approach allows us to analyze complex algebraic systems using categorical concepts. We'll explore chain complexes, homology, and derived categories, seeing how these ideas connect and apply to various mathematical fields.

Categories in Homological Algebra

Abelian Categories and Their Properties

- Abelian categories generalize the notion of abelian groups to a broader setting
- Objects in an abelian category have a group structure compatible with morphisms
- Morphisms between objects in an abelian category form abelian groups under addition
- Abelian categories have kernels and cokernels for every morphism
- Kernel of a morphism $f: A \rightarrow B$ is an object K with a morphism $k: K \rightarrow A$ such that for any object X and morphism $x: X \rightarrow A$ with $f \circ x = 0$, there exists a unique morphism $x': X \rightarrow K$ with $x = k \circ x'$
- Cokernel of a morphism $f: A \rightarrow B$ is an object C with a morphism $c: B \rightarrow C$ such that for any object Y and morphism $y: B \rightarrow Y$ with $y \circ f = 0$, there exists a unique morphism $y': C \rightarrow Y$ with $y = y' \circ c$
- Every monomorphism in an abelian category is the kernel of its cokernel
- Every epimorphism in an abelian category is the cokernel of its kernel

Additive Categories and Universal Properties

- Additive categories are categories in which the morphism sets are abelian groups and composition of morphisms is bilinear
- Objects in an additive category have a zero object and biproducts (direct sums)
- Morphisms between objects in an additive category can be added and have additive inverses
- Universal properties characterize objects and morphisms in terms of their relationships with other objects and morphisms
- Examples of universal properties include the universal property of the product (uniqueness of morphisms into the product) and the universal property of the coproduct (uniqueness of morphisms out of the coproduct)
- Universal properties allow for the construction of objects and morphisms satisfying certain conditions without explicitly specifying their internal structure

Derived Categories and Localization

- Derived categories are obtained from categories of chain complexes by formally inverting quasi-isomorphisms (morphisms inducing isomorphisms on homology)

- Derived categories provide a way to study homological invariants (e.g., Ext and Tor functors) in a more intrinsic setting
- Objects in a derived category are chain complexes, and morphisms are equivalence classes of chain maps modulo homotopy
- Localization is a process of adding formal inverses to a category with respect to a collection of morphisms
- Derived categories can be viewed as the localization of the category of chain complexes with respect to quasi-isomorphisms
- Localization allows for the construction of new categories with desired properties (e.g., the homotopy category of a model category)

Chain Complexes and Homology

Chain Complexes and Their Properties

- A chain complex is a sequence of abelian groups (or modules) connected by homomorphisms (differentials) such that the composition of any two consecutive differentials is zero
- A chain complex C^* is denoted as $\dots \rightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \rightarrow \dots$, where $d_n \circ d_{n+1} = 0$ for all n
- The condition $d_n \circ d_{n+1} = 0$ ensures that the image of each differential is contained in the kernel of the next differential
- Chain complexes can be used to model various algebraic and topological objects, such as simplicial complexes, CW complexes, and resolutions of modules
- Morphisms between chain complexes are chain maps, which are collections of homomorphisms between the corresponding abelian groups that commute with the differentials

Homology and Cohomology of Chain Complexes

- The homology of a chain complex C^* at degree n is defined as the quotient group $H_n(C^*) = \ker(d_n) / \text{im}(d_{n+1})$
- Elements of $\ker(d_n)$ are called n -cycles, and elements of $\text{im}(d_{n+1})$ are called n -boundaries
- Homology measures the extent to which the sequence of differentials fails to be exact
- The cohomology of a chain complex C^* at degree n is defined as the homology of the dual cochain complex C^n , where $C^n = \text{Hom}(C_n, A)$ for some fixed abelian group A
- The differentials in the cochain complex are the duals of the differentials in the original chain complex
- Cohomology can be interpreted as a contravariant functor from the category of chain complexes to the category of abelian groups
- Homology and cohomology are functorial, meaning that chain maps between chain complexes induce homomorphisms between their homology and cohomology groups

Applications and Examples of Homology

- Singular homology is a homology theory for topological spaces, constructed using the chain complex of singular simplices

- Singular homology groups are topological invariants that capture information about the "holes" in a space
- The singular homology of a sphere S^n is $H_k(S^n) = \mathbb{Z}$ for $k = 0, n$ and $H_k(S^n) = 0$ otherwise
- Group homology is a homology theory for groups, constructed using the chain complex of the group ring with coefficients in a module
- Group homology measures the failure of a module to be projective over the group ring
- The group homology of a cyclic group C_n with coefficients in $\mathbb{Z}$ is $H_0(C_n; \mathbb{Z}) = \mathbb{Z}$, $H_1(C_n; \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, and $H_k(C_n; \mathbb{Z}) = 0$ for $k \geq 2$

Unit 2 – Chain Complexes and Homology

2.1 Definition and properties of chain complexes

Chain complexes are sequences of abelian groups connected by boundary maps. They're crucial in homological algebra, helping us study algebraic structures and topological spaces. The key property is that composing two consecutive boundary maps gives zero.

This section dives into the definition and properties of chain complexes. We'll explore cycles, boundaries, and homology groups, which measure "holes" in algebraic objects. Understanding these concepts is essential for grasping the broader ideas in homological algebra.

Chain Complexes and Graded Modules

Definition and Structure

- A chain complex is a sequence of abelian groups or modules C_n connected by homomorphisms $d_n : C_n \rightarrow C_{n-1}$ called boundary maps or differentials
- The boundary maps satisfy the condition $d_{n-1} \circ d_n = 0$ for all n , meaning the composition of any two consecutive boundary maps is the zero map
- A graded module is a module that can be expressed as a direct sum of submodules indexed by the integers, i.e., $M = \bigoplus_{n \in \mathbb{Z}} M_n$
- The elements of M_n are said to have degree n , and the degree of a homogeneous element $x \in M_n$ is denoted by $|x| = n$

Boundary Operators and Differentials

- The boundary operator or differential $d_n : C_n \rightarrow C_{n-1}$ lowers the degree of the elements by 1
- The condition $d_{n-1} \circ d_n = 0$ implies that the image of d_n is contained in the kernel of d_{n-1} , i.e., $\text{im } d_n \subseteq \ker d_{n-1}$
- This condition is necessary for the homology groups of the chain complex to be well-defined
- The differential can be thought of as a generalization of the boundary operator in simplicial or cellular homology, which maps a chain to its boundary

Exactness and Homology

Exact Sequences

- A sequence of abelian groups or modules and homomorphisms between them is called exact if the image of each homomorphism is equal to the kernel of the next homomorphism
- In the context of chain complexes, a short exact sequence is a sequence of the form $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$, where f is injective, g is surjective, and $\text{im } f = \ker g$
- Exact sequences are important tools in homological algebra, as they allow for the study of relationships between different chain complexes and their homology groups

Cycles, Boundaries, and Homology

- A cycle in a chain complex (C_n, d_n) is an element $x \in C_n$ such that $d_n(x) = 0$, i.e., an element in the kernel of the boundary map
- A boundary in a chain complex (C_n, d_n) is an element $x \in C_n$ such that there exists a $y \in C_{n+1}$ with $d_{n+1}(y) = x$, i.e., an element in the image of the boundary map
- The n -th homology group of a chain complex (C_n, d_n) is defined as the quotient group $H_n(C) = \ker d_n / \operatorname{im} d_{n+1}$, i.e., the cycles modulo the boundaries
- Homology groups measure the "holes" or "voids" in a topological space or algebraic object, with different dimensions of holes corresponding to different homology groups

Chain Homotopy

Definition and Properties

- A chain homotopy between two chain maps $f, g : C \rightarrow D$ is a sequence of homomorphisms $h_n : C_n \rightarrow D_{n+1}$ such that $f_n - g_n = d_{n+1}^D \circ h_n + h_{n-1} \circ d_n^C$ for all n
- If there exists a chain homotopy between f and g , then f and g are said to be chain homotopic, denoted by $f \simeq g$
- Chain homotopy is an equivalence relation on the set of chain maps between two chain complexes
- Chain homotopic maps induce the same homomorphisms on homology, i.e., if $f \simeq g$, then $f_* = g_* : H_n(C) \rightarrow H_n(D)$ for all n , where f_* and g_* are the induced maps on homology
- The concept of chain homotopy is analogous to the notion of homotopy between continuous maps in topology, but adapted to the algebraic setting of chain complexes

2.2 Homology groups and their computation

Homology groups are a key concept in algebraic topology, measuring the "holes" in a topological space. They're defined as quotient modules of cycles and boundaries, capturing essential structural information about the space.

Computing homology groups involves tools like free resolutions and long exact sequences. These techniques allow us to break down complex structures into simpler components, making it easier to calculate and understand the homology of various mathematical objects.

Homology Groups and Their Properties

Definition and Components of Homology Groups

- Homology group $H_n(C_*)$ defined as the quotient module $\ker d_n / \operatorname{im} d_{n+1}$
- Kernel $\ker d_n$ consists of n -cycles, elements of C_n that are mapped to zero by the boundary map d_n
- Image $\operatorname{im} d_{n+1}$ consists of n -boundaries, elements of C_n that are the image of the boundary map d_{n+1} from C_{n+1}
- Quotient module $\ker d_n / \operatorname{im} d_{n+1}$ identifies n -cycles that differ by an n -boundary, capturing the "essential" n -dimensional holes in the complex

Invariants of Homology Groups

- Betti numbers β_n represent the rank of the free part of the n -th homology group $H_n(C_*)$
- β_0 counts the number of connected components
- β_1 counts the number of 1-dimensional holes or "loops"
- β_2 counts the number of 2-dimensional voids or "cavities"
- Torsion of $H_n(C_*)$ captures the non-free part of the homology group
- Torsion elements have finite order, i.e., they become zero when multiplied by some non-zero scalar
- Torsion can be represented by a list of the orders of the torsion subgroups (e.g., $\mathbb{Z}_2, \mathbb{Z}_3$)

Computational Tools for Homology

Free Resolutions

- Free resolution of a module M is an exact sequence of free modules F_* with a map $\varepsilon : F_0 \rightarrow M$ such that the sequence $\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \xrightarrow{\varepsilon} M \rightarrow 0$ is exact
- Free resolutions can be used to compute homology groups by applying the homology functor H_* to the resolution
- $H_n(M) \cong H_n(F_*)$ for all $n \geq 0$
- Constructing a free resolution involves finding a generating set for the module and relations among the generators

Long Exact Sequences

- Long exact sequence is a tool for computing homology groups of a chain complex from the homology groups of simpler complexes
- Given a short exact sequence of chain complexes $0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$ there is an associated long exact sequence in homology $\cdots \rightarrow H_n(A_*) \rightarrow H_n(B_*) \rightarrow H_n(C_*) \xrightarrow{\partial} H_{n-1}(A_*) \rightarrow \cdots$
- The connecting homomorphism ∂ relates the homology groups of the complexes A_* , B_* , and C_*
- Long exact sequences can be used to compute unknown homology groups from known ones by exploiting the exactness property
- Exactness means that the kernel of each map is equal to the image of the previous map

2.3 Chain maps and induced homomorphisms

Chain maps are the backbone of homological algebra, connecting different chain complexes. They preserve the structure of chain complexes and allow us to compare them. Understanding chain maps is crucial for grasping how information flows between different algebraic structures.

Induced homomorphisms on homology groups are a key consequence of chain maps. They let us relate the homology of different chain complexes, giving us a powerful tool for studying topological and algebraic properties. This connection is fundamental for many applications in algebra and topology.

Chain Maps and Induced Homomorphisms

Definition and Properties of Chain Maps

- Chain map $f: C_\bullet \rightarrow D_\bullet$ between chain complexes $C_\bullet$ and $D_\bullet$ consists of a family of homomorphisms $f_n: C_n \rightarrow D_n$ for each $n \in \mathbb{Z}$ that commute with the boundary operators
- Commuting with boundary operators means $f_{n-1} \circ \partial_n^C = \partial_n^D \circ f_n$ for all $n \in \mathbb{Z}$, where ∂_n^C and ∂_n^D are the boundary operators of $C_\bullet$ and $D_\bullet$ respectively
- Commutative diagram illustrates the relationship between chain maps and boundary operators, with vertical arrows representing f_n and horizontal arrows representing boundary operators
- Composition of chain maps $f: C_\bullet \rightarrow D_\bullet$ and $g: D_\bullet \rightarrow E_\bullet$ defined by $(g \circ f)_n = g_n \circ f_n$ for each $n \in \mathbb{Z}$, resulting in a chain map $g \circ f: C_\bullet \rightarrow E_\bullet$.
- Identity chain map $\text{id}_{C_\bullet}: C_\bullet \rightarrow C_\bullet$ defined by $(\text{id}_{C_\bullet})_n = \text{id}_{C_n}$ for each $n \in \mathbb{Z}$, serving as the identity element for composition of chain maps

Induced Homomorphisms on Homology Groups

- Chain map $f: C_\bullet \rightarrow D_\bullet$ induces homomorphisms $H_n(f): H_n(C_\bullet) \rightarrow H_n(D_\bullet)$ on homology groups for each $n \in \mathbb{Z}$
- Induced homomorphism $H_n(f)$ maps homology class $[c] \in H_n(C_\bullet)$ to homology class $[f_n(c)] \in H_n(D_\bullet)$
- Well-defined since chain maps preserve cycles and boundaries, ensuring that homologous cycles in $C_\bullet$ are mapped to homologous cycles in $D_\bullet$.
- Functoriality of induced homomorphisms states that $H_n(g \circ f) = H_n(g) \circ H_n(f)$ for chain maps $f: C_\bullet \rightarrow D_\bullet$ and $g: D_\bullet \rightarrow E_\bullet$, and $H_n(\text{id}_{C_\bullet}) = \text{id}_{H_n(C_\bullet)}$ for the identity chain map
- Functoriality allows the study of homology groups and their relationships through the lens of category theory, treating chain complexes as objects and chain maps as morphisms

Naturality of Induced Homomorphisms

- Naturality of induced homomorphisms expresses the compatibility between chain maps and the connecting homomorphisms in long exact sequences of homology groups
- Given a short exact sequence of chain complexes $0 \rightarrow A_\bullet \rightarrow B_\bullet \rightarrow C_\bullet \rightarrow 0$ and chain maps $f_A: A_\bullet \rightarrow A'_\bullet$, $f_B: B_\bullet \rightarrow B'_\bullet$, and $f_C: C_\bullet \rightarrow C'_\bullet$ that commute with the maps in the short exact sequence, the induced homomorphisms on homology groups commute with the connecting homomorphisms in the corresponding long exact sequence of homology groups
- Naturality is a powerful tool for studying the behavior of homology groups under various constructions and transformations in homological algebra
- Allows the transfer of information between different chain complexes and their homology groups, facilitating the computation and comparison of homology groups in various contexts

Chain Homotopy Equivalence and Quasi-Isomorphisms

Chain Homotopy and Chain Homotopy Equivalence

- Chain homotopy between chain maps $f, g : C \cdot \rightarrow D \cdot$ is a family of homomorphisms $h_n : C_n \rightarrow D_{n+1}$ for each $n \in \mathbb{Z}$ satisfying $f_n - g_n = \partial_{n+1}^D \circ h_n + h_{n-1} \circ \partial_n^C$
- Chain homotopy measures the "difference" between chain maps f and g , with the homomorphisms h_n providing a way to "deform" one chain map into the other
- Existence of a chain homotopy between f and g implies that they induce the same homomorphisms on homology groups, i.e., $H_n(f) = H_n(g)$ for all $n \in \mathbb{Z}$
- Chain homotopy equivalence between chain complexes $C \cdot$ and $D \cdot$ consists of chain maps $f : C \cdot \rightarrow D \cdot$ and $g : D \cdot \rightarrow C \cdot$ such that $g \circ f$ is chain homotopic to id_C , and $f \circ g$ is chain homotopic to id_D .
- Chain homotopy equivalence is a weaker notion than isomorphism of chain complexes, as it allows for non-invertible chain maps that still induce isomorphisms on homology groups
- Chain complexes that are chain homotopy equivalent have isomorphic homology groups, i.e., $H_n(C \cdot) \cong H_n(D \cdot)$ for all $n \in \mathbb{Z}$

Quasi-Isomorphisms and Their Properties

- Quasi-isomorphism is a chain map $f : C \cdot \rightarrow D \cdot$ that induces isomorphisms $H_n(f) : H_n(C \cdot) \rightarrow H_n(D \cdot)$ on homology groups for all $n \in \mathbb{Z}$
- Quasi-isomorphisms are a stronger notion than chain homotopy equivalences, as they directly induce isomorphisms on homology groups without the need for a chain homotopy inverse
- Composition of quasi-isomorphisms is again a quasi-isomorphism, allowing for the study of chain complexes and their homology groups up to quasi-isomorphism
- Quasi-isomorphisms play a central role in the theory of derived categories and derived functors, where they are used to localize categories of chain complexes with respect to quasi-isomorphisms
- Localization process allows for the construction of derived categories, where quasi-isomorphic chain complexes are identified and morphisms are formally inverted, leading to a powerful framework for studying homological invariants and constructions
- Quasi-isomorphisms between chain complexes of modules over a ring R induce isomorphisms between the corresponding derived functors, such as Ext and Tor , computed using these chain complexes
- Allows for the computation and comparison of derived functors using simpler or more convenient chain complex resolutions, providing a valuable tool in homological algebra

2.4 Exact sequences of chain complexes

Exact sequences of chain complexes are crucial tools in homological algebra. They help us understand relationships between different chain complexes and their homology groups. These sequences form the backbone of many important results in the field.

This topic builds on our understanding of chain complexes and introduces key concepts like short and long exact sequences. We'll explore powerful tools like the snake lemma and mapping cones, which are

essential for analyzing complex algebraic structures.

Exact Sequences

Short Exact Sequences and Their Properties

- A short exact sequence is a sequence of three objects and two morphisms in an abelian category where the image of the first morphism equals the kernel of the second morphism
- Short exact sequences have the form $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ where f is injective, g is surjective, and $\text{im}(f) = \ker(g)$
- The injectivity of f and the surjectivity of g imply that A can be viewed as a subobject of B and C is isomorphic to the quotient object B/A
- Short exact sequences are used to study the relationships between objects in an abelian category and can provide information about the structure of the objects involved

Long Exact Sequences and Connecting Homomorphisms

- A long exact sequence is an infinite exact sequence of objects and morphisms in an abelian category
- Long exact sequences are often obtained by applying a covariant or contravariant functor to a short exact sequence
- The connecting homomorphism is a morphism that relates the homology groups of the objects in a long exact sequence
- Given a short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$, the connecting homomorphism $\delta_n : H_n(C) \rightarrow H_{n-1}(A)$ is defined by the composition $H_n(C) \rightarrow H_n(B, A) \rightarrow H_{n-1}(A)$, where the first map is induced by the quotient map $B \rightarrow C$ and the second map is the boundary map in the long exact sequence of relative homology
- The connecting homomorphism provides a way to relate the homology groups of the objects in a short exact sequence and is a crucial component in the construction of long exact sequences

Lemmas

The Snake Lemma and Its Applications

- The snake lemma is a powerful tool in homological algebra that relates the kernels, cokernels, and homology groups of morphisms between short exact sequences
- Given a commutative diagram of short exact sequences with vertical morphisms f , g , and h , the snake lemma provides a long exact sequence connecting the kernels, cokernels, and homology groups of f , g , and h
- The long exact sequence obtained from the snake lemma has the form $0 \rightarrow \ker(f) \rightarrow \ker(g) \rightarrow \ker(h) \xrightarrow{\delta} \text{coker}(f) \rightarrow \text{coker}(g) \rightarrow \text{coker}(h) \rightarrow 0$, where δ is the connecting homomorphism
- The snake lemma is often used to prove the functoriality of homology and cohomology theories and to study the relationships between morphisms in abelian categories

The Five Lemma and the Horseshoe Lemma

- The five lemma states that if a commutative diagram of exact sequences has isomorphisms for the first, second, fourth, and fifth vertical arrows, then the middle vertical arrow is also an isomorphism
- The five lemma is useful for proving that certain morphisms are isomorphisms by verifying that the surrounding morphisms in a commutative diagram are isomorphisms
- The horseshoe lemma is a result in homological algebra that relates the homology of a complex to the homology of its subcomplexes and quotient complexes
- Given a short exact sequence of chain complexes $0 \rightarrow A_{\bullet} \rightarrow B_{\bullet} \rightarrow C_{\bullet} \rightarrow 0$, the horseshoe lemma provides a long exact sequence connecting the homology groups of $A_{\bullet}$, $B_{\bullet}$, and $C_{\bullet}$.

Constructions

The Mapping Cone and Its Properties

- The mapping cone is a construction in homological algebra that associates a chain complex to a chain map between two chain complexes
- Given a chain map $f: A_{\bullet} \rightarrow B_{\bullet}$, the mapping cone of f is the chain complex $\text{Cone}(f)_n = A_{n-1} \oplus B_n$ with differential $d(a, b) = (-d_A(a), f(a) + d_B(b))$
- The mapping cone fits into a short exact sequence $0 \rightarrow B_{\bullet} \rightarrow \text{Cone}(f)_{\bullet} \rightarrow A_{\bullet}[-1] \rightarrow 0$, where $A_{\bullet}[-1]$ denotes the chain complex $A_{\bullet}$ shifted by one degree
- The homology of the mapping cone is related to the homology of the source and target complexes by a long exact sequence $\dots \rightarrow H_n(A_{\bullet}) \xrightarrow{f_*} H_n(B_{\bullet}) \rightarrow H_n(\text{Cone}(f)_{\bullet}) \rightarrow H_{n-1}(A_{\bullet}) \rightarrow \dots$
- The mapping cone construction is used to study the relationships between chain maps and their induced morphisms on homology groups

Unit 3 – Exact Sequences and Diagram Chasing

3.1 Short and long exact sequences

Exact sequences are powerful tools in algebra, connecting objects through precise relationships. They reveal structural connections between groups, rings, and modules, helping us understand complex mathematical structures.

