

# Metric Differential Geometry

## Archived study guides

These archived materials are no longer updated. This archive was exported from a saved copy of these guides.

Export date: 2026-10-05T19:26:30.244Z

## Contents

Unit 1 – Manifolds and Coordinate Systems in Geometry - 6 guides

Unit 2 – Curves and surfaces - 6 guides

Unit 3 – Riemannian metrics - 6 guides

Unit 4 – Geodesics and Distance in Metric Geometry - 6 guides

Unit 5 – Riemann Curvature Tensor and Its Applications - 6 guides

Unit 6 – Isometries and Killing Vector Fields - 6 guides

Unit 7 – Connections and Parallel Transport - 6 guides

Unit 8 – Submanifolds and Hypersurfaces in Geometry - 6 guides

Unit 9 – Variational Principles & Geodesic Equations - 6 guides

Unit 10 – Jacobi Fields and Conjugate Points - 6 guides

Unit 11 – Comparison Theorems & Geometric Inequalities - 6 guides

Unit 12 – Physics Applications in Differential Geometry - 6 guides

# Unit 1 – Manifolds and Coordinate Systems in Geometry

## 1.1 Topological spaces

Topological spaces form the foundation of modern geometry, providing a framework to study continuity and nearness without relying on distance. They generalize metric spaces, allowing for more abstract notions of proximity and connectedness.

Open and closed sets, bases, and subbases are key concepts in topological spaces. These structures define the topology and enable the study of continuous functions, homeomorphisms, and important properties like compactness and connectedness in a wide range of mathematical contexts.

### Definition of topological spaces

- A topological space is a set  $X$  together with a collection  $\tau$  of subsets of  $X$ , called open sets, satisfying certain axioms
- The open sets in a topological space form a topology on the set, which encodes the notion of nearness or proximity between points without relying on a metric or distance function
- The axioms that define a topology ensure that the collection of open sets is closed under arbitrary unions and finite intersections, and that both the empty set and the entire space are open

### Open sets in topological spaces

- A subset  $U$  of a topological space  $(X, \tau)$  is called open if it belongs to the topology  $\tau$
- Open sets are the fundamental building blocks of a topological space and are used to define various topological properties
- Examples of open sets in common topological spaces:
  - In the standard topology on  $\mathbb{R}$ , open intervals  $(a, b)$  and their unions are open sets
  - In the discrete topology on any set  $X$ , every subset of  $X$  is open

### Closed sets in topological spaces

- A subset  $F$  of a topological space  $(X, \tau)$  is called closed if its complement  $X \setminus F$  is open
- Closed sets are the complements of open sets and play a crucial role in the study of topological spaces
- Examples of closed sets in common topological spaces:
  - In the standard topology on  $\mathbb{R}$ , closed intervals  $[a, b]$  and their intersections are closed sets
  - In the discrete topology on any set  $X$ , every subset of  $X$  is closed

### Basis for a topology

- A basis for a topology on a set  $X$  is a collection  $\mathcal{B}$  of subsets of  $X$  such that every open set in the topology can be expressed as a union of elements from  $\mathcal{B}$

- A basis generates a unique topology on  $X$  by taking all possible unions of its elements
- Examples of bases for common topological spaces:
- The collection of open intervals  $(a, b)$  forms a basis for the standard topology on  $\mathbb{R}$
- The collection of all singleton sets  $\{x\}$  forms a basis for the discrete topology on any set  $X$

## Subbasis for a topology

- A subbasis for a topology on a set  $X$  is a collection  $S$  of subsets of  $X$  such that the collection of all finite intersections of elements from  $S$  forms a basis for the topology
- A subbasis generates a unique topology on  $X$  by first forming a basis from finite intersections of its elements and then taking all possible unions of the basis elements
- Example of a subbasis for a common topological space:
- The collection of all open intervals of the form  $(-\infty, a)$  and  $(b, \infty)$  forms a subbasis for the standard topology on  $\mathbb{R}$

## Hausdorff spaces

- A topological space  $(X, \tau)$  is called a Hausdorff space (or  $T_2$  space) if for any two distinct points  $x, y \in X$ , there exist disjoint open sets  $U, V \in \tau$  such that  $x \in U$  and  $y \in V$
- Hausdorff spaces are a fundamental class of topological spaces that ensure a certain level of separation between distinct points
- Many important topological spaces, such as metric spaces and manifolds, are Hausdorff spaces