Short exact sequences link three objects, while long exact sequences extend infinitely. Both types use homomorphisms to preserve structure and provide insights into algebraic relationships. They're essential for computing invariants and analyzing object properties.

Exact Sequences and Homomorphisms

Short Exact Sequences

- A short exact sequence is a sequence of homomorphisms between three objects $A \xrightarrow{f} B \xrightarrow{g} C$ such that f is injective, g is surjective, and $\text{im } f = \ker g$
- The injectivity of f implies that A can be viewed as a subobject of B , while the surjectivity of g implies that C is a quotient object of B
- Short exact sequences provide a way to relate three objects in a category, often revealing important structural relationships
- Example: In the category of abelian groups, the sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ is a short exact sequence

Long Exact Sequences

- A long exact sequence is an infinite sequence of homomorphisms between objects $\cdots \rightarrow A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \rightarrow \cdots$ such that the composition of any two consecutive homomorphisms is zero and the kernel of each homomorphism is equal to the image of the previous one
- Long exact sequences often arise from short exact sequences through the application of functors, such as the homology or cohomology functors
- They provide a powerful tool for computing invariants of objects by relating them to the invariants of other objects in the sequence
- Example: The long exact sequence in homology associated to a pair of topological spaces (X, A) is given by $\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots$

Exactness and Homomorphisms

- A sequence of homomorphisms is called exact if the image of each homomorphism is equal to the kernel of the next homomorphism
- Exactness is a crucial property in the study of exact sequences, as it ensures that the objects and homomorphisms fit together in a precise way

- Homomorphisms are structure-preserving maps between algebraic objects, such as groups, rings, or modules
- In the context of exact sequences, homomorphisms play a central role in relating the objects and their invariants
- Example: In the short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$, the homomorphisms f and g preserve the algebraic structure of the objects A , B , and C

Kernels, Images, and Cokernels

Kernels

- The kernel of a homomorphism $f: A \rightarrow B$ is the subobject of A consisting of all elements that are mapped to the zero element in B
- Kernels capture the notion of the "zero-set" of a homomorphism and play a crucial role in the construction of exact sequences
- The kernel of a homomorphism is always a normal subobject, which allows for the formation of quotient objects
- Example: In the category of groups, the kernel of a group homomorphism $f: G \rightarrow H$ is the normal subgroup $\ker f = \{g \in G \mid f(g) = e_H\}$

Images

- The image of a homomorphism $f: A \rightarrow B$ is the subobject of B consisting of all elements that are the result of applying f to some element in A
- Images capture the notion of the "range" of a homomorphism and are essential in the study of surjectivity
- In an exact sequence, the image of each homomorphism is equal to the kernel of the next homomorphism
- Example: In the category of vector spaces, the image of a linear map $f: V \rightarrow W$ is the subspace $\text{im } f = \{f(v) \mid v \in V\}$

Cokernels

- The cokernel of a homomorphism $f: A \rightarrow B$ is the quotient object of B by the image of f
- Cokernels capture the notion of the "coimage" of a homomorphism and are dual to the concept of kernels
- In an exact sequence, the cokernel of a homomorphism can be identified with the kernel of the next homomorphism
- Example: In the category of abelian groups, the cokernel of a group homomorphism $f: A \rightarrow B$ is the quotient group $\text{coker } f = B/\text{im } f$

Splitting Lemma

Splitting Exact Sequences

- The splitting lemma states that a short exact sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ splits if and only if there exists a homomorphism $h : C \rightarrow B$ such that $g \circ h = \text{id}_C$
- A split exact sequence implies that B is isomorphic to the direct sum of A and C , i.e., $B \cong A \oplus C$
- The splitting lemma provides a way to decompose an object in an exact sequence into simpler components
- Example: In the category of vector spaces, a short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ splits if and only if there exists a linear map $s : W \rightarrow V$ such that the composition $W \rightarrow V \rightarrow W$ is the identity on W

Applications of the Splitting Lemma

- The splitting lemma is a powerful tool for understanding the structure of objects in an exact sequence
- It allows for the classification of extensions of objects, which has applications in various areas of algebra and topology
- The splitting lemma is often used in conjunction with other techniques, such as the snake lemma, to compute invariants and study the properties of algebraic objects
- Example: In the study of group extensions, the splitting lemma can be used to determine whether an extension is trivial or not, providing insights into the structure of the groups involved

3.2 The snake lemma and its applications

The snake lemma is a powerful tool in homological algebra. It connects kernels and cokernels in commutative diagrams with exact rows, creating a long exact sequence. This lemma is crucial for understanding how different parts of algebraic structures relate to each other.

The snake lemma has wide-ranging applications in algebra. It's used to prove other important results like the five lemma and can be extended to more complex situations. Understanding this lemma is key to grasping how short and long exact sequences work together.

The Snake Lemma and Connecting Homomorphisms

Snake Lemma and its Components

- States that given a commutative diagram with exact rows, there exists a long exact sequence connecting the kernels and cokernels of the vertical maps
- Involves the connecting homomorphism, which maps the cokernel of one vertical map to the kernel of the next vertical map
- Utilizes the ker-coker exact sequence, which is a long exact sequence that alternates between kernels and cokernels of the vertical maps in the commutative diagram
- Applies the zig-zag lemma, which states that if a composition of two maps is zero, then the image of the first map is contained in the kernel of the second map

Applications and Extensions

- Used to prove the five lemma, which states that if the first two and last two vertical maps are isomorphisms in a commutative diagram with exact rows, then the middle vertical map is also an isomorphism
- Can be generalized to the 3x3 lemma, which involves a 3x3 commutative diagram with exact rows and columns
- Applies to various algebraic structures, such as modules over a ring, abelian groups, and chain complexes
- Helps in understanding the relationship between short exact sequences and long exact sequences in homological algebra

Homological Algebra Fundamentals

Commutative Diagrams and Chain Complexes

- A commutative diagram is a diagram of objects and morphisms where all directed paths between two objects lead to the same result
- Commutativity is a crucial property in homological algebra, as it allows for the study of functorial properties and the construction of exact sequences
- A chain complex is a sequence of abelian groups or modules connected by homomorphisms (differentials) such that the composition of any two consecutive homomorphisms is zero
- Chain complexes are used to define homology groups, which measure the "holes" or "cycles modulo boundaries" in the complex

Homology and its Properties

- Homology is a functor that assigns to each chain complex a sequence of abelian groups (homology groups), which are the quotients of the kernel of one differential by the image of the previous differential
- Homology groups are invariants of the chain complex and provide information about its algebraic structure
- The homology functor is a covariant functor from the category of chain complexes to the category of graded abelian groups
- Homology satisfies the Eilenberg-Steenrod axioms, which characterize homology theories in algebraic topology and allow for the comparison of different homology theories

3.3 Diagram chasing techniques

Diagram chasing is a key technique in homological algebra, helping us navigate complex relationships between objects and morphisms. It's all about following paths through diagrams to prove statements and transfer information between different parts.

Universal constructions like pullbacks and pushouts are essential tools in category theory. They capture important concepts like fiber products and amalgamated sums, giving us a powerful way to describe and work with mathematical structures.

Diagram Chasing Techniques

Navigating Commutative Diagrams

- Diagram chasing involves navigating commutative diagrams to prove statements about the objects and morphisms in the diagram
- Relies on the commutativity of certain squares or triangles within the diagram
- Allows for the transfer of information between different parts of the diagram
- Element chasing is a specific technique within diagram chasing
- Involves following the path of a particular element through the diagram
- Utilizes the commutativity of the diagram to show that different compositions of morphisms yield the same result for the chosen element
- Commutative squares are fundamental building blocks in diagram chasing
- A square is commutative if the composition of morphisms along one path equals the composition along the other path
- Enables the substitution of one path for another when working with elements or morphisms

Applying the Diagram Lemma

- The diagram lemma is a powerful tool in diagram chasing
- States that if certain squares in a diagram are commutative and the rows are exact, then the remaining squares are also commutative
- Applying the diagram lemma simplifies proofs by reducing the number of squares that need to be directly verified for commutativity
- Allows for the deduction of commutativity in squares that may be difficult to prove directly
- The diagram lemma is particularly useful in scenarios involving long exact sequences or complex diagrams with multiple interconnected squares
- Enables the propagation of commutativity throughout the diagram
- Simplifies the overall proof structure by leveraging the relationships between the squares

Universal Constructions

Pullbacks and Pushouts

- Pullbacks are universal constructions that capture the notion of a "fiber product" in category theory
- Given morphisms $f: A \rightarrow C$ and $g: B \rightarrow C$, the pullback is an object P with morphisms $p_1: P \rightarrow A$ and $p_2: P \rightarrow B$ such that $f \circ p_1 = g \circ p_2$
- The pullback satisfies a universal property: for any object Q with morphisms $q_1: Q \rightarrow A$ and $q_2: Q \rightarrow B$ such that $f \circ q_1 = g \circ q_2$, there exists a unique morphism $u: Q \rightarrow P$ such that $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$
- Pushouts are dual to pullbacks and capture the notion of a "amalgamated sum" in category theory

- Given morphisms $f : C \rightarrow A$ and $g : C \rightarrow B$, the pushout is an object P with morphisms $p_1 : A \rightarrow P$ and $p_2 : B \rightarrow P$ such that $p_1 \circ f = p_2 \circ g$
- The pushout satisfies a universal property: for any object Q with morphisms $q_1 : A \rightarrow Q$ and $q_2 : B \rightarrow Q$ such that $q_1 \circ f = q_2 \circ g$, there exists a unique morphism $u : P \rightarrow Q$ such that $u \circ p_1 = q_1$ and $u \circ p_2 = q_2$

Uniqueness of Maps in Universal Constructions

- Universal constructions, such as pullbacks and pushouts, are characterized by the existence and uniqueness of certain morphisms
- The uniqueness of maps in universal constructions is crucial for their well-definedness and their role in categorical reasoning
- Ensures that the universal object is determined up to unique isomorphism
- Allows for the unambiguous definition of constructions and the derivation of their properties
- The uniqueness of maps is often proved using the universal property of the construction
- Suppose there are two morphisms u_1, u_2 satisfying the universal property
- By applying the universal property to each morphism, one can show that $u_1 = u_2$, establishing uniqueness

3.4 Five lemma and nine lemma

The five lemma and nine lemma are powerful tools for proving isomorphisms in commutative diagrams with exact rows. They extend the concept of exact sequences, allowing us to analyze more complex relationships between objects and morphisms.

These lemmas build on our understanding of exact sequences, providing a framework for tackling intricate diagrams. By applying these tools, we can simplify proofs and gain deeper insights into the structure of algebraic objects and their connections.

Five and Nine Lemmas

The Five Lemma and its Variations

- Five lemma states that given a commutative diagram with exact rows, if four out of five vertical arrows are isomorphisms, then the fifth one is also an isomorphism
- Weak five lemma relaxes the conditions of the five lemma by requiring only that the four vertical arrows are either epimorphisms or monomorphisms, concluding that the fifth arrow has the same property
- These lemmas are powerful tools for proving the existence of isomorphisms between objects in a commutative diagram with exact rows ($A \rightarrow B \rightarrow C \rightarrow D \rightarrow E$)
- Applying the five lemma or weak five lemma can simplify the process of proving isomorphisms in complex diagrams

The Nine Lemma and 3x3 Lemma

- Nine lemma extends the five lemma to a 3x3 grid of commutative diagrams with exact rows and columns

- It states that if the three objects in the top left corner, the three objects in the bottom right corner, and the object in the center are zero, and the remaining two objects in the middle row and column are isomorphic, then the objects in the four corners are also isomorphic
- 3x3 lemma is a special case of the nine lemma where all the objects in the diagram are zero, except for the four corner objects
- These lemmas are useful for proving isomorphisms in more intricate commutative diagrams ($A \rightarrow B \rightarrow C, D \rightarrow E \rightarrow F, G \rightarrow H \rightarrow I$) where the five lemma alone may not be sufficient

Exact Sequences and Isomorphism Theorems

Exact Sequences and their Properties

- Exact sequences are sequences of objects and morphisms ($A \rightarrow B \rightarrow C$) where the image of each morphism is equal to the kernel of the next morphism
- Short exact sequences ($0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$) are exact sequences where the first object is the zero object, the morphism $A \rightarrow B$ is injective, and the morphism $B \rightarrow C$ is surjective
- Splitting lemma states that a short exact sequence splits (i.e., $B \cong A \oplus C$) if and only if there exists a morphism $C \rightarrow B$ such that the composition $C \rightarrow B \rightarrow C$ is the identity morphism on C
- Exact sequences provide a way to study the relationships between objects and morphisms in a category, and they are fundamental in homological algebra

Isomorphism Theorems and Long Exact Sequences

- Isomorphism theorems relate quotients, subobjects, and homomorphisms in a category
- First isomorphism theorem (or fundamental theorem of homomorphisms) states that for a homomorphism $f: A \rightarrow B$, there is an isomorphism between the quotient $A/\ker(f)$ and the image $\text{im}(f)$
- Second isomorphism theorem states that for a subobject A and a normal subobject B of an object C , there is an isomorphism between $(A + B)/B$ and $A/(A \cap B)$
- Third isomorphism theorem states that for normal subobjects A and B of an object C with $A \subseteq B$, there is an isomorphism between $(C/A)/(B/A)$ and C/B
- Exact sequences of complexes are sequences of chain complexes and chain maps between them, where the composition of any two consecutive chain maps is zero
- Long exact sequence in homology is an exact sequence that relates the homology groups of a short exact sequence of chain complexes ($0 \rightarrow A_{\bullet} \rightarrow B_{\bullet} \rightarrow C_{\bullet} \rightarrow 0$)
- Connects homology groups of the three complexes:
 $\cdots \rightarrow H_n(A_{\bullet}) \rightarrow H_n(B_{\bullet}) \rightarrow H_n(C_{\bullet}) \rightarrow H_{n-1}(A_{\bullet}) \rightarrow \cdots$
- Useful for computing homology groups of complexes by relating them to the homology groups of simpler complexes

Unit 4 – Functors and Natural Transformations

4.1 Covariant and contravariant functors

Functors are the backbone of category theory, bridging different mathematical structures. They map objects and morphisms between categories, preserving essential properties. Understanding functors is crucial for grasping how mathematical concepts relate across diverse fields.

Covariant and contravariant functors are two key types that differ in how they handle morphism direction. Covariant functors maintain direction, while contravariant functors reverse it. This distinction is vital for understanding how information flows between categories.

Functors and Categories

Defining Functors and Their Components

- Functor maps objects and morphisms from one category to another category while preserving the structure and composition of morphisms
- Category consists of a collection of objects and morphisms between those objects, along with a composition operation that satisfies associativity and identity laws
- Morphism is an arrow between two objects in a category representing a structure-preserving map or transformation (functions between sets)
- Object is a basic element or entity within a category upon which morphisms act (sets in the category of sets)
- Identity functor maps each object and morphism in a category to itself, serving as the identity element under functor composition

Functor Properties and Identity Functor

- Functors preserve the identity morphism of each object, mapping it to the identity morphism of the corresponding object in the target category
- Functors respect the composition of morphisms, ensuring that the composition of mapped morphisms equals the mapping of the composed morphisms: $F(g \circ f) = F(g) \circ F(f)$
- Identity functor, often denoted as 1_C or Id_C , maps every object and morphism in a category C to itself
- The identity functor serves as the identity element under functor composition, meaning for any functor $F : C \rightarrow D$, we have $F \circ 1_C = F$ and $1_D \circ F = F$

Types of Functors

Covariant and Contravariant Functors

- Covariant functor preserves the direction of morphisms, mapping morphisms from $A \rightarrow B$ in the source category to morphisms $F(A) \rightarrow F(B)$ in the target category

- Covariant functors are the most common type of functor and are often referred to simply as "functors" when the context is clear
- Contravariant functor reverses the direction of morphisms, mapping morphisms from $A \rightarrow B$ in the source category to morphisms $F(B) \rightarrow F(A)$ in the target category
- Contravariant functors are sometimes called "cofunctors" and can be thought of as functors that "go against the grain" of the original category

Functor Composition and Examples

- Functor composition allows functors to be composed together, creating a new functor that combines their actions
- Given functors $F : C \rightarrow D$ and $G : D \rightarrow E$, their composition $G \circ F : C \rightarrow E$ is defined by $(G \circ F)(A) = G(F(A))$ for objects and $(G \circ F)(f) = G(F(f))$ for morphisms
- Functor composition is associative, meaning $(H \circ G) \circ F = H \circ (G \circ F)$ for functors $F : C \rightarrow D$, $G : D \rightarrow E$, and $H : E \rightarrow F$
- Examples of covariant functors include the power set functor $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$ and the fundamental group functor $\pi_1 : \mathbf{Top}^* \rightarrow \mathbf{Grp}$
- An example of a contravariant functor is the dual vector space functor $(-)^* : \mathbf{Vect}_K^{op} \rightarrow \mathbf{Vect}_K$, which maps a vector space to its dual space and reverses the direction of linear transformations

4.2 Additive and exact functors

Functors are key players in category theory, bridging different mathematical structures. Additive functors preserve the addition of morphisms, while exact functors maintain the exactness of sequences. These concepts are crucial for understanding how information flows between categories.

Left exact functors preserve kernels, while right exact functors preserve cokernels. This distinction helps us analyze how functors interact with short exact sequences, shedding light on the relationships between different mathematical objects and structures.

Additive and Exact Functors

Additive Functors and Their Properties

- Additive functors preserve the additive structure of categories
- Map objects to objects and morphisms to morphisms
- Respect addition of morphisms and the zero morphism
- Additive functors commute with finite direct sums
- $F(A \oplus B) \cong F(A) \oplus F(B)$
- Examples of additive functors include:
 - Identity functor $1_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$
 - Forgetful functor from the category of abelian groups to the category of sets

Exact Functors and Sequences

- Exact functors preserve exact sequences
- Map short exact sequences to short exact sequences
- Preserve kernels and cokernels
- Left exact functors preserve exactness at the beginning of a sequence
- Preserve kernels and monomorphisms
- Example: Hom functor $\text{Hom}(A, -) : \mathcal{A} \rightarrow \mathbf{Ab}$
- Right exact functors preserve exactness at the end of a sequence
- Preserve cokernels and epimorphisms
- Example: Tensor product functor $- \otimes B : \mathbf{Ab} \rightarrow \mathbf{Ab}$

Short Exact Sequences and Abelian Categories

Short Exact Sequences

- A short exact sequence is a sequence of objects and morphisms in an abelian category
- $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$
- f is a monomorphism (injective), g is an epimorphism (surjective)
- $\text{im } f = \ker g$
- Short exact sequences capture the idea of B being an extension of A by C
- Examples of short exact sequences in the category of abelian groups:
 - $0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \rightarrow 0$
 - $0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$

Abelian Categories and Their Properties

- Abelian categories are categories with additional structure
- Have a zero object, binary products and coproducts, and kernels and cokernels
- Satisfy certain axioms related to the behavior of morphisms and exact sequences
- Kernels are the categorical generalization of the kernel of a group homomorphism
- The kernel of a morphism $f : A \rightarrow B$ is the equalizer of f and the zero morphism
- Cokernels are the dual notion of kernels
- The cokernel of a morphism $f : A \rightarrow B$ is the coequalizer of f and the zero morphism
- Examples of abelian categories include:
 - The category of abelian groups **Ab**
 - The category of modules over a ring **$R\text{-Mod}$**

4.3 Natural transformations and functor categories

Natural transformations are the secret sauce of functors, letting us compare them smoothly. They're like bridges between functors, with components that play nice with morphisms. Think of them as the glue that holds the functor world together.

Functor categories take things up a notch, treating functors as objects and natural transformations as morphisms. It's like a meta-category where functors themselves become the stars of the show. This setup lets us study functors in a whole new light.

Natural Transformations

Definition and Components

- Natural transformation $\eta : F \rightarrow G$ consists of a family of morphisms $\eta_X : F(X) \rightarrow G(X)$ for each object X in the source category
- Components of a natural transformation are the individual morphisms η_X that make up the natural transformation
- Components must satisfy the naturality condition for every morphism $f : X \rightarrow Y$ in the source category:
 $G(f) \circ \eta_X = \eta_Y \circ F(f)$
- Naturality condition ensures that the components of the natural transformation are compatible with the morphisms in the source and target categories
- Commutative diagram can be used to visualize the naturality condition

Natural Isomorphisms and Properties

- Natural isomorphism is a natural transformation $\eta : F \rightarrow G$ where each component η_X is an isomorphism in the target category
- Inverse of a natural isomorphism is also a natural transformation, denoted as $\eta^{-1} : G \rightarrow F$
- Identity natural transformation $1_F : F \rightarrow F$ has components $(1_F)_X = 1_{F(X)}$ for each object X in the source category
- Composition of natural transformations $\eta : F \rightarrow G$ and $\mu : G \rightarrow H$ is defined componentwise:
 $(\mu \circ \eta)_X = \mu_X \circ \eta_X$
- Natural transformations can be used to define equivalence of categories (if there exist natural isomorphisms between two functors)

Functor Categories

Definition and Objects

- Functor category $\mathbf{C}^{\mathbf{D}}$ has functors $F : \mathbf{D} \rightarrow \mathbf{C}$ as objects
- Morphisms in the functor category are natural transformations between the functors
- Identity functor $1_{\mathbf{D}} : \mathbf{D} \rightarrow \mathbf{D}$ serves as the identity object in the functor category
- Constant functor $\Delta_X : \mathbf{D} \rightarrow \mathbf{C}$ sends every object in $\mathbf{D}$ to a fixed object X in $\mathbf{C}$ and every morphism in $\mathbf{D}$ to the identity morphism 1_X

Composition of Natural Transformations

- Vertical composition of natural transformations $\eta : F \rightarrow G$ and $\mu : G \rightarrow H$ is defined as $\mu \circ \eta : F \rightarrow H$
- Horizontal composition of natural transformations $\eta : F \rightarrow G$ in $\mathbf{C}^{\mathbf{D}}$ and $\theta : H \rightarrow K$ in $\mathbf{D}^{\mathbf{E}}$ is a natural transformation $\eta * \theta : F \circ H \rightarrow G \circ K$ in $\mathbf{C}^{\mathbf{E}}$
- Horizontal composition is defined componentwise: $(\eta * \theta)_X = \eta_{K(X)} \circ F(\theta_X)$
- Interchange law holds for vertical and horizontal composition: $(\mu \circ \eta) * (\theta \circ \phi) = (\mu * \theta) \circ (\eta * \phi)$

Yoneda Lemma

Statement and Consequences

- Yoneda lemma states that for any functor $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ and any object X in $\mathbf{C}$, there is a bijection between the set of natural transformations $\text{Nat}(\text{Hom}_{\mathbf{C}}(X, -), F)$ and the set $F(X)$
- Bijection is given by evaluating a natural transformation at the identity morphism of X
- Yoneda lemma implies that the functor $\text{Hom}_{\mathbf{C}}(X, -)$ (represented functor) completely determines the object X up to isomorphism
- Yoneda embedding $\mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ sends each object X to its represented functor $\text{Hom}_{\mathbf{C}}(X, -)$ and each morphism $f : X \rightarrow Y$ to the natural transformation $\text{Hom}_{\mathbf{C}}(f, -)$
- Yoneda embedding is fully faithful, meaning that it preserves and reflects isomorphisms between objects and morphisms

4.4 Adjoint functors

Adjoint functors are like matchmakers between categories, pairing up objects and morphisms in a special way. They help us see connections between different mathematical structures and give us tools to move between them.

Left and right adjoints have unique properties, preserving certain structures as they map between categories. Understanding adjoint functors is key to grasping how different mathematical worlds relate to each other.

Adjoint Functors and Adjoints

Defining Adjoint Functors

- Adjoint functors consist of a pair of functors $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{D} \rightarrow \mathbf{C}$ between categories $\mathbf{C}$ and $\mathbf{D}$
- Adjoint functors establish a relationship between the morphisms in the two categories
- The existence of an adjunction implies that the categories $\mathbf{C}$ and $\mathbf{D}$ share similar properties and structures
- Adjoint functors play a crucial role in understanding the connections and similarities between different mathematical structures

Left and Right Adjoints

- If $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ form an adjoint pair, F is called the left adjoint of G , and G is called the right adjoint of F
- The left adjoint F preserves colimits (coproducts, coequalizers) from $\mathcal{C}$ to $\mathcal{D}$
- The right adjoint G preserves limits (products, equalizers) from $\mathcal{D}$ to $\mathcal{C}$
- The existence of a left or right adjoint for a functor provides additional properties and insights into the functor's behavior (preservation of certain categorical constructions)

Adjunction Isomorphism

- An adjunction between functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ is characterized by a natural isomorphism: $\text{Hom}_{\mathcal{D}}(F(C), D) \cong \text{Hom}_{\mathcal{C}}(C, G(D))$ for all objects C in $\mathcal{C}$ and D in $\mathcal{D}$
- The adjunction isomorphism relates the morphisms between objects in the categories $\mathcal{C}$ and $\mathcal{D}$
- The adjunction isomorphism is natural, meaning it commutes with the composition of morphisms in both categories
- Examples of adjoint functors include:
 - The free functor $F : \mathbf{Set} \rightarrow \mathbf{Grp}$ (from sets to groups) and the forgetful functor $G : \mathbf{Grp} \rightarrow \mathbf{Set}$
 - The product functor $\Pi : \mathbf{Set}^I \rightarrow \mathbf{Set}$ (from the category of I -indexed sets to sets) and the diagonal functor $\Delta : \mathbf{Set} \rightarrow \mathbf{Set}^I$

Universal Properties and Units

Universal Properties

- A universal property characterizes an object or a morphism in a category by its relationship with other objects and morphisms
- Universal properties describe the unique existence of morphisms that satisfy certain conditions
- Objects or morphisms satisfying a universal property are unique up to unique isomorphism
- Universal properties provide a way to define and characterize mathematical structures categorically
- Examples of universal properties include:
 - The product of objects in a category (characterized by the existence of unique morphisms from any other object to the product)
 - The coproduct of objects in a category (characterized by the existence of unique morphisms from the coproduct to any other object)

Unit and Counit of Adjunction

- Given an adjunction between functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$, there exist natural transformations called the unit and counit of the adjunction
- The unit of the adjunction is a natural transformation $\eta : 1_{\mathcal{C}} \rightarrow GF$, where $1_{\mathcal{C}}$ is the identity functor on $\mathcal{C}$

- The counit of the adjunction is a natural transformation $\varepsilon : FG \rightarrow 1_D$, where 1_D is the identity functor on D
- The unit and counit satisfy the triangle identities:
 - $(\varepsilon F) \circ (F\eta) = 1_F$
 - $(G\varepsilon) \circ (\eta G) = 1_G$
- The unit and counit provide a way to relate the compositions of the adjoint functors F and G to the identity functors on their respective categories

Adjunctions and Universal Morphisms

- Adjunctions can be expressed in terms of universal morphisms
- The unit of an adjunction corresponds to a universal morphism from an object to its image under the right adjoint functor
- The counit of an adjunction corresponds to a universal morphism from the image of an object under the left adjoint functor to the object itself
- The universal property of the unit and counit characterizes the adjunction
- Examples of universal morphisms in adjunctions include:
 - The unit of the free-forgetful adjunction between sets and groups (the inclusion of a set into its free group)
 - The counit of the product-diagonal adjunction between sets and indexed sets (the projection from a product to its components)

Equivalence of Categories

Defining Equivalence of Categories

- An equivalence of categories is a stronger notion than an isomorphism of categories
- Two categories C and D are equivalent if there exist functors $F : C \rightarrow D$ and $G : D \rightarrow C$ such that:
 - There are natural isomorphisms $\alpha : GF \rightarrow 1_C$ and $\beta : FG \rightarrow 1_D$
 - The compositions GF and FG are naturally isomorphic to the identity functors on C and D , respectively
- Equivalent categories have the same categorical properties and structure, but may differ in their object and morphism sets
- Equivalence of categories preserves limits, colimits, and other categorical constructions
- Examples of equivalent categories include:
 - The category of finite-dimensional vector spaces over a field k and the category of $n \times n$ matrices over k
 - The category of sets and the category of complete atomic Boolean algebras

Adjoint Equivalence

- An adjoint equivalence is a special case of an equivalence of categories
- In an adjoint equivalence, the functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ form an adjoint pair
- The unit and counit of the adjunction are natural isomorphisms
- Adjoint equivalences provide a way to establish the equivalence of categories using the properties of adjoint functors
- Examples of adjoint equivalences include:
 - The equivalence between the category of finite-dimensional vector spaces over a field k and the category of finite-dimensional k -algebras
 - The equivalence between the category of compact Hausdorff spaces and the category of commutative C^* -algebras

Applications of Equivalence

- Equivalence of categories allows the transfer of results and properties between equivalent categories
- Equivalent categories can be used interchangeably in mathematical reasoning and problem-solving
- Equivalence of categories simplifies the study of complex structures by relating them to simpler or better-understood categories
- Equivalence of categories is used in various areas of mathematics, including algebra, topology, and geometry, to establish connections and similarities between different mathematical objects and structures

Unit 5 – Projective and Injective Modules

5.1 Definitions and properties of projective modules

Projective modules are a key concept in homological algebra, allowing us to "lift" along surjective homomorphisms. They're closely related to free modules, which have a basis. Understanding these modules helps us analyze algebraic structures and solve complex problems.

Projective modules have special properties like being direct summands of free modules and splitting exact sequences. These characteristics make them powerful tools for studying module theory and developing advanced algebraic techniques.

Projective and Free Modules

Properties and Definitions

- Projective module is a module P such that for every surjective homomorphism $f: M \rightarrow N$ and every homomorphism $g: P \rightarrow N$, there exists a homomorphism $h: P \rightarrow M$ such that $f \circ h = g$
- Free module is a module that has a basis, which means it is isomorphic to a direct sum of copies of the base ring R (denoted as $R^{(I)}$ for some index set I)
- Examples of free modules include $\mathbb{R}^n$ over $\mathbb{R}$ and $\mathbb{Z}^n$ over $\mathbb{Z}$
- Projectivity is a property of modules that allows them to be "lifted" along surjective homomorphisms
- Every free module is projective, but not every projective module is free (e.g., the $\mathbb{Z}$ -module $\mathbb{Q}$ is projective but not free)

Dual Basis Lemma

- Dual basis lemma states that if M is a free module with basis $\{x_i\}_{i \in I}$, then there exists a unique family of linear maps $\{f_i: M \rightarrow R\}_{i \in I}$ such that $f_i(x_j) = \delta_{ij}$ (Kronecker delta) for all $i, j \in I$
- The family $\{f_i\}_{i \in I}$ is called the dual basis of $\{x_i\}_{i \in I}$
- Dual basis lemma is useful for proving that every free module is projective and for constructing projective resolutions of modules

Direct Summands and Splitting

Direct Summands

- Direct summand is a submodule N of a module M such that there exists another submodule N' of M with $M = N \oplus N'$ (direct sum)
- Examples of direct summands include $\mathbb{Z}$ as a direct summand of $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Q}$ as a direct summand of $\mathbb{Q} \oplus \mathbb{Q}$
- A module M is projective if and only if it is a direct summand of a free module

Splitting Exact Sequences

- Splitting exact sequence is a short exact sequence $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$ such that the middle term M is isomorphic to the direct sum of N and P (i.e., $M \cong N \oplus P$)
- The sequence is said to split if there exists a homomorphism $s : P \rightarrow M$ such that $f \circ s = id_P$, where $f : M \rightarrow P$ is the surjective homomorphism in the sequence
- A module P is projective if and only if every short exact sequence $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$ splits

Lifting Property

- Lifting property is a characteristic of projective modules, stating that for every surjective homomorphism $f : M \rightarrow N$ and every homomorphism $g : P \rightarrow N$, there exists a homomorphism $h : P \rightarrow M$ such that $f \circ h = g$
- This property allows projective modules to be "lifted" along surjective homomorphisms
- The lifting property is equivalent to the splitting of short exact sequences and the existence of a direct summand in a free module

Key Lemmas

Schanuel's Lemma

- Schanuel's lemma states that if $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ and $0 \rightarrow K' \rightarrow P' \rightarrow M \rightarrow 0$ are two short exact sequences with P and P' projective modules, then $K \oplus P' \cong K' \oplus P$
- This lemma is useful for comparing the kernels of two projective resolutions of the same module
- Schanuel's lemma can be used to prove that the projective dimension of a module is well-defined (i.e., independent of the choice of projective resolution)
- The projective dimension of a module M is the smallest non-negative integer n such that there exists a projective resolution of M of length n , or ∞ if no such finite resolution exists

5.2 Definitions and properties of injective modules

Injective modules are crucial in homological algebra, offering a dual concept to projective modules. They're characterized by their ability to extend homomorphisms from submodules to larger modules, making them vital for constructing resolutions and studying cohomology.

Baer's criterion and essential extensions provide key ways to identify injective modules. The Matlis-Gabriel theorem reveals their structure over Noetherian rings, connecting them to prime ideals and opening up applications in local cohomology and Gorenstein ring theory.

Injective Modules and Divisible Modules

Definition and Properties of Injective Modules

- Injective module is a module Q over a ring R with the following property:
- For any module homomorphism $f : M \rightarrow Q$ and any injective module homomorphism $i : M \rightarrow N$, there exists a module homomorphism $g : N \rightarrow Q$ such that $g \circ i = f$

- Intuitively, this means that any homomorphism from a submodule of N into Q can be extended to a homomorphism from the entire module N into Q
- Divisible module is an important example of an injective module
- A module M over a ring R is divisible if for every $m \in M$ and every $r \in R$, there exists an element $x \in M$ such that $rx = m$
- In the case of abelian groups (modules over $\mathbb{Z}$), a divisible group is one in which every element can be divided by any non-zero integer
- Examples of divisible abelian groups include $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$
- Injectivity is a categorical notion dual to the concept of projectivity
- A module Q is injective if and only if the functor $\text{Hom}_R(-, Q)$ is exact
- This means that $\text{Hom}_R(-, Q)$ preserves short exact sequences

Extension Property and Applications

- Extension property is a key characteristic of injective modules
- If Q is an injective R -module, then for any R -module homomorphism $f: M \rightarrow Q$ and any injective R -module homomorphism $i: M \rightarrow N$, there exists an R -module homomorphism $g: N \rightarrow Q$ such that $g \circ i = f$
- This property allows for the extension of homomorphisms from submodules to the entire module
- Injective modules have important applications in homological algebra
- They are used in the construction of injective resolutions, which are crucial for computing derived functors
- Injective modules also play a role in the study of cohomology theories, such as group cohomology and Ext functors

Characterizations of Injectivity

Baer's Criterion and Essential Extensions

- Baer's criterion is a necessary and sufficient condition for a module to be injective
- An R -module Q is injective if and only if for every ideal I of R and every R -module homomorphism $f: I \rightarrow Q$, there exists an R -module homomorphism $g: R \rightarrow Q$ such that $g|_I = f$
- In other words, every homomorphism from an ideal of R into Q can be extended to a homomorphism from the entire ring R into Q
- Essential extension is another characterization of injectivity
- An R -module M is an essential extension of a submodule N if every non-zero submodule of M has a non-zero intersection with N
- A module Q is injective if and only if it has no proper essential extensions

Injective Envelopes and Their Properties

- Injective envelope (also called injective hull) is a minimal injective module containing a given module

- For any R -module M , there exists an injective R -module $E(M)$ containing M as an essential submodule
- The injective envelope $E(M)$ is unique up to isomorphism
- Properties of injective envelopes:
 - Every R -module can be embedded as an essential submodule of an injective R -module
 - If M is a submodule of an injective module Q , then M is essential in Q if and only if Q is an injective envelope of M
- Injective envelopes are used in the construction of minimal injective resolutions

Structure Theorems

Matlis-Gabriel Theorem and Its Implications

- Matlis-Gabriel theorem describes the structure of injective modules over a Noetherian ring
- Let R be a commutative Noetherian ring. Then every injective R -module is a direct sum of indecomposable injective modules
- Moreover, every indecomposable injective R -module is isomorphic to the injective envelope $E(R/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of R
- Implications of the Matlis-Gabriel theorem:
 - It provides a classification of injective modules over Noetherian rings
 - The theorem establishes a correspondence between indecomposable injective modules and prime ideals of the ring
 - It allows for the study of injective modules through the lens of prime ideals and localizations
 - The theorem has applications in local cohomology and the theory of Gorenstein rings

5.3 Characterizations and examples

Projective and injective modules are key players in homological algebra. They're like the superheroes of module theory, swooping in to solve problems and make life easier for algebraists.

This section dives into characterizing these modules and provides examples. We'll see how projective and injective modules relate to other concepts like flat modules, vector spaces, and abelian groups.

Projective and Injective Modules

Properties and Constructions of Projective and Injective Modules

- Projective cover
- Smallest projective module that maps onto a given module
- Unique up to isomorphism
- Can be used to measure the complexity of a module
- Injective hull

- Smallest injective module containing a given module as a submodule
- Unique up to isomorphism
- Dual notion to the projective cover
- Can be constructed using the injective envelope

Homological Dimensions

- Projective dimension
- Shortest length of a projective resolution of a module
- Measures how far a module is from being projective
- Projective modules have projective dimension 0
- Used to define the global dimension of a ring
- Injective dimension
- Shortest length of an injective resolution of a module
- Measures how far a module is from being injective
- Injective modules have injective dimension 0
- Used to define the global dimension of a ring

Special Module Types

Flat and Cotorsion Modules

- Flat module
- Module M such that the functor $- \otimes_R M$ is exact
- Generalizes the notion of a flat ring homomorphism
- Projective modules are flat, but the converse is not always true
- Example: Over a principal ideal domain, a module is flat if and only if it is torsion-free
- Cotorsion module
- Module C such that $\text{Ext}_R^1(F, C) = 0$ for all flat modules F
- Dual notion to flat modules
- Injective modules are cotorsion, but the converse is not always true

Vector Spaces

- Vector space
- Module over a field
- Has a basis, allowing for coordinate representation of elements
- Dimension of a vector space is the cardinality of its basis

- Example: $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ with the standard basis $\{e_1, \dots, e_n\}$

Ring Properties

Finiteness Conditions on Rings

- Noetherian ring
- Ring in which every ideal is finitely generated
- Equivalent to the ascending chain condition on ideals
- Examples: Fields, principal ideal domains, and polynomial rings over Noetherian rings
- Artinian ring
- Ring in which every descending chain of ideals stabilizes
- Implies the ring is Noetherian
- Examples: Fields and finite-dimensional algebras over fields

Abelian Groups

Examples and Properties

- Abelian group examples
- $\mathbb{Z}$ under addition
- $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ under addition
- $\mathbb{Z}/n\mathbb{Z}$ (cyclic groups) under addition
- The multiplicative group of nonzero elements of a field
- Abelian groups are $\mathbb{Z}$ -modules
- The ring action is given by integer multiplication
- Allows for the application of module-theoretic concepts to abelian groups
- Finitely generated abelian groups are direct sums of cyclic groups
- Fundamental theorem of finitely generated abelian groups
- Classification up to isomorphism

5.4 Existence theorems and applications

Projective and injective resolutions are powerful tools in homological algebra. They allow us to study modules through sequences of simpler objects, giving us insight into their structure and properties.

These resolutions are crucial for computing derived functors like Ext and Tor. They help us measure how far a module is from being projective or injective, and provide a way to understand the global structure of rings.

Projective and Injective Resolutions

Existence of Projective and Injective Modules

- Every module M over a ring R has a projective resolution
- Construct a projective resolution by taking a free resolution and replacing each free module with a projective module
- Every module M over a ring R has an injective resolution
- Construct an injective resolution by embedding M into an injective module and repeating the process
- A ring R has enough projectives if every R -module is a quotient of a projective module
- Equivalent to every R -module having a projective resolution
- A ring R has enough injectives if every R -module can be embedded into an injective module
- Equivalent to every R -module having an injective resolution

Properties of Projective and Injective Resolutions

- Projective resolutions are unique up to homotopy equivalence
- Two projective resolutions of the same module are chain homotopic
- Injective resolutions are unique up to homotopy equivalence
- Two injective resolutions of the same module are chain homotopic
- Projective resolutions can be used to compute derived functors of right exact functors
- Example: $\text{Ext}_R^n(M, N)$ is the n -th homology of $\text{Hom}_R(P_\bullet, N)$, where $P_\bullet$ is a projective resolution of M
- Injective resolutions can be used to compute derived functors of left exact functors
- Example: $\text{Ext}_R^n(M, N)$ is the n -th cohomology of $\text{Hom}_R(M, I^\bullet)$, where $I^\bullet$ is an injective resolution of N

Fundamental Homological Lemmas and Theorems

Horseshoe Lemma

- Given a short exact sequence of chain complexes, there exists a short exact sequence of the corresponding homology groups
- Constructs a long exact sequence in homology from a short exact sequence of chain complexes
- Provides a tool for computing homology groups of chain complexes
- Example: If $0 \rightarrow A_\bullet \rightarrow B_\bullet \rightarrow C_\bullet \rightarrow 0$ is a short exact sequence of chain complexes, then there is a long exact sequence $\cdots \rightarrow H_n(A_\bullet) \rightarrow H_n(B_\bullet) \rightarrow H_n(C_\bullet) \rightarrow H_{n-1}(A_\bullet) \rightarrow \cdots$

Comparison Theorem

- Given a morphism of chain complexes that induces isomorphisms on homology, the chain complexes are chain homotopic
- Useful for comparing resolutions and proving uniqueness up to homotopy
- Allows for the construction of chain maps between resolutions
- Example: If $P_\bullet$ and $Q_\bullet$ are projective resolutions of a module M , then there exists a chain map $f_\bullet : P_\bullet \rightarrow Q_\bullet$ lifting the identity on M , and $f_\bullet$ is a homotopy equivalence

Derived Functors

Ext and Tor Functors

- $\text{Ext}_R^n(M, N)$ is the n -th derived functor of $\text{Hom}_R(M, -)$ applied to N
- Measures the failure of the functor $\text{Hom}_R(M, -)$ to be exact
- $\text{Tor}_n^R(M, N)$ is the n -th derived functor of $- \otimes_R N$ applied to M
- Measures the failure of the functor $- \otimes_R N$ to be exact
- $\text{Ext}_R^n(M, N)$ can be computed using a projective resolution of M or an injective resolution of N
- $\text{Ext}_R^n(M, N) = H^n(\text{Hom}_R(P_\bullet, N))$, where $P_\bullet$ is a projective resolution of M
- $\text{Ext}_R^n(M, N) = H^n(\text{Hom}_R(M, I_\bullet))$, where $I_\bullet$ is an injective resolution of N
- $\text{Tor}_n^R(M, N)$ can be computed using a projective resolution of M or a flat resolution of N
- $\text{Tor}_n^R(M, N) = H_n(P_\bullet \otimes_R N)$, where $P_\bullet$ is a projective resolution of M
- $\text{Tor}_n^R(M, N) = H_n(M \otimes_R F_\bullet)$, where $F_\bullet$ is a flat resolution of N

Global Dimension

- The global dimension of a ring R is the supremum of the projective dimensions of all R -modules
- Measures how far the ring is from being semisimple
- A ring has finite global dimension if and only if every module has a finite projective resolution
- Example: Semisimple rings have global dimension 0, as every module is projective
- The global dimension of a ring is equal to the supremum of the injective dimensions of all R -modules
- Consequence of the balance of Ext functor: $\text{Ext}_R^n(M, N) \cong \text{Ext}_R^n(N, M)$ for all $n > \max(\text{pd } M, \text{id } N)$

Unit 6 – Derived Functors and Resolutions

6.1 Projective and injective resolutions

Projective and injective resolutions are key tools in homological algebra. They help us understand modules by breaking them down into simpler pieces, allowing us to compute important algebraic invariants and derived functors.