## Continuous functions between topological spaces

- A function  $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$  between two topological spaces is called continuous if the preimage of every open set in  $Y$  is open in  $X$
- Continuous functions preserve the topological structure and are the primary objects of study in topology
- Composition of continuous functions is continuous, and the identity function is always continuous

## Homeomorphisms of topological spaces

- A function  $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$  between two topological spaces is called a homeomorphism if it is a continuous bijection with a continuous inverse
- Homeomorphic spaces are considered topologically equivalent, as they have the same topological properties
- Examples of homeomorphic spaces:
- The open interval  $(0, 1)$  and the real line  $\mathbb{R}$  are homeomorphic
- The unit circle  $S^1$  and the one-point compactification of  $\mathbb{R}$  are homeomorphic

## Topological properties preserved by homeomorphisms

- Homeomorphisms preserve many important topological properties, such as:

- Compactness
- Connectedness
- Hausdorff property
- Countability axioms (first and second countability, separability, Lindelöf property)
- If a topological property is preserved by homeomorphisms, it is called a topological invariant

## Subspaces of topological spaces

- A subset  $A$  of a topological space  $(X, \tau)$  can be endowed with a subspace topology, making it a topological space in its own right
- The subspace topology on  $A$  is defined as  $\tau_A = \{U \cap A : U \in \tau\}$ , consisting of the intersections of open sets in  $X$  with  $A$

## Subspace topology

- The subspace topology is the coarsest topology on  $A$  that makes the inclusion map  $i : A \hookrightarrow X$  continuous
- Many topological properties, such as compactness and connectedness, are inherited by subspaces from the ambient space
- Examples of subspaces with the subspace topology:
  - The closed interval  $[0, 1]$  as a subspace of  $\mathbb{R}$  with the standard topology
  - The unit circle  $S^1$  as a subspace of  $\mathbb{R}^2$  with the standard topology

## Closure and interior in subspaces

- The closure of a subset  $B$  of a subspace  $A$  in the subspace topology is given by  $\overline{B}^A = \overline{B} \cap A$ , where  $\overline{B}$  is the closure of  $B$  in the ambient space  $X$
- The interior of a subset  $B$  of a subspace  $A$  in the subspace topology is given by  $\text{int}_A(B) = \text{int}(B) \cap A$ , where  $\text{int}(B)$  is the interior of  $B$  in the ambient space  $X$
- These properties allow for the study of topological concepts within subspaces and their relationship to the ambient space

## Product topology

- The product topology on the Cartesian product of topological spaces is a natural way to define a topology that is compatible with the topologies of the individual spaces
- Given a family of topological spaces  $\{(X_i, \tau_i)\}_{i \in I}$ , the product topology on  $\prod_{i \in I} X_i$  is generated by the basis consisting of products of open sets from each factor space

## Finite product topology

- For a finite product of topological spaces  $X_1 \times X_2 \times \cdots \times X_n$ , the product topology is generated by the basis consisting of sets of the form  $U_1 \times U_2 \times \cdots \times U_n$ , where each  $U_i$  is open in  $X_i$

- The product topology on a finite product is the coarsest topology that makes all projection maps continuous
- Example of a finite product space:
- The torus  $T^2 = S^1 \times S^1$  with the product topology

## Infinite product topology

- For an infinite product of topological spaces  $\prod_{i \in I} X_i$ , the product topology is generated by the basis consisting of sets of the form  $\prod_{i \in I} U_i$ , where each  $U_i$  is open in  $X_i$  and  $U_i = X_i$  for all but finitely many indices  $i$
- The infinite product topology is also known as the Tychonoff topology
- Example of an infinite product space:
- The Hilbert cube  $[0, 1]^{\mathbb{N}}$  with the product topology

## Projection maps in product spaces

- For each index  $j \in I$ , the projection map  $\pi_j : \prod_{i \in I} X_i \rightarrow X_j$  is defined by  $\pi_j((x_i)_{i \in I}) = x_j$
- Projection maps are always continuous with respect to the product topology
- Projection maps play a crucial role in the study of product spaces and their relationship to the factor spaces