These resolutions are built using special types of modules with nice lifting properties. They form the backbone of many calculations in homological algebra, connecting abstract concepts to concrete computations we can perform.

Projective and Injective Modules

Properties and Definitions of Projective and Injective Modules

- Projective module is a module P such that for any surjective homomorphism $f: M \rightarrow N$ and any homomorphism $g: P \rightarrow N$, there exists a homomorphism $h: P \rightarrow M$ with $f \circ h = g$
- Injective module is a module I such that for any injective homomorphism $f: M \rightarrow N$ and any homomorphism $g: M \rightarrow I$, there exists a homomorphism $h: N \rightarrow I$ with $h \circ f = g$
- Projective modules are direct summands of free modules ($P \oplus Q \cong F$ for some module Q and free module F)
- Injective modules are direct summands of divisible modules ($I \oplus J \cong D$ for some module J and divisible module D)

Resolutions using Projective and Injective Modules

- Projective resolution of a module M is an exact sequence $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ where each P_i is a projective module
- Used to compute derived functors of covariant functors ($\text{Tor}_n(M, N) = H_n(M \otimes_R P_\bullet)$ where $P_\bullet$ is a projective resolution of N)
- Injective resolution of a module M is an exact sequence $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ where each I^i is an injective module
- Used to compute derived functors of contravariant functors ($\text{Ext}^n(M, N) = H^n(\text{Hom}_R(M, I^\bullet))$ where $I^\bullet$ is an injective resolution of N)
- Every module has a projective resolution and an injective resolution
- Constructed using enough projectives/injectives property ($\text{Hom}_R(P, -)$ is exact for projective P , $\text{Hom}_R(-, I)$ is exact for injective I)

Complexes and Exact Sequences

Definitions and Properties of Complexes

- Chain complex is a sequence of modules and homomorphisms $\cdots \rightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \rightarrow \cdots$ such that $d_n \circ d_{n+1} = 0$ for all n

- Cochain complex has arrows going in the opposite direction ($M^n \xrightarrow{d^n} M^{n+1}$)
- Homology of a chain complex at M_n is defined as $H_n(M) = \ker d_n / \operatorname{im} d_{n+1}$
- Cohomology of a cochain complex at M^n is defined as $H^n(M) = \ker d^n / \operatorname{im} d^{n-1}$
- Acyclic complex is a complex with trivial homology ($H_n(M) = 0$ for all n)
- Exact sequence is an acyclic complex ($0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact iff $\operatorname{im} f = \ker g$ at each stage)

Minimal Resolutions and Applications

- Minimal resolution is a projective or injective resolution with the smallest possible terms
- Minimality condition: $d_n(P_n) \subseteq \mathfrak{m}P_{n-1}$ for all n where $\mathfrak{m}$ is the maximal ideal of the base ring
- Minimal resolutions are unique up to isomorphism
- Used to define invariants of modules (Betti numbers, Bass numbers)
- Projective dimension of M is the length of its minimal projective resolution ($\operatorname{pd} M = \sup \{n \mid P_n \neq 0\}$)
- Injective dimension of M is the length of its minimal injective resolution ($\operatorname{id} M = \sup \{n \mid I^n \neq 0\}$)

6.2 Left and right derived functors

Left and right derived functors measure how non-exact functors behave. They're built using projective or injective resolutions and taking homology. These tools help us understand functor behavior in complex algebraic structures.

Derived functors come in two flavors: left and right. Left derived functors use projective resolutions, while right derived functors use injective resolutions. They're key to studying important concepts like Ext and Tor in homological algebra.

Derived Functors

Constructing Derived Functors

- Left derived functor $\mathbf{L}F$ constructed by applying functor F to a projective resolution of an object and taking homology
- Measures the extent to which F fails to be exact when applied to the left
- Independent of the choice of projective resolution up to natural isomorphism
- Right derived functor $\mathbf{R}F$ constructed by applying functor F to an injective resolution of an object and taking homology
- Measures the extent to which F fails to be exact when applied to the right
- Independent of the choice of injective resolution up to natural isomorphism

Types of Functors

- Covariant functor $F : C \rightarrow D$ preserves the direction of morphisms
- For morphism $f : A \rightarrow B$ in C , $F(f) : F(A) \rightarrow F(B)$ in D

- Composes morphisms in the same order as the original category: $F(g \circ f) = F(g) \circ F(f)$
- Contravariant functor $G : C \rightarrow D$ reverses the direction of morphisms
- For morphism $f : A \rightarrow B$ in C , $G(f) : G(B) \rightarrow G(A)$ in D
- Composes morphisms in the opposite order of the original category: $G(g \circ f) = G(f) \circ G(g)$
- Examples include $\text{Hom}(-, A) : C^{\text{op}} \rightarrow \mathbf{Set}$ and $\text{Hom}(A, -) : C \rightarrow \mathbf{Set}$

Specific Derived Functors

Ext and Tor Functors

- Ext functor $\text{Ext}_R^n(A, B)$ is the n -th right derived functor of $\text{Hom}_R(A, -)$
- Measures the failure of $\text{Hom}_R(A, -)$ to be exact
- Can be computed using a projective resolution of A or an injective resolution of B
- $\text{Ext}_R^0(A, B) \cong \text{Hom}_R(A, B)$ and $\text{Ext}_R^1(A, B)$ classifies extensions of B by A
- Tor functor $\text{Tor}_n^R(A, B)$ is the n -th left derived functor of $- \otimes_R B$
- Measures the failure of $- \otimes_R B$ to be exact
- Can be computed using a projective resolution of either A or B
- $\text{Tor}_0^R(A, B) \cong A \otimes_R B$ and $\text{Tor}_1^R(A, B)$ measures the abelian group of relations between A and B

Cohomology and Satellites

- (Co)homology functors are derived functors in various settings
- Singular cohomology is the right derived functor of the global sections functor on sheaves
- Group cohomology is the right derived functor of the invariants functor on G -modules
- Lie algebra cohomology is the right derived functor of the invariants functor on Lie modules
- Satellite functors generalize the construction of derived functors
- Include local cohomology functors and local homology functors
- Defined using the language of triangulated categories and localization

6.3 Universal properties of derived functors

Derived functors are powerful tools in homological algebra, measuring how far a functor is from being exact. They help us understand complex structures by breaking them down into simpler parts we can analyze.

Universal properties of derived functors give us a way to characterize these tools uniquely. This approach lets us work with derived functors abstractly, without worrying about the specific constructions used to define them.

Universal Properties and Functors

Universal Properties and Natural Transformations

- A universal property defines an object in terms of its relationships with other objects, rather than by its internal structure
- Universal properties are used to characterize objects and morphisms in category theory
- Natural transformations provide a way to compare functors between categories
- A natural transformation $\eta : F \rightarrow G$ is a family of morphisms $\eta_X : F(X) \rightarrow G(X)$ for each object X in the source category, such that for every morphism $f : X \rightarrow Y$, the diagram commutes: $G(f) \circ \eta_X = \eta_Y \circ F(f)$
- Universal properties and natural transformations play a crucial role in the study of derived functors and homological algebra

Delta and Cohomological Functors

- A delta functor is a sequence of functors $T^n : \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories, along with connecting homomorphisms $\delta^n : T^n(A') \rightarrow T^{n+1}(A)$ for each short exact sequence $0 \rightarrow A \rightarrow A' \rightarrow A'' \rightarrow 0$ in $\mathcal{A}$, satisfying certain axioms
- The connecting homomorphisms form a long exact sequence:
$$\cdots \rightarrow T^n(A) \rightarrow T^n(A') \rightarrow T^n(A'') \xrightarrow{\delta^n} T^{n+1}(A) \rightarrow \cdots$$
- A cohomological functor is a contravariant delta functor, i.e., a delta functor with $T^n(f) : T^n(B) \rightarrow T^n(A)$ for each morphism $f : A \rightarrow B$ in $\mathcal{A}$
- Examples of cohomological functors include the Ext functors $\text{Ext}_R^n(A, -)$ and the cohomology functors $H^n(X; -)$ in algebraic topology

Derived Functors and Exact Sequences

Effacement and Derived Functors

- The effacement theorem states that for any additive functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories and any object A in $\mathcal{A}$, there exists an epimorphism $P \rightarrow A$ with P projective such that $F(P) \rightarrow F(A)$ is an epimorphism
- The effacement theorem is used to construct derived functors by taking projective resolutions of objects
- Given a left exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ and an object A in $\mathcal{A}$, the right derived functors $R^n F(A)$ are defined as the cohomology of the complex $F(P_\bullet)$, where $P_\bullet \rightarrow A$ is a projective resolution of A
- The derived functors measure the failure of F to be exact

Long Exact Sequences and Connecting Homomorphisms

- A long exact sequence is a sequence of homomorphisms between abelian groups or modules
$$\cdots \rightarrow A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \rightarrow \cdots$$
 such that the image of each homomorphism is equal to the kernel of the next: $\text{Im}(f_{n-1}) = \text{Ker}(f_n)$
- Long exact sequences often arise from short exact sequences of chain complexes by taking homology or cohomology

- The connecting homomorphism in a long exact sequence is a homomorphism $\delta_n : H_n(C'') \rightarrow H_{n-1}(C)$ induced by the short exact sequence of chain complexes $0 \rightarrow C \rightarrow C' \rightarrow C'' \rightarrow 0$
- The connecting homomorphism measures the obstruction to lifting cycles in C'' to cycles in C'
- Examples of long exact sequences include the long exact sequence in homology associated with a short exact sequence of chain complexes and the long exact sequence of a pair in singular homology

6.4 Computation and applications of derived functors

Derived functors are powerful tools for studying modules and sheaves. They extend familiar concepts like Hom and tensor product to capture more subtle relationships between objects. This section explores how to compute and apply these functors.

We'll dive into Ext and Tor groups, which measure extensions and tensor product failures. We'll also look at group and sheaf cohomology, along with advanced tools like spectral sequences and derived categories. These concepts are key to understanding homological algebra.

Ext and Tor Groups

Computing Ext and Tor

- Ext groups $\text{Ext}_R^n(M, N)$ measure extensions between R -modules M and N
- $\text{Ext}_R^0(M, N)$ is the set of module homomorphisms $\text{Hom}_R(M, N)$
- Higher Ext groups correspond to equivalence classes of long exact sequences
- Tor groups $\text{Tor}_n^R(M, N)$ measure the extent to which the tensor product of R -modules M and N fails to be exact
- $\text{Tor}_0^R(M, N)$ is the tensor product $M \otimes_R N$
- Higher Tor groups arise from the failure of the tensor product to be left-exact

Computational Techniques

- Dimension shifting allows computing higher Ext or Tor groups from lower ones
- For Ext: $\text{Ext}_R^{n+1}(M, N) \cong \text{Ext}_R^n(M, \Omega N)$ where ΩN is the first syzygy of N
- For Tor: $\text{Tor}_{n+1}^R(M, N) \cong \text{Tor}_n^R(\Omega M, N)$ where ΩM is the first syzygy of M
- Horseshoe lemma relates the projective resolutions of M , N , and $M \oplus N$
- Allows constructing a projective resolution of $M \oplus N$ from those of M and N
- Useful for computing Ext and Tor groups involving direct sums

Cohomology Theories

Group Cohomology

- Group cohomology $H^n(G, A)$ measures the cohomology of a group G with coefficients in a G -module A

- $H^0(G, A)$ is the set of G -invariant elements of A
- $H^1(G, A)$ classifies crossed homomorphisms modulo principal crossed homomorphisms
- Higher cohomology groups have interpretations in terms of extensions and obstructions
- Computed using the bar resolution or projective resolutions of the trivial G -module $\mathbb{Z}$

Sheaf Cohomology

- Sheaf cohomology $H^n(X, \mathcal{F})$ measures the cohomology of a sheaf $\mathcal{F}$ on a topological space X
- $H^0(X, \mathcal{F})$ is the set of global sections of $\mathcal{F}$
- Higher cohomology groups measure obstructions to extending local sections to global sections
- Computed using Čech cohomology or derived functors of the global sections functor
- Künneth formula relates the cohomology of a product space to the cohomology of its factors
- For sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y , there is a short exact sequence involving $H^n(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$ and $\bigoplus_{p+q=n} H^p(X, \mathcal{F}) \otimes H^q(Y, \mathcal{G})$

Advanced Tools

Spectral Sequences

- Spectral sequences are algebraic tools for computing homology or cohomology groups
- Consist of a sequence of pages $E_r^{p,q}$ with differentials $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$
- Each page is the homology of the previous page with respect to its differential
- The sequence converges to the desired homology or cohomology groups
- Examples include the Leray spectral sequence and the Grothendieck spectral sequence

Derived Categories

- The derived category $D(R)$ of an abelian category R is obtained by formally inverting quasi-isomorphisms
- Objects are chain complexes, morphisms are chain maps modulo homotopy
- Quasi-isomorphisms (chain maps inducing isomorphisms on homology) become isomorphisms
- Derived functors can be defined as functors on the derived category
- Examples: derived tensor product $\otimes^L$, derived Hom functor $RHom$
- Provide a unified framework for homological algebra constructions

Unit 7 – The Tor and Ext Functors

7.1 Definition and properties of Tor functor

The Tor functor measures how tensor products deviate from exactness. It takes two modules and produces a new module for each non-negative integer, with Tor_0 being the tensor product itself. Higher Tor functors capture the non-exactness.

Tor is derived from the tensor product functor using projective resolutions. It's additive, preserves direct sums, and vanishes for flat modules. Tor helps detect flatness and appears in long exact sequences, making it a powerful tool in homological algebra.

Definition and Properties

Tor as a Derived Functor

- Tor functor $\text{Tor}_n^R(M, N)$ measures the degree to which the tensor product $M \otimes_R N$ fails to be exact
- Takes two R -modules M and N as input and produces a new R -module $\text{Tor}_n^R(M, N)$ for each non-negative integer n
- $\text{Tor}_0^R(M, N)$ is isomorphic to the tensor product $M \otimes_R N$
- Higher Tor functors $\text{Tor}_n^R(M, N)$ for $n > 0$ measure the deviation from exactness
- Tor is a derived functor obtained by taking a left derived functor of the tensor product functor $- \otimes_R N$
- Derived functors are a way to extend a functor that is not exact to a sequence of functors that preserve exactness
- Left derived functors are computed using projective resolutions of the first argument (M in this case)
- Tor is an additive functor covariant in both arguments and preserves direct sums
- $\text{Tor}_n^R(M_1 \oplus M_2, N) \cong \text{Tor}_n^R(M_1, N) \oplus \text{Tor}_n^R(M_2, N)$
- $\text{Tor}_n^R(M, N_1 \oplus N_2) \cong \text{Tor}_n^R(M, N_1) \oplus \text{Tor}_n^R(M, N_2)$

Flatness and Vanishing of Tor

- An R -module M is flat if the functor $M \otimes_R -$ is exact
- Equivalently, M is flat if $\text{Tor}_n^R(M, N) = 0$ for all $n > 0$ and all R -modules N
- Examples of flat modules include free modules, projective modules, and localizations of flat modules
- If either M or N is a flat R -module, then $\text{Tor}_n^R(M, N) = 0$ for all $n > 0$
- In this case, the tensor product $M \otimes_R N$ is exact, and there is no need for higher Tor functors
- Tor can be used to detect flatness: an R -module M is flat if and only if $\text{Tor}_1^R(M, N) = 0$ for all R -modules N
- This provides a practical way to check flatness using a single Tor functor instead of all higher Tor functors

Computation and Applications

Computing Tor using Resolutions

- Tor functors can be computed using projective resolutions or flat resolutions
- To compute $\text{Tor}_n^R(M, N)$, take a projective resolution $P_\bullet \rightarrow M$ of M and tensor it with N to get a chain complex $P_\bullet \otimes_R N$
- The homology of this chain complex at the n -th position is $\text{Tor}_n^R(M, N)$
- Alternatively, Tor can be computed using flat resolutions of the second argument
- Take a flat resolution $F_\bullet \rightarrow N$ of N and tensor it with M to get a chain complex $M \otimes_R F_\bullet$.
- The homology of this chain complex at the n -th position is $\text{Tor}_n^R(M, N)$
- The choice of resolution (projective or flat) depends on the properties of the modules involved and the convenience of computation

Applications and Properties of Tor

- Tor is related to the concept of torsion in module theory
- An element $x \in M$ is a torsion element if $rx = 0$ for some nonzero $r \in R$
- The set of torsion elements in M forms a submodule called the torsion submodule $\text{Tor}(M)$
- $\text{Tor}_1^R(R/I, M)$ is isomorphic to the I -torsion submodule of M
- Tor appears in the long exact sequence of Tor associated with a short exact sequence of modules
- Given a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and an R -module M , there is a long exact sequence of Tor functors: $\cdots \rightarrow \text{Tor}_{n+1}^R(C, M) \rightarrow \text{Tor}_n^R(A, M) \rightarrow \text{Tor}_n^R(B, M) \rightarrow \text{Tor}_n^R(C, M) \rightarrow \cdots$
- This long exact sequence can be used to compute Tor functors and study the relationships between modules
- Tor satisfies the dimension shifting property: $\text{Tor}_n^R(M, N) \cong \text{Tor}_{n-1}^R(M, \Omega N)$, where ΩN is the first syzygy of N
- This property allows for the computation of higher Tor functors using lower ones and syzygies
- Tor is balanced, meaning that $\text{Tor}_n^R(M, N) \cong \text{Tor}_n^R(N, M)$ for all $n \geq 0$
- This symmetry property simplifies computations and proofs involving Tor functors

7.2 Definition and properties of Ext functor

The Ext functor is a powerful tool in homological algebra that measures how far the Hom functor is from being exact. It's derived from the Hom functor and provides insights into module structures and relationships.

Ext groups are computed using injective resolutions and play a crucial role in long exact sequences. They exhibit properties like dimension shifting and balance, connecting different modules and allowing for complex algebraic computations.

Definition and Basic Properties

Ext Functor and its Derivation

- Ext functor is a bifunctor that measures the deviation of the Hom functor from being exact
- Defined as the right derived functor of the Hom functor
- Derived functors are a way to extend a functor that is not exact to a family of functors that measure the degree to which the original functor fails to be exact
- The right derived functors are obtained by applying the original functor to an injective resolution of the second argument and taking homology
- Notation: $\text{Ext}_R^n(A, B)$ denotes the n -th Ext group of A and B over the ring R

Properties of the Hom Functor

- Hom functor $\text{Hom}_R(A, -)$ is a left exact functor for any R -module A
- Preserves exact sequences of the form $0 \rightarrow B \rightarrow C \rightarrow D$
- Does not necessarily preserve exactness at later terms in the sequence
- Hom functor is contravariant in the first argument and covariant in the second argument
- $\text{Hom}_R(-, B)$ is contravariant
- $\text{Hom}_R(A, -)$ is covariant

Resolutions and Long Exact Sequences

Injective and Projective Resolutions

- Injective resolution of an R -module A is an exact sequence of the form:
- $0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ where each I^i is an injective R -module
- Used to compute right derived functors (Ext)
- Projective resolution of an R -module A is an exact sequence of the form:
- $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$ where each P_i is a projective R -module
- Used to compute left derived functors (Tor)
- Existence of resolutions:
- Every module over a ring with enough injectives admits an injective resolution
- Every module over a ring with enough projectives admits a projective resolution

Long Exact Sequences

- Fundamental tool in homological algebra to study relationships between Ext groups
- For a short exact sequence of R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, there is a long exact sequence:
- $0 \rightarrow \text{Hom}_R(X, A) \rightarrow \text{Hom}_R(X, B) \rightarrow \text{Hom}_R(X, C) \rightarrow \text{Ext}_R^1(X, A) \rightarrow \text{Ext}_R^1(X, B) \rightarrow \dots$

- Allows the computation of Ext groups by breaking them into shorter exact sequences
- Connects Ext groups of different modules in a functorial way

Advanced Properties and Theorems

Dimension Shifting and Balance

- Dimension shifting: relationship between Ext groups of different dimensions
- For R -modules A and B and an integer n , there is an isomorphism:
- $\text{Ext}_R^{n+1}(A, B) \cong \text{Ext}_R^n(A, C)$ where C is the cokernel of an injective hull of B
- Allows computation of higher Ext groups from lower ones
- Balance: symmetry between left and right Ext functors
- For R -modules A and B , there is an isomorphism:
- $\text{Ext}_R^n(A, B) \cong \text{Ext}_{R^{op}}^n(B, A)$ where R^{op} is the opposite ring of R
- Reflects the duality between injective and projective modules

Universal Coefficient Theorem

- Relates Ext groups to homology and cohomology groups
- For a ring R , an R -module A , and a chain complex C of projective R -modules, there is a short exact sequence:
- $0 \rightarrow \text{Ext}_R^1(H_{n-1}(C), A) \rightarrow H^n(C; A) \rightarrow \text{Hom}_R(H_n(C), A) \rightarrow 0$
- Allows computation of cohomology groups using Ext groups and homology groups
- Particularly useful in algebraic topology and algebraic geometry

7.3 Computation techniques for Tor and Ext

Tor and Ext are crucial functors in homological algebra, and knowing how to compute them is key. This section covers various techniques, from resolutions to spectral sequences, that help us calculate these functors efficiently.

Understanding these computational methods not only makes working with Tor and Ext easier but also deepens our grasp of their properties. We'll explore how these tools connect and build upon each other to solve complex problems.

Resolution Techniques

Projective and Injective Resolutions

- Projective resolution constructs a resolution of a module using projective modules
- Allows for the computation of left derived functors (Tor)
- Starts with the module and builds a sequence of projective modules with maps between them
- The sequence is exact and the maps are surjective

- Injective resolution constructs a resolution of a module using injective modules
- Allows for the computation of right derived functors (Ext)
- Starts with the module and builds a sequence of injective modules with maps between them
- The sequence is exact and the maps are injective
- Resolutions are used to replace a module with a more structured sequence of modules
- This allows for the application of functors and the computation of derived functors

Horseshoe Lemma

- Horseshoe lemma is a tool for constructing resolutions and comparing them
- Given a short exact sequence of modules and a projective (or injective) resolution of one of the modules, the lemma allows for the construction of resolutions for the other two modules
- The resulting diagram has the shape of a horseshoe, hence the name
- Useful for proving the long exact sequence of Tor (or Ext) associated with a short exact sequence of modules
- Allows for the comparison of resolutions and the transfer of information between them

Computational Tools

Spectral Sequences

- Spectral sequences are a powerful computational tool in homological algebra
- They are a systematic way of computing homology (or cohomology) groups by successive approximations
- Consist of a sequence of pages, each containing a grid of groups (or modules) with differentials between them
- The differentials on each page satisfy certain properties (e.g., $d^2 = 0$)
- The homology (or cohomology) of one page forms the groups (or modules) on the next page
- The process continues until the differentials become zero, yielding the final result
- Spectral sequences can be used to compute Tor and Ext groups in various settings (e.g., group homology, sheaf cohomology)

Künneth Formula

- Künneth formula is a tool for computing the homology (or cohomology) of a tensor product of chain complexes (or modules)
- Relates the homology (or cohomology) of the tensor product to the homology (or cohomology) of the individual complexes (or modules)
- Involves a short exact sequence that splits, leading to a direct sum decomposition
- Can be used to compute Tor (or Ext) of a tensor product of modules

- Useful in algebraic topology for computing the homology (or cohomology) of product spaces

Dimension Shifting

- Dimension shifting is a technique for relating Tor (or Ext) groups in different dimensions
- Allows for the computation of higher Tor (or Ext) groups in terms of lower ones
- Involves the use of short exact sequences and the long exact sequence of Tor (or Ext)
- Can simplify computations by reducing the problem to a lower dimension
- Useful in conjunction with other computational tools (e.g., spectral sequences, Künneth formula)

Ring-Theoretic Considerations

Change of Rings

- Change of rings is a technique for relating Tor (or Ext) groups over different rings
- Given a ring homomorphism $f: R \rightarrow S$ and an S -module M , one can compute $\text{Tor}_*^R(M, N)$ in terms of $\text{Tor}_*^S(M, S \otimes_R N)$ for an R -module N
- Similarly, for Ext, one can compute $\text{Ext}_R^*(M, N)$ in terms of $\text{Ext}_S^*(M, \text{Hom}_R(S, N))$
- Useful when the Tor (or Ext) groups are easier to compute over one ring than the other
- Can be used to transfer results between different rings (e.g., from a ring to its quotient, or from a ring to its localization)
- Involves the use of spectral sequences (e.g., the Grothendieck spectral sequence) to relate the Tor (or Ext) groups over the different rings

7.4 Applications in algebra and topology

The Tor and Ext functors play crucial roles in algebra and topology. They help us understand relationships between homology and cohomology, compute homology of product spaces, and classify group extensions.

These functors appear in key theorems like the Universal Coefficient Theorem and Künneth formula. They also show up in cohomology ring structures and spectral sequences, providing powerful tools for analyzing topological spaces.

Universal Coefficient and Künneth Theorems

Relationship between Homology and Cohomology

- Universal coefficient theorem establishes a relationship between homology and cohomology groups
- States that for a chain complex C of free abelian groups, there is a short exact sequence:

$$0 \rightarrow \text{Ext}(H_{n-1}(C), G) \rightarrow H^n(C; G) \rightarrow \text{Hom}(H_n(C), G) \rightarrow 0$$
- Allows computation of cohomology groups from homology groups and vice versa (singular homology, cellular homology)
- Künneth formula computes the homology groups of a tensor product of two chain complexes
- For chain complexes C and D , and a principal ideal domain R , there is a short exact sequence:

$$0 \rightarrow \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(D) \rightarrow H_n(C \otimes_R D) \rightarrow \bigoplus_{i+j=n-1} \text{Tor}_1^R(H_i(C), H_j(D)) \rightarrow 0$$

- Allows computation of homology groups of product spaces (torus, Klein bottle)

Algebraic Interpretations of Ext and Tor

- Ext functor can be interpreted as classifying group extensions
- For abelian groups A and B , $\text{Ext}(A, B)$ classifies extensions of the form: $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$
- Elements of $\text{Ext}(A, B)$ correspond to equivalence classes of such extensions (central extensions, split extensions)
- Tor functor appears in the homology of groups
- For a group G and a G -module M , there is a long exact sequence:

$$\dots \rightarrow H_n(G; M) \rightarrow H_n(G) \otimes M \rightarrow \text{Tor}_1^{\mathbb{Z}G}(H_{n-1}(G), M) \rightarrow H_{n-1}(G; M) \rightarrow \dots$$
- Tor measures the failure of the homology of G to be a flat module over the group ring $\mathbb{Z}G$ (group homology, Lyndon-Hochschild-Serre spectral sequence)

Cohomology and Spectral Sequences

Cohomology Ring Structure

- Cohomology groups $H^n(X; R)$ of a space X with coefficients in a ring R form a graded ring
- The product is given by the cup product: $\smile : H^i(X; R) \otimes H^j(X; R) \rightarrow H^{i+j}(X; R)$
- The cup product is induced by the diagonal map $X \rightarrow X \times X$ (Poincaré duality, Künneth formula for cohomology)
- The cohomology ring encodes important topological information about the space
- For example, the cohomology ring of a compact oriented manifold determines its homotopy type (rational homotopy theory)

Spectral Sequences and Local Cohomology

- Eilenberg-Moore spectral sequence relates the cohomology of a fiber bundle to the cohomology of its base and fiber
- For a fibration $F \rightarrow E \rightarrow B$ with B simply connected, there is a spectral sequence:

$$E_2^{p,q} = \text{Tor}_{p,q}^{H^*(B)}(H^*(E), \mathbb{Z}) \Rightarrow H^{p+q}(F)$$
- The spectral sequence converges to the cohomology of the fiber F (Serre spectral sequence, Leray-Serre spectral sequence)
- Local cohomology is a cohomology theory that captures local properties of a space near a subspace
- For a space X and a subspace Y , the local cohomology groups $H_Y^i(X; R)$ fit into a long exact sequence: $\dots \rightarrow H_Y^i(X; R) \rightarrow H^i(X; R) \rightarrow H^i(X - Y; R) \rightarrow H_Y^{i+1}(X; R) \rightarrow \dots$
- Local cohomology is related to sheaf cohomology and has applications in algebraic geometry (Grothendieck's local cohomology theory)

Unit 8 – Spectral Sequences

8.1 Basic theory of spectral sequences

Spectral sequences are powerful tools in homological algebra, linking different levels of algebraic structures. They consist of pages of bigraded modules with differentials, where each page's homology determines the next page, ultimately converging to some limit.

Filtrations, exact couples, and edge homomorphisms are key concepts in understanding spectral sequences. These ideas help us analyze complex algebraic structures by breaking them down into simpler pieces and tracking how information flows between different levels of computation.

Spectral Sequences and Filtrations

Definition and Properties of Spectral Sequences

- Spectral sequence consists of a collection of pages $\{E_r^{p,q}\}$ indexed by non-negative integers r called the page number
- Each page is a differential bigraded module over a ring R with a differential d_r of bidegree $(-r, r-1)$ satisfying $d_r \circ d_r = 0$
- The pages are related by isomorphisms $H(E_r, d_r) \cong E_{r+1}$ between the homology of the r -th page and the $(r+1)$ -th page
- Spectral sequences often arise from filtrations on a chain complex or from double complexes

Filtrations and Convergence

- Filtration on a module M is a sequence of submodules $\cdots \subseteq F_{p-1}M \subseteq F_pM \subseteq F_{p+1}M \subseteq \cdots$ indexed by integers p
- Filtration gives rise to a spectral sequence by taking the associated graded module $\text{gr}_p M = F_pM / F_{p-1}M$ and defining the pages using the filtration
- Spectral sequence is said to converge to a graded module H^* if there exists a filtration on H^* such that $E_\infty^{p,q} \cong \text{gr}_p H_{p+q}^*$ (E_∞ page isomorphic to associated graded of the limit)
- Convergence allows us to extract information about the limit module H^* from the spectral sequence (E_2 and E_∞ pages often of particular interest)

Exact and Derived Couples

Exact Couples and Spectral Sequences

- Exact couple is a diagram of R -modules and homomorphisms $A \xrightarrow{i} A \xrightarrow{j} E \xrightarrow{k} A$ where the composition of any two consecutive maps is zero
- Exact couples give rise to spectral sequences by repeatedly taking homology to form derived couples
- Derived couple of an exact couple (A, E, i, j, k) is the exact couple (A', E', i', j', k') where $A' = i(A)$, $E' = H(E) = \ker(k)/\text{im}(j)$, and the maps i', j', k' are induced by i, j, k

- Spectral sequence obtained from an exact couple has pages given by $E_r = E^{(r)}$ (the E -module of the r -th derived couple) and differentials induced by the map k

Relationship between Exact Couples and Filtrations

- Filtrations on a module M give rise to exact couples by setting $A_p = F_p M$ and $E_p = F_p M / F_{p-1} M$ with appropriate maps between them
- Spectral sequence of this exact couple recovers the spectral sequence associated to the filtration
- Exact couples provide a convenient way to construct and study spectral sequences, particularly in the context of filtered complexes and double complexes

Additional Concepts

Edge Homomorphisms

- Edge homomorphisms in a spectral sequence are maps from the limit module H^* to certain terms on the E_2 page or from certain E_∞ terms to the limit module
- For a first-quadrant spectral sequence $\{E_r^{p,q}\}$ converging to H^* , there are edge homomorphisms $H_n \rightarrow E_2^{n,0}$ and $E_\infty^{p,n-p} \rightarrow \text{gr}_p H_n$ for each n and p
- Edge homomorphisms allow us to extract additional information about the limit module from the spectral sequence (E_2 and E_∞ pages)
- Example: In the Serre spectral sequence for a fibration $F \rightarrow E \rightarrow B$, the edge homomorphisms relate the homology of the base B and the fiber F to the homology of the total space E

8.2 Spectral sequence of a filtered complex

Spectral sequences are powerful tools for computing homology groups of complex algebraic structures. They work by filtering a complex and analyzing the resulting graded objects, providing a systematic approach to unraveling complicated homological information.

This section focuses on the spectral sequence of a filtered complex, introducing key concepts like filtrations, graded objects, and spectral sequence pages. We'll explore how these ideas come together to create a powerful computational framework for homological algebra.

Filtered Complexes and Graded Objects

Filtered Complexes

- Filtered complex consists of a chain complex C and a family of subcomplexes $F^p C$ indexed by integers p
- Subcomplexes satisfy $F^p C \subseteq F^{p+1} C$ for all p
- Entire complex is the union of all subcomplexes $C = \bigcup_p F^p C$
- Filtration gives a notion of "size" or "degree" to elements of the complex
- Elements in $F^p C$ are considered to have filtration degree at least p
- Morphisms between filtered complexes $f: (C, F) \rightarrow (D, G)$ preserve the filtration

- Require $f(F^p C) \subseteq G^p D$ for all p

Graded Modules and Associated Graded Objects

- Graded module M is a direct sum of submodules $M = \bigoplus_{n \in \mathbb{Z}} M_n$
- Elements in M_n are homogeneous of degree n
- Homomorphisms between graded modules preserve the grading
- Associated graded object $\text{gr}(C, F)$ of a filtered complex (C, F) is a graded module
- Defined as $\text{gr}(C, F)_p = F^p C / F^{p-1} C$
- Measures the "jump" in filtration degree from $p-1$ to p
- Associated graded object captures the structure of the filtration
- Loses some information about the complex C itself
- Useful for studying properties that depend on the filtration (gr functor)

Spectral Sequence Pages

The E_0 and E_1 Pages

- E_0 page of a spectral sequence is the associated graded object of the filtered complex
- $E_0^{p,q} = \text{gr}(C, F)_p^q = F^p C^q / F^{p-1} C^q$
- Differential $d_0 : E_0^{p,q} \rightarrow E_0^{p,q-1}$ induced by the differential of C
- E_1 page is the homology of the E_0 page with respect to d_0
- $E_1^{p,q} = H^q(E_0^{p,*}, d_0) = \ker(d_0 : E_0^{p,q} \rightarrow E_0^{p,q-1}) / \text{im}(d_0 : E_0^{p,q+1} \rightarrow E_0^{p,q})$
- Differential $d_1 : E_1^{p,q} \rightarrow E_1^{p-1,q}$ induced by d_0 (zigzag homomorphism)

The Higher Pages and Differentials

- E_r page for $r \geq 1$ is obtained by taking homology of the previous page E_{r-1}
- $E_r^{p,q} = H^q(E_{r-1}^{p,*}, d_{r-1}) = \ker(d_{r-1} : E_{r-1}^{p,q} \rightarrow E_{r-1}^{p-r+1, q+r-2}) / \text{im}(d_{r-1} : E_{r-1}^{p+r-1, q-r+2} \rightarrow E_{r-1}^{p,q})$
- Differential $d_r : E_r^{p,q} \rightarrow E_r^{p-r, q+r-1}$ has bidegree $(-r, r-1)$
- Spectral sequence pages form a sequence of successive approximations to the homology of the original complex C
- Each page E_r is a bigraded module with a differential d_r
- Homology of (E_r, d_r) gives the next page E_{r+1}

Convergence and Degeneration

Convergence of Spectral Sequences

- Spectral sequence $\{E_r^{p,q}, d_r\}$ of a filtered complex (C, F) is said to converge to $H^*(C)$ if there exists an r_0 such that for all $r \geq r_0$:
 - $E_r^{p,q} \cong E_{r_0}^{p,q}$ (isomorphic as bigraded modules)
 - $d_r = 0$ (differentials vanish)
- Convergence theorem states that under certain conditions (bounded or exhaustive filtration), the spectral sequence converges to the associated graded object of $H^*(C)$ with respect to the induced filtration
- $E_\infty^{p,q} \cong \text{gr}(H^*(C), F)_p^q = F^p H^{p+q}(C) / F^{p-1} H^{p+q}(C)$
- Convergence allows us to compute the homology of the original complex C from the limit term E_∞ of the spectral sequence
- Requires knowledge of the induced filtration on $H^*(C)$

Degeneration and Collapsing

- Spectral sequence is said to degenerate at the E_r page if $d_r = 0$ and all subsequent differentials also vanish
- Implies $E_r^{p,q} \cong E_{r+1}^{p,q} \cong \dots \cong E_\infty^{p,q}$ for all p, q
- Spectral sequence collapses at the E_r page if it degenerates at E_r and $E_r = E_\infty$
- Stronger condition than degeneration
- Collapsing at E_1 or E_2 is particularly useful for computations
- Degeneration and collapsing simplify the calculation of the limit term E_∞
- Avoid the need to compute higher differentials
- Provide a direct relationship between the pages and the homology of the original complex

8.3 Spectral sequences of a double complex

Spectral sequences of double complexes are powerful tools for computing homology. They allow us to break down complex calculations into manageable steps, using filtrations to analyze the structure piece by piece.

By studying the pages of a spectral sequence, we can uncover hidden relationships between different parts of a double complex. This approach often leads to surprising connections and simplifies seemingly difficult homological problems.

Double Complex and Filtrations

Definition and Structure of Double Complexes

- Double complex consists of a collection of abelian groups or modules $C_{p,q}$ arranged in a grid indexed by integers p and q

- Equipped with horizontal differentials $d_h : C_{p,q} \rightarrow C_{p+1,q}$ and vertical differentials $d_v : C_{p,q} \rightarrow C_{p,q+1}$
- Satisfies the conditions $d_h \circ d_h = 0$, $d_v \circ d_v = 0$, and $d_h \circ d_v + d_v \circ d_h = 0$ (anticommutativity)
- Can be visualized as a commutative diagram with rows and columns connected by differentials

Total Complex and Its Properties

- Total complex $\text{Tot}(C)$ of a double complex C is a single complex constructed by "flattening" the double complex
- Defined as $\text{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$ with differential $d = d_h + d_v$
- Differential d satisfies $d \circ d = 0$ due to the anticommutativity condition in the double complex
- Allows studying the homology of the double complex by considering the homology of the total complex

Filtrations on Double Complexes

- Row filtration on a double complex C is a sequence of subcomplexes $F_p C = \bigoplus_{i \geq p} C_{i,*}$
- Obtained by considering the rows of the double complex from the p -th row onwards
- Satisfies $F_p C \subseteq F_{p-1} C$ for all p
- Column filtration on a double complex C is a sequence of subcomplexes $F_q C = \bigoplus_{j \geq q} C_{*,j}$
- Obtained by considering the columns of the double complex from the q -th column onwards
- Satisfies $F_q C \subseteq F_{q-1} C$ for all q
- Filtrations provide a way to study the double complex by considering its "slices" along rows or columns

Spectral Sequences and Convergence

First Quadrant Spectral Sequence

- First quadrant spectral sequence is a spectral sequence $\{E_r^{p,q}, d_r\}$ with $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$
- Arises naturally from a double complex C with $C_{p,q} = 0$ for $p < 0$ or $q < 0$ (first quadrant double complex)
- Pages of the spectral sequence are computed iteratively using the differentials $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$
- Each page E_r is the homology of the previous page E_{r-1} with respect to the differential d_{r-1}

Convergence of Double Complex Spectral Sequence

- Spectral sequence of a first quadrant double complex C converges to the homology of the total complex $\text{Tot}(C)$
- Convergence means that there exists an r_0 such that for all $r \geq r_0$, the pages E_r stabilize (i.e., $E_r \cong E_{r+1} \cong \dots$)

- The stable page E_∞ is isomorphic to the associated graded module of the homology of $\text{Tot}(C)$ with respect to a certain filtration
- Convergence provides a way to compute the homology of the total complex by studying the spectral sequence pages

Collapse of Spectral Sequence

- Spectral sequence of a double complex is said to collapse at the r -th page if $d_r = 0$ and $d_i = 0$ for all $i > r$
- Collapsing implies that $E_r \cong E_{r+1} \cong \cdots \cong E_\infty$
- If the spectral sequence collapses at the r -th page, the homology of the total complex can be read off directly from the r -th page
- Collapsing at an early page simplifies the computation of the homology of the total complex
- Occurs in various situations, such as when the double complex has only a few non-zero terms or when certain vanishing conditions are satisfied (e.g., $C_{p,q} = 0$ for $p > p_0$ or $q > q_0$)

8.4 Applications in homological algebra

Spectral sequences are powerful tools in homological algebra, connecting different cohomology theories. They allow us to compute complex structures by breaking them down into simpler pieces and using successive approximations.

From Leray and Serre to Eilenberg-Moore and Adams, these sequences tackle various algebraic and topological problems. They're essential for understanding group cohomology, fibrations, and even stable homotopy groups of spheres.

Spectral Sequences in Homological Algebra

Key Spectral Sequences for Homology and Cohomology

- Leray spectral sequence relates the cohomology of a space with the cohomology of a sheaf on that space
- Enables computing cohomology groups using a fibration $F \rightarrow E \rightarrow B$ by successive approximations
- Starts with $E_2^{p,q} = H^p(B; H^q(F))$ and converges to $H^{p+q}(E)$
- Serre spectral sequence is a spectral sequence that relates the homology of the base space and the homology of the fiber to the homology of the total space in a fibration
- For a fibration $F \rightarrow E \rightarrow B$, it has $E_{p,q}^2 = H_p(B; H_q(F))$ and converges to $H_{p+q}(E)$
- Powerful tool for computing homology groups of a space when it's realized as a fibration (Hopf fibration of spheres)
- Atiyah-Hirzebruch spectral sequence relates a generalized cohomology theory of a space to its ordinary cohomology
- For a spectrum h and CW complex X , it starts with $E_2^{p,q} = H^p(X; \pi_q(h))$ and converges to $h^{p+q}(X)$
- Allows computing extraordinary cohomology groups like K-theory or cobordism from ordinary cohomology (stable homotopy groups of spheres)

Spectral Sequences for Ring and Module Structures

- Eilenberg-Moore spectral sequence relates the cohomology of a fiber space to the cohomology of the base and total space for a fibration of spaces with additional structure
- Applies when cohomology of fiber, total space and base have compatible ring or module structures
- Starts with $E_2^{p,q} = \text{Tor}_{p,q}^{H^*(B)}(H^*(E), k)$ and converges to $H^{p+q}(F)$ as modules over $H^*(B)$
- Adams spectral sequence computes stable homotopy groups of spheres from Ext groups in the category of modules over the Steenrod algebra
- $E_2^{s,t} = \text{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{t-s}^S(S^0)_{(p)}$
- Connects stable homotopy theory to homological algebra of the Steenrod algebra (graded Hopf algebra encoding cohomology operations)

Group Cohomology Spectral Sequences

Spectral Sequences for Group Extensions

- Hochschild-Serre spectral sequence relates the cohomology of a group extension to the cohomology of the subgroup and quotient group
- For a group extension $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$ it has $E_2^{p,q} = H^p(K; H^q(H; M))$ converging to $H^{p+q}(G; M)$
- Allows computing group cohomology using the cohomology of smaller pieces H and K (when G is a semidirect product)
- Lyndon-Hochschild-Serre spectral sequence is a specific case of the Hochschild-Serre spectral sequence
- Applies to group cohomology with coefficients in a trivial module
- Has $E_2^{p,q} = H^p(K; H^q(H; \mathbb{Z}))$ and converges to $H^{p+q}(G; \mathbb{Z})$
- Useful for analyzing the cohomological dimension of solvable groups (successive extensions by abelian groups)

Unit 9 – Cohomology Theories

9.1 Sheaf cohomology

Sheaf cohomology is a powerful tool in algebraic topology that measures how local data on a space fits together globally. It extends the idea of sheaves, which track local information, to capture more complex relationships between local and global properties.

This topic introduces key concepts like derived functors and Čech cohomology, which are essential for computing sheaf cohomology. We'll explore how these methods reveal deep insights about topological spaces and their associated sheaves.

Sheaf Theory Fundamentals

Sheaf and Presheaf Concepts

- Sheaf is a tool for systematically tracking locally defined data attached to the open sets of a topological space
- Consists of a topological space X and a sheaf F on X associating data to each open set $U \subseteq X$ in a way that is compatible with restriction
- Sheaf axioms ensure local properties can be uniquely glued together
- Presheaf is a similar concept to a sheaf but without the gluing axiom
- Associates data to open sets of a topological space and is compatible with restriction
- May not allow for unique gluing of local sections

Étalé Space and Sheaf of Sections

- Étalé space is a topological space E along with a local homeomorphism $p : E \rightarrow X$
- Provides a useful perspective for studying sheaves ($p^{-1}(U)$ corresponds to $F(U)$)
- Allows for visualizing how the data varies over the base space X
- Sheaf of sections is a fundamental example of a sheaf
- Given a continuous map $f : Y \rightarrow X$, the sheaf of sections associates to each open set $U \subseteq X$ the set of continuous sections of f over U
- Encodes the idea of continuously varying data over a space

Cohomology Constructions

Derived Functor and Čech Cohomology

- Derived functor cohomology is a general method for constructing cohomology theories
- Starts with a left exact functor F (such as the global section functor Γ) and derives right derived functors $R^i F$
- The i -th derived functor $R^i F$ applied to a sheaf F gives the i -th cohomology group $H^i(X, F)$

- Čech cohomology is a concrete realization of sheaf cohomology using open covers
- Given an open cover $\mathcal{U} = \{U_i\}$ of X , defines cochains as functions on intersections of open sets
- The Čech complex gives a flasque resolution of the sheaf, allowing for computation of cohomology

Injective Resolutions and Cohomology Groups

- Injective resolution is a tool for computing derived functor cohomology
- An injective resolution of a sheaf F is an exact sequence $0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ where each I^i is an injective sheaf
- Existence of injective resolutions allows for the computation of $R^i F$ as the cohomology of the complex $F(I^\bullet)$
- Cohomology groups $H^i(X, F)$ are the central objects of study in sheaf cohomology
- Measure the global obstruction to solving a geometric problem related to the sheaf F (such as finding global sections)
- Encode topological and geometric information about the space X and the sheaf F

Special Sheaves

Flasque and Acyclic Sheaves

- Flasque sheaf is a special type of sheaf with vanishing higher cohomology
- Defined by the property that any section over an open set can be extended to a global section
- Flasque sheaves are used in constructing resolutions to compute cohomology
- Examples include the sheaf of locally constant functions and the sheaf of sections of a vector bundle
- Acyclic sheaf is a sheaf that is "invisible" to the cohomology functor
- A sheaf F is acyclic if $H^i(X, F) = 0$ for all $i > 0$
- Acyclic sheaves play a role in the machinery of spectral sequences and hypercohomology
- Examples include flasque sheaves and soft sheaves (sheaves where sections extend over closed subsets)

9.2 Group cohomology

Group cohomology extends homological algebra to study groups through their actions on modules. It uses cochain complexes and cohomology groups to measure obstructions and analyze group structures.

This approach connects to broader cohomology theories by providing tools for understanding group extensions, representations, and algebraic invariants. It also introduces key concepts like spectral sequences for computing complex cohomological information.

Cochain Complexes and Cohomology Groups

Defining Cochain Complexes

- Cochain complex consists of a sequence of abelian groups or modules C^n and homomorphisms (coboundary operators) $d^n : C^n \rightarrow C^{n+1}$
- Coboundary operators satisfy the condition $d^{n+1} \circ d^n = 0$ for all n , meaning the composition of any two consecutive coboundary operators is the zero map
- Elements of C^n are called n -cochains
- Cocycles are elements $x \in C^n$ such that $d^n(x) = 0$, forming the kernel of d^n
- Coboundaries are elements $x \in C^n$ such that $x = d^{n-1}(y)$ for some $y \in C^{n-1}$, forming the image of d^{n-1}

Computing Cohomology Groups

- Cohomology groups $H^n(C^\bullet)$ are defined as the quotient of the kernel of d^n (cocycles) by the image of d^{n-1} (coboundaries)
- $H^n(C^\bullet) = \ker(d^n) / \text{im}(d^{n-1})$
- Cohomology groups measure the "obstruction" to a cochain being a coboundary, similar to how homology groups measure the "holes" in a chain complex
- Bar resolution is a specific type of resolution used to compute the cohomology of a group with coefficients in a module
- Normalized cochains are a subcomplex of the cochains in the bar resolution, obtained by considering only those cochains that vanish on degenerate elements

Group Actions and Cohomological Operations

Group Actions on Cochain Complexes

- Group action on a cochain complex $C^\bullet$ is a collection of group homomorphisms $G \rightarrow \text{Aut}(C^n)$ for each n , compatible with the coboundary operators
- Compatibility means that for any $g \in G$ and $x \in C^n$, $g \cdot d^n(x) = d^n(g \cdot x)$
- Group action induces a group action on the cohomology groups $H^n(C^\bullet)$
- Cup product is a bilinear operation on cochains, $\smile : C^p \times C^q \rightarrow C^{p+q}$, that is compatible with the coboundary operators and induces a graded-commutative product on cohomology

Cohomological Operations

- Restriction is a map from the cohomology of a group G to the cohomology of a subgroup H , induced by the inclusion $H \hookrightarrow G$
- Inflation is a map from the cohomology of a quotient group G/N to the cohomology of G , induced by the quotient map $G \twoheadrightarrow G/N$
- Transfer map is a homomorphism from the cohomology of a subgroup H to the cohomology of the ambient group G , induced by averaging over cosets of H in G

- These operations allow for the comparison of cohomology groups of related groups and the study of the behavior of cohomology under group homomorphisms

Spectral Sequences

The Lyndon-Hochschild-Serre Spectral Sequence

- Spectral sequence is a tool for computing cohomology groups by successively approximating them with "pages" $E_r^{p,q}$, where each page is obtained from the previous one by taking cohomology
- Lyndon-Hochschild-Serre spectral sequence relates the cohomology of a group extension $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ to the cohomology of N and Q
- $E_2^{p,q} = H^p(Q; H^q(N; A)) \Rightarrow H^{p+q}(G; A)$, where A is a G -module and the action of Q on $H^q(N; A)$ is induced by the action of G on A and the conjugation action of G on N
- Spectral sequence converges to the cohomology of G with coefficients in A , providing a way to compute it from the cohomology of the subgroup N and quotient Q
- Differentials on each page are homomorphisms $d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ that square to zero, and the cohomology of d_r gives the next page E_{r+1}

9.3 Lie algebra cohomology

Lie algebra cohomology is a powerful tool for understanding the structure of Lie algebras and their representations. It combines ideas from exterior algebra and homological algebra to provide insights into central extensions, deformations, and invariant forms.

This section explores the Chevalley-Eilenberg complex, which computes Lie algebra cohomology, and relates it to other cohomology theories. We'll see how these concepts apply to physics, geometry, and algebraic topology.

Lie Algebra and Exterior Algebra

Fundamental Algebraic Structures

- Lie algebra consists of a vector space equipped with a binary operation called the Lie bracket that satisfies bilinearity, alternating identity, and the Jacobi identity
- Exterior algebra constructs an associative algebra from a given vector space where the product of two elements is antisymmetric
- Invariant forms are multilinear forms on a Lie algebra that are invariant under the adjoint action of the Lie algebra on itself
- Cohomology ring represents the collection of cohomology groups of a topological space or algebraic object with a ring structure induced by the cup product

Applications and Properties

- Lie algebras describe the infinitesimal symmetries of differential equations and play a crucial role in physics (quantum mechanics, gauge theories)
- Exterior algebra finds applications in differential geometry, where differential forms are elements of the exterior algebra of the cotangent space

- Invariant forms on a Lie algebra are closely related to the structure of the corresponding Lie group and its homogeneous spaces
- Cohomology ring encodes important topological invariants and provides a way to study the global structure of a space or algebraic object (de Rham cohomology, group cohomology)

Chevalley-Eilenberg and Koszul Complexes

Definitions and Constructions

- Chevalley-Eilenberg complex is a cochain complex associated with a Lie algebra and a module over it, used to define Lie algebra cohomology
- Koszul complex is a chain complex associated with a commutative ring and a sequence of elements, used to study the depth and regularity of ideals
- Cartan's formula expresses the Lie derivative of a differential form in terms of the exterior derivative and interior product, connecting Lie theory with differential geometry

Cohomological Computations

- Chevalley-Eilenberg complex allows the computation of Lie algebra cohomology, which classifies central extensions, abelian extensions, and deformations of Lie algebras
- Koszul complex provides a resolution of a module over a polynomial ring, enabling the computation of Tor functors and Betti numbers
- Cartan's formula facilitates the computation of the cohomology of homogeneous spaces and the study of invariant differential forms on Lie groups (Maurer-Cartan forms, Killing forms)

Relative Cohomology and Whitehead's Theorem

Relative Cohomology Theory

- Relative cohomology extends the notion of cohomology to pairs of spaces or algebras, allowing the study of relative invariants and exact sequences
- Relative Lie algebra cohomology considers cochains that vanish on a given subalgebra, leading to long exact sequences relating absolute and relative cohomology groups
- Relative cohomology arises naturally in the study of fiber bundles, where the cohomology of the total space is related to the cohomology of the base and fiber via the Serre spectral sequence

Whitehead's Theorem and Applications

- Whitehead's theorem for Lie algebras states that a morphism of Lie algebras inducing an isomorphism on cohomology is a quasi-isomorphism, providing a cohomological criterion for equivalence
- Whitehead's theorem has important consequences in deformation theory, allowing the classification of infinitesimal deformations of Lie algebras via cohomology
- Whitehead's theorem is a key tool in the study of homotopy theory of Lie algebras and the formality of differential graded Lie algebras (Quillen's rational homotopy theory)

9.4 De Rham cohomology

De Rham cohomology connects differential forms to the topology of smooth manifolds. It uses the exterior derivative to build a complex of forms, measuring how closed forms differ from exact ones.

This approach reveals geometric and topological properties of manifolds through analytic tools. It bridges the gap between calculus on manifolds and algebraic topology, providing insights into the structure of spaces.

Differential Forms and Exterior Derivative

Constructing Differential Forms

- Differential forms generalize the concept of functions and differential operators to arbitrary dimensions
- Construct differential k -forms on a smooth manifold M by taking the wedge product of k differential 1-forms (dx^i)
- Differential forms provide a way to integrate over submanifolds of M (curves, surfaces, etc.)
- Allow for a coordinate-free description of geometric structures on manifolds (volume forms, symplectic forms)

Properties of the Exterior Derivative

- Exterior derivative d maps k -forms to $(k + 1)$ -forms, generalizing the concept of the differential of a function
- Satisfies the property $d^2 = 0$, which is analogous to the equality of mixed partial derivatives
- Closed forms defined as differential forms ω satisfying $d\omega = 0$ (ω has vanishing exterior derivative)
- Exact forms defined as differential forms ω that can be written as $\omega = d\eta$ for some $(k - 1)$ -form η
- Exact forms are always closed due to the property $d^2 = 0$ (closed forms may not be exact)

De Rham Cohomology

De Rham Complex and Cohomology Groups

- De Rham complex is the sequence of vector spaces of differential forms connected by the exterior derivative d
- Cohomology groups $H_{dR}^k(M)$ measure the failure of the Poincaré lemma globally on the manifold M
- $H_{dR}^k(M)$ defined as the quotient of the space of closed k -forms by the space of exact k -forms ($\ker d / \text{im } d$)
- Intuitively, cohomology groups capture the "holes" in the manifold that prevent closed forms from being exact

Poincaré Lemma and De Rham Theorem

- Poincaré lemma states that on a contractible open subset U of M , every closed form is exact ($H_{dR}^k(U) = 0$ for $k > 0$)
- Provides a local characterization of the cohomology groups (they vanish on contractible sets)

- De Rham theorem establishes an isomorphism between the de Rham cohomology groups and the singular cohomology groups with real coefficients
- Connects the purely analytic theory of differential forms with the topological invariants of the manifold (Betti numbers)

Cohomological Tools

Mayer-Vietoris Sequence

- Mayer-Vietoris sequence relates the cohomology of a manifold M to the cohomology of two open subsets U and V covering M
- Provides a tool for computing cohomology groups by decomposing the manifold into simpler pieces (e.g., contractible sets)
- Sequence involves the cohomology groups of U , V , $U \cap V$, and M , connected by restriction and difference maps
- Useful for proving the Poincaré lemma by induction on the dimension of the manifold

Künneth Formula and Hodge Theory

- Künneth formula relates the cohomology of a product manifold $M \times N$ to the cohomology of the factors M and N
- Allows for the computation of cohomology groups of higher-dimensional manifolds from lower-dimensional ones (e.g., tori from circles)
- Hodge theory studies the relationship between de Rham cohomology and the Laplacian operator on differential forms
- Establishes a decomposition of differential forms into harmonic, exact, and co-exact components (Hodge decomposition)
- Harmonic forms represent the cohomology classes and satisfy both $d\omega = 0$ and $d^*\omega = 0$ (d^* is the adjoint of d)

Unit 10 – Applications in Algebraic Topology

10.1 Singular homology and cohomology

Singular homology and cohomology are powerful tools in algebraic topology. They help us understand the structure of topological spaces by studying their "holes" and other geometric features. These concepts build on the ideas of chain complexes and homology groups.

Homology groups capture information about cycles and boundaries in a space, while cohomology groups provide a dual perspective. Together, they offer insights into the topology of spaces and allow us to compute important invariants.

Singular Homology

Simplicial Complexes and Chain Complexes

- Singular simplex maps standard n -simplex into a topological space X
- Collection of singular simplices forms a simplicial complex
- Chain complex constructed from simplicial complex by taking free abelian groups generated by n -simplices ($C_n(X)$)
- Boundary operator $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ defined by mapping each n -simplex to its oriented boundary (alternating sum of its $(n-1)$ -dimensional faces)
- Boundary operator satisfies $\partial_{n-1} \circ \partial_n = 0$, forming a chain complex

Homology Groups and Their Properties

- n -th homology group $H_n(X)$ defined as kernel of ∂_n modulo image of ∂_{n+1}
- Elements of $H_n(X)$ are equivalence classes of n -cycles (elements of kernel of ∂_n) modulo boundaries (elements of image of ∂_{n+1})
- Homology groups are topological invariants, independent of the choice of simplicial complex
- Functorial properties allow for the study of induced homomorphisms between homology groups
- Homology groups capture "holes" in a topological space (connected components for H_0 , loops for H_1 , voids for H_2 , etc.)

Singular Cohomology

Cohomology Groups and Cochains

- Cohomology groups $H^n(X)$ defined as the dual of homology groups $H_n(X)$
- Cochain complex formed by taking $\text{Hom}(C_n(X), G)$ for each n , where G is an abelian group (coefficients)

- Coboundary operator $\delta^n : \text{Hom}(C_n(X), G) \rightarrow \text{Hom}(C_{n+1}(X), G)$ defined as the dual of the boundary operator
- n -th cohomology group $H^n(X; G)$ defined as kernel of δ^n modulo image of δ^{n-1}
- Elements of $H^n(X; G)$ are equivalence classes of n -cocycles modulo n -coboundaries

Universal Coefficient Theorem and Künneth Formula

- Universal Coefficient Theorem relates homology and cohomology groups via a short exact sequence involving *Ext* and *Tor* functors
- Allows for the computation of cohomology groups from homology groups and vice versa
- Künneth Formula expresses the homology (or cohomology) of a product space in terms of the homology (or cohomology) of its factors and their tensor products
- Useful for computing homology and cohomology of product spaces (torus, product of spheres, etc.)

Computational Tools

Mayer-Vietoris Sequence

- Mayer-Vietoris sequence is a long exact sequence relating homology groups of a space X to homology groups of subspaces A and B , where $X = A \cup B$
- Sequence involves homology groups of A , B , $A \cap B$, and X , connected by boundary operators and inclusion-induced homomorphisms
- Useful for computing homology groups of spaces that can be decomposed into simpler subspaces (CW complexes, simplicial complexes, etc.)
- Provides a way to break down the computation of homology groups into smaller, more manageable pieces
- Can be applied iteratively to compute homology groups of spaces with multiple decompositions

10.2 Cellular homology

Cellular homology is a powerful tool for calculating homology groups of spaces built from simple pieces. It simplifies computations by focusing on the structure of cell attachments, making it easier to understand complex topological spaces.

This approach connects to the broader applications of algebraic topology by providing a concrete method to analyze spaces. It bridges the gap between abstract concepts and practical calculations, showcasing how algebraic tools can reveal geometric properties.

CW Complexes and Cell Attachment

Construction and Properties of CW Complexes

- CW complexes are built by attaching cells of increasing dimension to a discrete set of points (0-cells)
- Cells are attached via continuous maps from the boundary of an n -dimensional ball to the $(n-1)$ -skeleton
- The resulting space has the weak topology with respect to the cell decomposition

- CW complexes are Hausdorff, second-countable, and locally contractible
- The dimension of a CW complex is the highest dimension of its cells

Cell Attachment and Homotopy Type

- Cell attachment is the process of building a CW complex by attaching cells of increasing dimension
- The homotopy type of a CW complex is determined by the homotopy classes of the attaching maps
- Attaching an n -cell to an $(n-1)$ -connected space results in an n -connected space
- Two CW complexes are homotopy equivalent if they have the same homotopy type
- Cell attachment can be used to construct spaces with desired homotopy properties (spheres, tori)

Cellular Chain Complexes and Boundary Operators

Construction of Cellular Chain Complexes

- The cellular chain complex $C_*(X)$ of a CW complex X is a sequence of abelian groups $C_n(X)$ connected by boundary operators $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$
- $C_n(X)$ is the free abelian group generated by the n -cells of X
- The boundary operator ∂_n encodes the incidence relations between n -cells and $(n-1)$ -cells
- The composition of two consecutive boundary operators is always zero: $\partial_{n-1} \circ \partial_n = 0$
- The cellular chain complex is a chain complex, satisfying the property $\partial \circ \partial = 0$

Properties of Cellular Boundary Operators

- The cellular boundary operator ∂_n measures the degree of the attaching map of each n -cell
- The degree of an attaching map is the number of times the boundary of the n -cell wraps around the $(n-1)$ -cell, with orientation taken into account
- The boundary of a sum of cells is the sum of their boundaries: $\partial(a + b) = \partial(a) + \partial(b)$
- The boundary operator satisfies the Leibniz rule: $\partial(a \cdot b) = \partial(a) \cdot b + (-1)^{\deg(a)} a \cdot \partial(b)$
- The cellular boundary operators completely determine the homology of the CW complex

Cellular Homology and Cohomology

Cellular Homology Groups and Their Properties

- The n -th cellular homology group $H_n(X)$ of a CW complex X is the quotient group $\ker(\partial_n)/\text{im}(\partial_{n+1})$
- Elements of $H_n(X)$ are equivalence classes of n -cycles (elements of $\ker(\partial_n)$) modulo n -boundaries (elements of $\text{im}(\partial_{n+1})$)
- Cellular homology groups are topological invariants, independent of the cell decomposition
- Cellular homology satisfies the Eilenberg-Steenrod axioms for a homology theory
- The dimension of $H_n(X)$ as a vector space is the n -th Betti number $\beta_n(X)$, which counts the number of " n -dimensional holes" in X

Cellular Cohomology and the Universal Coefficient Theorem

- The n -th cellular cohomology group $H^n(X; G)$ of a CW complex X with coefficients in an abelian group G is the group of homomorphisms from $H_n(X)$ to G
- Cellular cohomology groups are related to cellular homology groups by the Universal Coefficient Theorem: $H^n(X; G) \cong \text{Hom}(H_n(X), G) \oplus \text{Ext}(H_{n-1}(X), G)$
- The Universal Coefficient Theorem allows for the computation of cohomology groups from homology groups
- Cellular cohomology satisfies the Eilenberg-Steenrod axioms for a cohomology theory
- The cup product gives cellular cohomology the structure of a graded ring, with the product $H^p(X; R) \otimes H^q(X; R) \rightarrow H^{p+q}(X; R)$ induced by the diagonal map $X \rightarrow X \times X$

10.3 Homological algebra in homotopy theory

Homological algebra plays a crucial role in homotopy theory, bridging algebraic and topological concepts. It provides tools to study spaces through algebraic structures, enabling deeper insights into their properties and relationships.

This topic explores key theorems and techniques in homological algebra applied to homotopy theory. We'll examine how these tools help analyze topological spaces and solve complex problems in algebraic topology.

Homotopy Groups and Theorems

Fundamental Concepts of Homotopy Groups

- Homotopy groups $\pi_n(X)$ describe the different ways to map an n -dimensional sphere S^n into a topological space X
- Two mappings are considered equivalent if they are homotopic, meaning they can be continuously deformed into each other
- The group operation is defined by concatenating the mappings, which results in a group structure on the set of homotopy classes
- The first homotopy group $\pi_1(X)$ is the fundamental group of X , which captures information about loops in the space (paths that start and end at the same point)
- Higher homotopy groups $\pi_n(X)$ for $n > 1$ capture higher-dimensional hole-like features of the space

Hurewicz Theorem and Its Implications

- The Hurewicz theorem relates the homotopy groups of a space to its homology groups, which are algebraic invariants that capture information about the space's structure
- For a simply connected space X (i.e., $\pi_1(X) = 0$), the Hurewicz theorem states that the first non-trivial homotopy group $\pi_n(X)$ is isomorphic to the corresponding homology group $H_n(X)$
- This theorem provides a way to compute homotopy groups using homology, which is often easier to calculate
- The Hurewicz theorem is a powerful tool for understanding the relationship between the geometric properties of a space (captured by homotopy groups) and its algebraic properties (captured by

homology groups)

Whitehead Theorem and Its Applications

- The Whitehead theorem states that a continuous map between two connected CW complexes is a homotopy equivalence if and only if it induces isomorphisms on all homotopy groups
- A homotopy equivalence is a continuous map that has a continuous inverse up to homotopy, meaning the spaces are essentially the same from a homotopy theory perspective
- This theorem is useful for determining whether two spaces are homotopy equivalent without explicitly constructing a homotopy equivalence
- The Whitehead theorem is particularly useful in algebraic topology, where CW complexes are commonly used to model topological spaces
- It allows for the classification of spaces up to homotopy equivalence using algebraic invariants (homotopy groups) rather than geometric properties

Spectral Sequences

Introduction to Spectral Sequences

- Spectral sequences are algebraic tools used to compute homology, cohomology, and other invariants of topological spaces
- They provide a systematic way to break down complex computations into a sequence of simpler steps
- A spectral sequence consists of a series of pages, each containing a grid of groups or modules, with differentials (maps) between them
- The pages are denoted by $E_r^{p,q}$, where r is the page number, and p and q are the indices of the group or module on the page
- The differentials on each page satisfy certain properties, such as $d_r \circ d_r = 0$, which allows for the computation of the next page using homology

Serre Spectral Sequence

- The Serre spectral sequence is a powerful tool for computing the homology of a fibration (a continuous map satisfying the homotopy lifting property)
- It relates the homology of the base space, the fiber, and the total space of a fibration
- The Serre spectral sequence has its E_2 page determined by the homology of the base space and the homology of the fiber
- The differentials on the E_2 page are induced by the twisting in the fibration, which captures how the fibers are "twisted" over the base space
- Convergence of the Serre spectral sequence to the homology of the total space allows for the computation of the latter using the homology of the base and fiber (under suitable conditions)

Eilenberg-Moore Spectral Sequence

- The Eilenberg-Moore spectral sequence is another tool for computing homology, specifically in the context of pullbacks (a limit in the category of topological spaces)

- It relates the homology of the pullback to the homology of the spaces involved in the pullback diagram
- The E_2 page of the Eilenberg-Moore spectral sequence is determined by the homology of the spaces in the pullback diagram and their tensor product
- The differentials on the E_2 page are induced by the maps in the pullback diagram
- Convergence of the Eilenberg-Moore spectral sequence to the homology of the pullback allows for its computation using the homology of the spaces in the diagram (under suitable conditions)
- This spectral sequence is particularly useful in algebraic topology when working with fiber squares (pullback squares in which one map is a fibration)

Obstruction Theory

Fundamentals of Obstruction Theory

- Obstruction theory is a framework for determining the existence of continuous maps between topological spaces satisfying certain conditions
- It provides a systematic way to identify and analyze the obstructions (algebraic or geometric) that prevent the existence of such maps
- The main idea behind obstruction theory is to build a continuous map step by step, starting with a map on the skeleton (a subspace consisting of cells up to a certain dimension) of the domain space
- At each stage, the obstruction to extending the map to the next dimensional skeleton is measured by an element in a cohomology group
- If the obstruction element is zero (or more generally, if it vanishes), then the map can be extended to the next skeleton
- The process continues until either a non-zero obstruction is encountered (implying the map cannot be extended) or the map is defined on the entire domain space

Applications and Examples of Obstruction Theory

- Obstruction theory is particularly useful in the study of vector bundles and principal bundles, where it helps determine the existence of cross-sections (continuous maps from the base space to the total space that are right inverses to the projection map)
- For example, the hairy ball theorem (which states that there is no non-vanishing continuous tangent vector field on an even-dimensional sphere) can be proved using obstruction theory
- In this case, the obstruction to the existence of a non-vanishing tangent vector field is measured by the Euler class, a characteristic class in the second cohomology group of the sphere
- Another application of obstruction theory is in the classification of maps between complex projective spaces, where the obstructions lie in certain cohomology groups of the domain space
- Obstruction theory also plays a role in the study of embeddings and immersions of manifolds, where it helps determine the codimension (the difference between the dimension of the ambient space and the dimension of the manifold) in which such maps exist

Cohomology and Its Role in Obstruction Theory

- Cohomology groups, which are algebraic invariants dual to homology groups, play a central role in obstruction theory
- The obstructions to extending continuous maps are typically elements of certain cohomology groups of the domain space
- The dimension of the cohomology group where the obstruction lies depends on the dimension of the skeleton being considered and the codimension of the target space
- For example, when studying the existence of cross-sections of a vector bundle, the obstructions lie in the cohomology groups of the base space with coefficients in the homotopy groups of the fiber
- The vanishing of these cohomology groups (or more precisely, the vanishing of the obstruction elements) is a necessary and sufficient condition for the existence of a cross-section
- Cohomology operations, such as the cup product and the Steenrod squares, can be used to analyze the structure of the obstruction elements and derive further conditions for the existence of maps

10.4 Eilenberg-Steenrod axioms

The Eilenberg-Steenrod axioms are the foundation of homology theory in algebraic topology. They define how homology groups behave under various operations, allowing us to calculate and understand the structure of topological spaces.

These axioms connect different aspects of topology, from homotopy equivalence to the decomposition of spaces. They provide a powerful framework for analyzing spaces by breaking them down into simpler parts and relating their homology groups.

Axioms of Homology Theory

Homotopy Invariance and Dimension

- Homotopy axiom states that if two continuous maps between topological spaces are homotopic, then they induce the same homomorphism on homology groups
- Implies that homology groups are invariants of homotopy type (spaces that are homotopy equivalent have isomorphic homology groups)
- Allows for the calculation of homology groups of a space by considering a simpler space homotopy equivalent to it
- Dimension axiom specifies the homology groups of a single point space
- States that the homology group $H_n(pt)$ is isomorphic to $\mathbb{Z}$ for $n = 0$ and is trivial for $n > 0$
- Provides a starting point for calculating homology groups using the other axioms

Excision and Additivity

- Excision axiom relates the homology of a space X to the homology of a subspace $A \subset X$ and its closure
- States that if the closure of A is contained in the interior of a subspace $U \subset X$, then the inclusion $(X \setminus A, U \setminus A) \hookrightarrow (X, U)$ induces isomorphisms on homology groups

- Allows for the calculation of homology groups by decomposing a space into smaller, simpler pieces
- Additivity axiom states that the homology of a disjoint union of spaces is isomorphic to the direct sum of the homology of each space
- Formally, if $X = \bigsqcup_{\alpha} X_{\alpha}$, then $H_n(X) \cong \bigoplus_{\alpha} H_n(X_{\alpha})$ for all n
- Enables the computation of homology groups of a space by breaking it down into its connected components

Exactness and Long Exact Sequence

- Exactness axiom relates the homology of a space, a subspace, and the corresponding quotient space
- For a pair (X, A) with $A \subset X$, there is a long exact sequence of homology groups:
 $\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow \cdots$
- Connects the homology groups of different spaces and allows for their computation using the properties of exact sequences
- The connecting homomorphism $H_n(X, A) \rightarrow H_{n-1}(A)$ is induced by the boundary operator in the chain complex

Fundamental Results

Homology Theory and Uniqueness

- Homology theory refers to a collection of functors from the category of topological spaces (or a suitable subcategory) to the category of abelian groups, satisfying the Eilenberg-Steenrod axioms
- Different homology theories may arise from different choices of chain complexes or different methods of construction
- Examples include simplicial homology, singular homology, and cellular homology
- Uniqueness theorem states that any two homology theories satisfying the Eilenberg-Steenrod axioms and agreeing on the homology groups of a point are naturally isomorphic
- Implies that the homology groups of a space are independent of the choice of homology theory, as long as the axioms are satisfied
- Allows for the use of different homology theories depending on the context and the available data about the space (simplicial complexes, CW complexes, etc.)

Unit 11 – Applications in Commutative Algebra

11.1 Local cohomology

Local cohomology connects algebra and geometry by studying modules with support in ideals. It uses Čech complexes to measure how deeply an ideal "cuts into" a module, revealing important structural information about rings and modules.

Local cohomology has deep connections to duality theories. Matlis duality links finitely generated and artinian modules, while local duality relates local cohomology to Ext modules. These tools provide powerful insights into module structure and ring properties.

Local Cohomology and Čech Complex

Definition and Construction of Local Cohomology

- Local cohomology functor $H_I^i(M)$ associates to an R -module M and an ideal $I \subset R$ the i -th local cohomology module of M with support in I
- Constructed using the Čech complex, a cochain complex associated to a cover of a topological space
- For an ideal $I = (f_1, \dots, f_n)$, the Čech complex is $C^\bullet(f_1, \dots, f_n; M)$
- The i -th local cohomology module is the i -th cohomology of this complex
 $H_I^i(M) = H^i(C^\bullet(f_1, \dots, f_n; M))$
- Local cohomology measures the depth of a module, the length of a maximal M -sequence in I
- $\text{depth}_I(M) = \inf \{i \mid H_I^i(M) \neq 0\}$

Properties and Vanishing Theorems

- Local cohomology modules $H_I^i(M)$ are I -torsion modules annihilated by a power of I
- Vanishing theorems give conditions for the local cohomology modules to be zero
- For a noetherian ring R , $H_I^i(M) = 0$ for $i > \dim_R(M)$ (Grothendieck's Vanishing Theorem)
- If I is generated by n elements, then $H_I^i(M) = 0$ for $i > n$ regardless of the dimension of M
- Computing examples of local cohomology using the Čech complex for specific ideals and modules (polynomial ring $k[x, y]$ and the ideal (x, y))

Duality in Local Cohomology

Support and Matlis Duality

- The support of an R -module M is the set of primes $\text{Supp}(M) = \{P \in \text{Spec}(R) \mid M_P \neq 0\}$
- Equivalently, the support is the variety of the annihilator ideal $\text{Ann}(M)$
- Matlis duality establishes a duality between finitely generated modules over a complete local ring and artinian modules

- For a complete local ring $(R, \mathfrak{m})$ and a finitely generated R -module M , the Matlis dual is $M^\vee = \text{Hom}_R(M, E(R/\mathfrak{m}))$ where $E(R/\mathfrak{m})$ is the injective hull of the residue field
- Matlis duality gives an equivalence of categories between finitely generated R -modules and artinian R -modules

Local Duality Theorem

- Local duality relates the local cohomology of a finitely generated module M over a local ring $(R, \mathfrak{m})$ with the Matlis dual of the local cohomology of the Matlis dual $M^\vee$
- For a Cohen-Macaulay local ring of dimension d , there are isomorphisms $H_{\mathfrak{m}}^i(M) \cong \text{Ext}_R^{d-i}(M, \omega_R)^\vee$ where ω_R is the canonical module
- Consequences and applications of local duality
- Relates the depth of a module to the smallest non-vanishing Ext module (Auslander-Buchsbaum formula)
- Gives a duality between the local cohomology of a ring and its canonical module (Grothendieck's Local Duality Theorem)
- Examples illustrating local duality for specific local rings and modules (regular local rings, Gorenstein local rings)

11.2 Koszul complexes

Koszul complexes are powerful tools in commutative algebra, linking sequences of ring elements to homological properties. They're built using exterior algebras and help us understand regular sequences, which are crucial for studying ideals and modules.

These complexes shine when it comes to minimal free resolutions, especially for modules defined by regular sequences. They let us calculate important stuff like depth, projective dimension, and Betti numbers, giving us a deeper look into the structure of rings and modules.

Koszul Complexes and Regular Sequences

Construction and Properties of Koszul Complexes

- Koszul complex is a chain complex associated with a sequence of elements $x_1, \dots, x_n$ in a commutative ring R
- Constructed using the exterior algebra $\Lambda_R(R^n)$ on a free R -module of rank n
- Differential is defined using the sequence elements and the exterior algebra multiplication
- Regular sequence is a sequence $x_1, \dots, x_n$ in a commutative ring R such that x_i is a non-zero-divisor in $R/(x_1, \dots, x_{i-1})$ for each i
- Important property for studying the homological behavior of ideals and modules
- Examples include the variables in a polynomial ring (regular sequence) and the maximal ideal in a regular local ring (regular sequence)
- Exterior algebra $\Lambda_R(M)$ over an R -module M is the quotient of the tensor algebra by the ideal generated by elements of the form $m \otimes m$ for $m \in M$

- Graded algebra with a wedge product that is alternating and satisfies the graded Leibniz rule
- Used in the construction of the Koszul complex and its differentials

Minimal Free Resolutions and Koszul Complexes

- Minimal free resolution of an R -module M is an exact sequence of free R -modules that resolves M with the property that each differential matrix has entries in the maximal ideal
- Encodes homological information about the module, such as its projective dimension and Betti numbers
- Koszul complex provides a minimal free resolution for cyclic modules defined by a regular sequence
- Koszul complex associated with a regular sequence $x_1, \dots, x_n$ is a minimal free resolution of the quotient ring $R/(x_1, \dots, x_n)$
- Acyclicity of the Koszul complex is equivalent to the sequence being regular
- Allows for the computation of invariants like depth, projective dimension, and Betti numbers for quotients by regular sequences

Koszul Homology and Cohomology

Definitions and Basic Properties

- Koszul homology $H_*(x_1, \dots, x_n; R)$ is the homology of the Koszul complex associated with a sequence $x_1, \dots, x_n$ in a commutative ring R
- Measures the deviation of the sequence from being a regular sequence
- Vanishes if and only if the sequence is regular
- Koszul cohomology $H^*(x_1, \dots, x_n; R)$ is the cohomology of the Koszul complex, obtained by applying the functor $\text{Hom}_R(-, R)$ to the Koszul complex
- Dual notion to Koszul homology, with analogous properties and interpretations
- Provides a cohomological characterization of regular sequences

Applications and Interpretations

- Betti numbers of a module M over a local ring R are the ranks of the free modules appearing in a minimal free resolution of M
- Measure the complexity of the module and its homological behavior
- Can be computed using the Koszul homology or cohomology when the module is defined by a regular sequence
- Koszul homology and cohomology have connections to other invariants and constructions in commutative algebra, such as:
 - Depth and projective dimension of modules
 - Local cohomology and local duality
 - Gorenstein rings and complete intersections (Koszul homology vanishes for complete intersection ideals)

- Koszul complexes and their (co)homology provide a bridge between the algebraic properties of sequences and the homological properties of modules and rings
- Enable the study of regular sequences, which play a central role in commutative algebra
- Allow for the computation of important invariants and the understanding of structural properties of rings and modules

11.3 Homological dimensions

Homological dimensions measure the complexity of modules and rings using resolutions. They connect algebraic properties to homological ones, providing insights into module and ring structures.

Projective, injective, and flat dimensions quantify how close modules are to having these properties. Global dimension measures a ring's homological complexity, while Krull dimension relates to its algebraic structure.

Projective and Injective Dimensions

Measuring Resolutions with Projective and Injective Dimensions

- Projective dimension measures the length of the shortest projective resolution of a module
- Defined as the smallest integer n such that there exists a projective resolution of length n
- If no finite projective resolution exists, the projective dimension is infinite
- Projective dimension of 0 means the module is projective itself
- Injective dimension measures the length of the shortest injective resolution of a module
- Defined as the smallest integer n such that there exists an injective resolution of length n
- If no finite injective resolution exists, the injective dimension is infinite
- Injective dimension of 0 means the module is injective itself
- Projective and injective dimensions provide a way to quantify how far a module is from being projective or injective respectively
- Higher dimensions indicate a module is further from having the respective property
- Dimensions give insight into the structure and complexity of a module

Relating Projective and Injective Dimensions

- Auslander-Buchsbaum formula relates the projective dimension of a finitely generated module M over a local ring R to the depth of M
- Formula states: $\text{projective dimension}(M) + \text{depth}(M) = \text{depth}(R)$
- Connects the homological notion of projective dimension to the algebraic notion of depth
- Allows for the computation of one invariant from the other
- Highlights the interplay between homological and algebraic properties of modules and rings

Global and Flat Dimensions

Measuring Rings with Global Dimension

- Global dimension is an invariant of a ring that measures the maximum projective dimension of its modules
- Defined as the supremum of the projective dimensions of all modules over the ring
- If the global dimension is finite, it is the maximum projective dimension attained by any module
- Rings with global dimension 0 are called semisimple rings (all modules are projective)
- Rings with finite global dimension are called regular rings
- Global dimension provides a measure of the homological complexity of a ring
- Lower global dimension indicates a simpler homological structure
- Hereditary rings are rings with global dimension at most 1 (every submodule of a projective module is projective)
- Global dimension can be used to classify and study rings based on their homological properties

Flat Dimension and its Properties

- Flat dimension is another invariant that measures the length of the shortest flat resolution of a module
- Defined as the smallest integer n such that there exists a flat resolution of length n
- If no finite flat resolution exists, the flat dimension is infinite
- Flat dimension of 0 means the module is flat itself
- Flat modules are modules that preserve injectivity of maps when tensored
- Flat resolutions are resolutions by flat modules
- Flat dimension measures how far a module is from being flat
- Flat dimension is related to the projective and injective dimensions
- Projective modules are flat, so projective dimension is always less than or equal to flat dimension
- If a ring has finite global dimension, then the flat dimension of any module is also finite

Dimension Invariants

Bass Numbers and their Applications

- Bass numbers are a sequence of invariants associated to a module over a local ring
- Defined as the ranks of the free modules in a minimal injective resolution of the module
- Provide a measure of the complexity of the injective resolution
- Can be used to study the structure of modules and local rings
- Bass numbers have connections to other invariants and properties

- The zeroth Bass number is the minimal number of generators of the module
- The first Bass number is related to the indecomposability of the injective hull of the module
- Higher Bass numbers are linked to the structure of the syzygies in the minimal injective resolution
- Applications of Bass numbers include:
 - Computing the injective dimension of a module (it is the index of the last nonzero Bass number)
 - Studying the local cohomology modules of a ring
 - Investigating the structure of artinian modules over a complete local ring

Krull Dimension and its Relationship with other Invariants

- Krull dimension is a notion of dimension for commutative rings and their modules
- Defined as the supremum of lengths of chains of prime ideals in the ring
- Measures the "size" or "complexity" of the ring
- For a module, it is defined as the Krull dimension of its support (the set of prime ideals containing its annihilator)
- Krull dimension is related to other dimension invariants
- For a local ring, the Krull dimension is always less than or equal to the global dimension
- If a local ring is Cohen-Macaulay, then its Krull dimension equals its depth
- For finitely generated modules over a local ring, the Auslander-Buchsbaum formula relates Krull dimension, projective dimension, and depth
- Krull dimension is a fundamental invariant in commutative algebra
- Used to classify and study commutative rings and their modules
- Plays a role in many important results and theorems (e.g., Krull's Principal Ideal Theorem, dimension theory)

11.4 Cohen-Macaulay rings and Gorenstein rings

Cohen-Macaulay and Gorenstein rings are special types of commutative rings with nice algebraic properties. They're important because they balance complexity and structure, making them useful in many areas of algebra and geometry.

These rings have specific depth and dimension relationships, and their properties are often described using homological tools. Understanding them helps us tackle problems in algebraic geometry, commutative algebra, and even some areas of physics.

Cohen-Macaulay Rings

Definition and Properties

- Cohen-Macaulay ring is a commutative Noetherian local ring R whose depth equals its Krull dimension, $\text{depth}(R) = \dim(R)$

- Regular local ring is a Cohen-Macaulay ring with the additional property that its maximal ideal can be generated by a regular sequence (sequence of elements $x_1, \dots, x_n$ such that x_i is a non-zero divisor in $R/(x_1, \dots, x_{i-1})$ for all i)
- Cohen-Macaulay rings have a canonical module ω_R , a finitely generated module with special homological properties (e.g., $\text{Ext}_R^i(\omega_R, R) = 0$ for $i \neq \dim(R)$ and $\text{Ext}_R^{\dim(R)}(\omega_R, R) \cong R$)

Type and Canonical Module

- Type of a Cohen-Macaulay ring R is the dimension of the socle of the ring $R/\mathfrak{m}$ as a vector space over the residue field $k = R/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal
- Canonical module ω_R plays a crucial role in the study of Cohen-Macaulay rings
- For a Gorenstein ring, the canonical module is isomorphic to the ring itself, $\omega_R \cong R$
- For a complete intersection ring, the canonical module is isomorphic to the top exterior power of the cotangent space, $\omega_R \cong \wedge^{\dim(R)} \mathfrak{m}/\mathfrak{m}^2$

Gorenstein Rings

Definition and Properties

- Gorenstein ring is a Cohen-Macaulay ring of type 1, meaning the socle of $R/\mathfrak{m}$ is a one-dimensional vector space over the residue field $k = R/\mathfrak{m}$
- Gorenstein rings have a symmetric dualizing complex, a complex of modules that generalizes the notion of a canonical module
- Local cohomology characterization states that a local ring R is Gorenstein if and only if $H_{\mathfrak{m}}^i(R) = 0$ for all $i \neq \dim(R)$ and $H_{\mathfrak{m}}^{\dim(R)}(R) \cong E_R(k)$, the injective hull of the residue field k

Dualizing Complex

- Dualizing complex is a complex of modules $D^\bullet$ over a local ring R with special homological properties (e.g., the cohomology of $D^\bullet$ is finitely generated and the natural map $R \rightarrow \mathbf{R}Hom_R(D^\bullet, D^\bullet)$ is a quasi-isomorphism)
- For a Gorenstein ring, the dualizing complex is isomorphic to a shift of the ring itself, $D^\bullet \cong R[-\dim(R)]$
- Existence of a dualizing complex is a key property that distinguishes Gorenstein rings from general Cohen-Macaulay rings

Properties and Characterizations

Serre's Conditions and Regular Local Rings

- "Serre's conditions" (S_n) and (R_n) are properties of a local ring R related to the depth of prime ideals and the regularity of the ring
- (S_n) : $\text{depth}(R_{\mathfrak{p}}) \geq \min(n, \dim(R_{\mathfrak{p}}))$ for all prime ideals $\mathfrak{p}$ of R
- (R_n) : $R_{\mathfrak{p}}$ is regular for all prime ideals $\mathfrak{p}$ with $\dim(R_{\mathfrak{p}}) \leq n$

- Regular local ring satisfies both (S_n) and (R_n) for all n , making it a Cohen-Macaulay ring with additional regularity properties

Homological Characterizations

- Canonical module and local cohomology provide homological characterizations of Cohen-Macaulay and Gorenstein rings
- For a Cohen-Macaulay ring R , the canonical module ω_R satisfies $\text{Ext}_R^i(\omega_R, R) = 0$ for $i \neq \dim(R)$ and $\text{Ext}_R^{\dim(R)}(\omega_R, R) \cong R$
- Local cohomology characterization of Gorenstein rings states that R is Gorenstein if and only if $H_{\mathfrak{m}}^i(R) = 0$ for all $i \neq \dim(R)$ and $H_{\mathfrak{m}}^{\dim(R)}(R) \cong E_R(k)$, the injective hull of the residue field k
- These characterizations highlight the deep connections between the homological properties of a ring and its Cohen-Macaulay or Gorenstein nature

Unit 12 – Advanced Topics in Homological Algebra

12.1 Derived categories and triangulated categories

Derived categories and triangulated categories are powerful tools in homological algebra. They provide a framework for studying chain complexes and their relationships, allowing us to work with quasi-isomorphisms as actual isomorphisms.

These structures generalize ideas from homological algebra to a broader setting. They're crucial for understanding advanced topics like derived functors, t-structures, and applications in algebraic geometry and topology.

Derived Categories

Constructing the Derived Category

- Derived category $D(A)$ formed by localizing the category of chain complexes $Ch(A)$ with respect to the class of quasi-isomorphisms
- Localization process adds formal inverses to quasi-isomorphisms, allowing them to become isomorphisms in the derived category
- Quasi-isomorphism a chain map $f: X_{\bullet} \rightarrow Y_{\bullet}$ inducing isomorphisms on all homology groups $H_n(f): H_n(X_{\bullet}) \rightarrow H_n(Y_{\bullet})$ for all $n \in \mathbb{Z}$
- Objects in the derived category are chain complexes, but morphisms are obtained by inverting quasi-isomorphisms (homotopy classes of chain maps)

Properties and Applications

- Derived functors (Tor, Ext) arise naturally in the derived category framework, as they are obtained by applying the localization functor to the original functor
- Verdier quotient construction used to form the derived category, where the localization is performed with respect to the multiplicative system of quasi-isomorphisms
- Derived category $D(A)$ has a triangulated structure, with distinguished triangles corresponding to short exact sequences of chain complexes (up to quasi-isomorphism)
- Derived categories play a central role in homological algebra and algebraic geometry, providing a powerful tool for studying homological invariants and derived functors

Triangulated Categories

Axioms and Structure

- Triangulated category $(\mathcal{T}, \Sigma)$ consists of an additive category $\mathcal{T}$ and an autoequivalence $\Sigma: \mathcal{T} \rightarrow \mathcal{T}$ called the shift or suspension functor
- Distinguished triangles in $\mathcal{T}$ are sequences of the form $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ satisfying certain axioms (rotation, morphism, octahedral)

- Octahedral axiom relates distinguished triangles and ensures the existence of a commutative diagram (octahedron) involving compositions of morphisms in distinguished triangles
- Triangulated categories axiomatize the properties of derived categories and stable homotopy categories in algebraic topology

t-Structures and Applications

- t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on a triangulated category $\mathcal{T}$ consists of two full subcategories satisfying certain orthogonality and stability conditions
- Heart of a t-structure $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is an abelian category, allowing the study of homological algebra within the triangulated framework
- Examples of t-structures include the standard t-structure on the derived category $D(A)$ (with heart A) and the perverse t-structure on the derived category of sheaves (with heart the category of perverse sheaves)
- t-structures provide a way to construct abelian categories from triangulated categories and are used in the study of perverse sheaves, intersection cohomology, and the Riemann-Hilbert correspondence

12.2 A-infinity algebras and operads

A-infinity algebras expand on associative algebras, allowing for higher homotopies. They consist of graded vector spaces with multilinear maps satisfying generalized associativity conditions. The Stasheff associahedron encodes these relations, with faces corresponding to the maps in an A-infinity algebra.

Operads provide a framework for studying algebraic structures with multiple operations. They consist of sets representing operations with composition maps. The bar construction assigns differential graded coalgebras to operads, while Koszul duality establishes correspondences between certain operad pairs.

A-infinity Algebras and Homotopy

Generalizing Associative Algebras

- A-infinity algebras generalize the notion of associative algebras by allowing for higher homotopies
- Consist of a graded vector space A equipped with a family of multilinear maps $m_n : A^{\otimes n} \rightarrow A$ of degree $2 - n$ for each $n \geq 1$
- The maps m_n satisfy a sequence of relations that generalize the associativity condition for ordinary algebras
- The first relation states that m_1 is a differential, meaning $m_1^2 = 0$, and the second relation is the Leibniz rule for m_2 with respect to m_1

Higher Homotopies and the Stasheff Associahedron

- Higher homotopies in an A-infinity algebra are encoded by the maps m_n for $n \geq 3$
- These higher homotopies measure the failure of associativity up to homotopy
- The Stasheff associahedron is a polytope whose vertices correspond to the different ways of bracketing n elements

- The faces of the associahedron correspond to the relations satisfied by the maps m_n in an A -infinity algebra
- For example, the pentagonal face corresponds to the relation involving m_2 and m_3

Transferring A -infinity Structures

- The homotopy transfer theorem states that an A -infinity structure can be transferred along a homotopy equivalence
- More precisely, if A and B are homotopy equivalent chain complexes and A has an A -infinity structure, then B inherits an A -infinity structure as well
- This theorem is useful for constructing A -infinity algebras from simpler data
- For instance, it can be used to transfer an A -infinity structure from a differential graded algebra to its homology

Operads and Bar Construction

Operads as a Unified Framework

- Operads provide a unified framework for studying algebraic structures with multiple operations
- An operad $\mathcal{O}$ consists of a collection of sets $\mathcal{O}(n)$ for each $n \geq 0$, representing operations with n inputs
- These sets are equipped with composition maps that allow for substitution of operations into each other
- Operads encode many common algebraic structures, such as associative algebras, commutative algebras, and Lie algebras
- For example, the associative operad Ass has $Ass(n) = \Sigma_n$, the symmetric group on n letters

The Bar Construction

- The bar construction is a functor that assigns to each operad $\mathcal{O}$ a differential graded coalgebra $B(\mathcal{O})$
- It is defined as the free coalgebra generated by the suspension of $\mathcal{O}$, with a differential induced by the composition maps
- The bar construction is a key tool in the study of operads and their algebras
- For instance, the bar construction of an A -infinity algebra is a differential graded coalgebra, and morphisms of A -infinity algebras correspond to coalgebra morphisms between their bar constructions

Koszul Duality for Operads

- Koszul duality is a correspondence between certain pairs of operads
- An operad $\mathcal{O}$ is Koszul if its bar construction is quasi-isomorphic to the suspension of another operad $\mathcal{O}^!$, called the Koszul dual of $\mathcal{O}$
- Koszul duality provides a way to study an operad $\mathcal{O}$ in terms of its Koszul dual $\mathcal{O}^!$, which often has a simpler structure
- Many important operads, such as the associative, commutative, and Lie operads, are Koszul

- For example, the Koszul dual of the associative operad is the shifted Lie operad

Minimal Models

Minimal A-infinity Algebras

- A minimal A-infinity algebra is an A-infinity algebra $(A, \{m_n\})$ where $m_1 = 0$
- In other words, the differential on the underlying chain complex of A vanishes
- Minimal A-infinity algebras are important because they serve as "simplified models" for general A-infinity algebras
- Every A-infinity algebra is quasi-isomorphic to a minimal one, called its minimal model

Constructing Minimal Models

- The existence of minimal models for A-infinity algebras is a consequence of the homotopy transfer theorem
- Given an A-infinity algebra $(A, \{m_n\})$, one can construct a minimal model by the following steps:
 1. Choose a homotopy equivalence between A and its homology $H(A)$
 2. Transfer the A-infinity structure from A to $H(A)$ using the homotopy transfer theorem
- The resulting A-infinity algebra on $H(A)$ is a minimal model for A
- The structure maps of the minimal model can be described explicitly in terms of the homotopy equivalence data

Applications of Minimal Models

- Minimal models are a powerful tool for studying the homotopy theory of A-infinity algebras
- They provide a way to reduce questions about general A-infinity algebras to questions about simpler, minimal ones
- For example, two A-infinity algebras are quasi-isomorphic if and only if their minimal models are isomorphic
- Minimal models also play a key role in the construction of spectral sequences that compute the Hochschild cohomology of A-infinity algebras
- These spectral sequences are analogous to the classical Adams spectral sequence in algebraic topology

12.3 Hochschild and cyclic homology

Hochschild and cyclic homology are powerful tools for studying associative algebras. They capture algebraic and geometric properties, with applications in noncommutative geometry and K-theory.

These theories use complexes built from tensor powers of algebras. The Connes' exact sequence links them, while operators like Connes' boundary and the Dennis trace map connect them to other areas of math.

Hochschild and Cyclic Homology

Defining Hochschild and Cyclic Homology

- Hochschild homology is a homology theory for associative algebras that captures algebraic and geometric properties
- Defined using the Hochschild complex, which is constructed from the tensor powers of the algebra
- Encodes information about the algebra's structure and its modules
- Cyclic homology is a refinement of Hochschild homology that incorporates the action of the cyclic group
- Defined using the cyclic complex, which is a subcomplex of the Hochschild complex
- Captures additional cyclic symmetries and has connections to noncommutative geometry
- Negative cyclic homology and periodic cyclic homology are variations of cyclic homology
- Negative cyclic homology is defined using a certain homotopy limit of cyclic complexes
- Periodic cyclic homology is obtained by inverting a certain operator in the cyclic complex
- These variations have important applications in algebraic K-theory and index theory

Relationships and Applications

- Hochschild and cyclic homology are related through the Connes' exact sequence
- Connects Hochschild homology, cyclic homology, and negative cyclic homology
- Provides a way to study the relationships between these theories
- Hochschild and cyclic homology have applications in various areas of mathematics
- Noncommutative geometry uses cyclic homology as a key tool for studying noncommutative spaces
- Algebraic K-theory relies on Hochschild and cyclic homology for important computations and results
- Deformation theory and quantization often involve Hochschild and cyclic cohomology, the dual theories

Important Operators and Maps

Connes' Boundary Operator

- Connes' boundary operator, denoted by B , is a key operator in cyclic homology
- Defined on the Hochschild complex and maps it to itself, increasing the degree by one
- Satisfies the important relation $B^2 = 0$, making it a differential operator
- The operator B plays a crucial role in the construction of cyclic homology
- The cyclic complex is defined as the subcomplex of the Hochschild complex annihilated by $1 - \bar{t}$, where $\bar{t}$ is the cyclic permutation
- B descends to the cyclic complex and induces a map on cyclic homology

- Connes' boundary operator has connections to important concepts in noncommutative geometry
- Related to the Chern character in K-theory and the index theory of noncommutative spaces
- Captures certain noncommutative analogues of classical geometric notions

Dennis Trace Map

- The Dennis trace map is a natural transformation from algebraic K-theory to Hochschild homology
- Connects the K-groups of an associative algebra to its Hochschild homology groups
- Provides a way to study algebraic K-theory using homological methods
- The Dennis trace is defined using the Chern character in K-theory and the Hochschild complex
- Maps a K-theory class represented by a matrix to a Hochschild cycle given by its trace
- Compatible with the structure maps in K-theory and Hochschild homology
- The Dennis trace has important applications in algebraic topology and algebraic geometry
- Allows for the computation of K-theory groups using Hochschild homology
- Relates the K-theory of schemes to their cyclic homology via the Chern character

Key Theorems

Hochschild-Kostant-Rosenberg Theorem

- The Hochschild-Kostant-Rosenberg (HKR) theorem is a fundamental result in Hochschild homology
- Computes the Hochschild homology of smooth commutative algebras
- States that for a smooth commutative algebra A over a field k , there is an isomorphism: $HH_*(A) \cong \Omega_{A/k}^*$ where $\Omega_{A/k}^*$ denotes the module of Kähler differentials
- The HKR theorem has several important consequences and generalizations
- Provides a connection between Hochschild homology and differential forms in the commutative case
- Generalizes to the noncommutative setting, relating Hochschild homology to cyclic homology
- Plays a key role in the study of deformation quantization and the Deligne conjecture

Loday-Quillen Theorem

- The Loday-Quillen theorem is a landmark result in the theory of cyclic homology
- Computes the cyclic homology of the algebra of stable matrices over a field
- States that for a field k , there is an isomorphism: $HC_*(\mathcal{M}_\infty(k)) \cong k[u]$ where $\mathcal{M}_\infty(k)$ denotes the algebra of stable matrices and $k[u]$ is a polynomial algebra
- The Loday-Quillen theorem has significant implications in algebraic K-theory
- Provides a way to compute the K-theory of stable matrices using cyclic homology
- Relates the algebraic K-theory of a field to its cyclic homology

- Generalizes to other contexts, such as the K-theory of group rings and Lie algebras
- The theorem is a key ingredient in the proof of the Milnor conjecture on the K-theory of fields
- Milnor conjecture states that the graded ring associated to the K-theory of a field is a polynomial ring
- Loday-Quillen theorem, combined with results of Suslin, led to the proof of this conjecture

12.4 Current research trends in homological algebra

Homological algebra is evolving, with current research exploring higher structures and homotopy theory. These areas delve into complex algebraic structures and their applications in topology, representation theory, and physics.

Derived geometry and deformation theory are pushing the boundaries of algebraic geometry. Meanwhile, homological mirror symmetry is uncovering deep connections between different mathematical realms, impacting string theory and symplectic geometry.

Higher Structures and Homotopy Theory

Higher Category Theory and Homotopical Algebra

- Higher category theory studies the algebraic structures arising from higher-dimensional categories, where morphisms between objects can have their own morphisms, and so on
- Includes the study of ∞ -categories, which are categories enriched over topological spaces or simplicial sets
- Allows for the formalization of homotopy-coherent structures and the encoding of homotopical information
- Homotopical algebra is the study of algebraic structures in the context of homotopy theory
- Involves the development of model categories, which provide a framework for studying homotopy theories in an abstract setting
- Enables the study of derived functors and derived categories, which are essential tools in homological algebra and algebraic topology
- Spectral sequences are powerful computational tools in homotopy theory and homological algebra
- They are used to compute homology and cohomology groups of topological spaces, chain complexes, and other algebraic objects
- In chromatic homotopy theory, spectral sequences are employed to study the stable homotopy groups of spheres and other spectra, which are central objects in stable homotopy theory

Applications and Connections

- Higher structures and homotopy theory have found applications in various areas of mathematics and mathematical physics
- In algebraic topology, they provide a language for describing and studying homotopy types of spaces and spectra
- In representation theory, higher category theory is used to study categorified representations and their connections to quantum groups and knot invariants

- Homotopical methods have also been applied to the study of operads and their algebras, which play important roles in algebraic topology, algebraic geometry, and mathematical physics
- Operads encode algebraic structures with multiple inputs and outputs, such as associative algebras, commutative algebras, and Lie algebras
- Homotopy theory provides a framework for studying homotopy-coherent algebras over operads and their deformations

Derived Geometry and Deformation Theory

Derived Algebraic Geometry

- Derived algebraic geometry is an extension of classical algebraic geometry that incorporates homotopical and homological methods
- It studies derived schemes and derived stacks, which are generalizations of schemes and stacks that allow for non-trivial derived structures
- Derived schemes are defined using simplicial commutative rings or differential graded algebras, which encode higher-order infinitesimal information
- Derived intersection theory is a central topic in derived algebraic geometry
- It provides a refined version of intersection theory that takes into account the derived structure of schemes and stacks
- Allows for the definition of virtual fundamental classes and virtual cycles, which play important roles in enumerative geometry and mirror symmetry

Derived Deformation Theory and Motivic Cohomology

- Derived deformation theory studies the infinitesimal deformations of algebraic and geometric objects using derived methods
- It extends classical deformation theory by considering derived structures and higher-order infinitesimal deformations
- Allows for the study of obstructions and deformation spaces in a homotopical setting
- Motivic cohomology is a cohomology theory for algebraic varieties that takes into account the motivic nature of algebraic cycles
- It is defined using algebraic K-theory and provides a bridge between algebraic geometry and algebraic topology
- Motivic cohomology groups have important connections to special values of L-functions and the Beilinson conjectures in arithmetic geometry

Homological Mirror Symmetry

Homological Mirror Symmetry and Its Applications

- Homological mirror symmetry is a conjectural relationship between the derived category of coherent sheaves on a Calabi-Yau manifold and the Fukaya category of its mirror manifold
- It states that these two categories should be equivalent as triangulated categories

- Provides a powerful tool for studying the geometry and topology of Calabi-Yau manifolds and their mirror partners
- Homological mirror symmetry has been proven in several important cases, such as for certain classes of toric varieties and for the quintic threefold
- The proofs often involve the use of tropical geometry and the study of Lagrangian submanifolds and their Floer homology
- The equivalence between the derived category and the Fukaya category has led to new insights into the structure of these categories and their associated invariants
- Homological mirror symmetry has found applications in various areas of mathematics and mathematical physics
- In string theory, it provides a mathematical framework for understanding the relationship between different types of string theories and their compactifications
- In symplectic geometry, it has led to the development of new invariants and the study of Lagrangian submanifolds and their deformations