## Continuity in product spaces

- A function  $f : Y \rightarrow \prod_{i \in I} X_i$  is continuous with respect to the product topology if and only if each component function  $\pi_i \circ f : Y \rightarrow X_i$  is continuous
- This characterization simplifies the study of continuity in product spaces by reducing it to continuity in the factor spaces
- Example of a continuous function in a product space:
- The diagonal map  $\Delta : X \rightarrow X \times X$  defined by  $\Delta(x) = (x, x)$  is continuous with respect to the product topology

## Quotient topology

- The quotient topology is a way to define a topology on a set that is obtained by identifying or "gluing together" certain points in a given topological space
- Given a topological space  $(X, \tau_X)$  and an equivalence relation  $\sim$  on  $X$ , the quotient topology on the quotient set  $X/\sim$  is the finest topology that makes the quotient map  $q : X \rightarrow X/\sim$  continuous

## Quotient maps

- A surjective function  $q : (X, \tau_X) \rightarrow (Y, \tau_Y)$  is called a quotient map if the topology  $\tau_Y$  is the quotient topology induced by  $q$

- Quotient maps are characterized by the property that a subset  $U \subseteq Y$  is open in  $Y$  if and only if  $q^{-1}(U)$  is open in  $X$
- Quotient maps are a powerful tool for constructing new topological spaces from existing ones

## Continuity of quotient maps

- Quotient maps are always continuous by definition
- A function  $f: (Y, \tau_Y) \rightarrow (Z, \tau_Z)$  is continuous if and only if the composition  $f \circ q: (X, \tau_X) \rightarrow (Z, \tau_Z)$  is continuous, where  $q: (X, \tau_X) \rightarrow (Y, \tau_Y)$  is a quotient map
- This property allows for the study of continuity in quotient spaces by lifting functions to the original space

## Examples of quotient spaces

- The torus  $T^2$  can be obtained as a quotient space of the square  $[0, 1] \times [0, 1]$  by identifying opposite sides
- The projective plane  $\mathbb{RP}^2$  can be obtained as a quotient space of the sphere  $S^2$  by identifying antipodal points
- The Klein bottle can be obtained as a quotient space of the square  $[0, 1] \times [0, 1]$  by identifying opposite sides with a twist

## Connectedness in topological spaces

- Connectedness is a fundamental topological property that captures the idea of a space being in one piece
- A topological space  $(X, \tau)$  is connected if it cannot be expressed as the union of two disjoint non-empty open sets

## Connected spaces

- Examples of connected spaces:
  - The real line  $\mathbb{R}$  with the standard topology
  - The unit circle  $S^1$  with the subspace topology
  - Any interval  $[a, b]$  with the subspace topology
- Examples of disconnected spaces:
  - The set  $\{0, 1\}$  with the discrete topology
  - The set  $\mathbb{Q}$  of rational numbers with the subspace topology inherited from  $\mathbb{R}$

## Path-connected spaces

- A topological space  $(X, \tau)$  is path-connected if for any two points  $x, y \in X$ , there exists a continuous function  $\gamma: [0, 1] \rightarrow X$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$
- Path-connectedness is a stronger property than connectedness, as every path-connected space is connected, but the converse is not true

- Example of a space that is connected but not path-connected:
- The topologist's sine curve  $\{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$

## Components and path components

- The (connected) components of a topological space  $(X, \tau)$  are the maximal connected subsets of  $X$
- The path components of a topological space  $(X, \tau)$  are the maximal path-connected subsets of  $X$
- Every path component is contained in a unique component, but a component may contain multiple path components

## Local connectedness

- A topological space  $(X, \tau)$  is locally connected if for every point  $x \in X$  and every open set  $U$  containing  $x$ , there exists a connected open set  $V$  such that  $x \in V \subseteq U$
- Local connectedness is a property that relates the global notion of connectedness to the local structure of the space
- Examples of locally connected spaces:
  - The real line  $\mathbb{R}$  with the standard topology
  - The unit circle  $S^1$  with the subspace topology

## Compactness in topological spaces

- Compactness is a fundamental topological property that generalizes the notion of closed and bounded subsets in Euclidean spaces
- A topological space  $(X, \tau)$  is compact if every open cover of  $X$  has a finite subcover

## Open cover of a topological space

- An open cover of a topological space  $(X, \tau)$  is a collection  $\{U_i\}_{i \in I}$  of open sets such that  $X = \bigcup_{i \in I} U_i$
- A subcover of an open cover  $\{U_i\}_{i \in I}$  is a subcollection  $\{U_j\}_{j \in J}$ , where  $J \subseteq I$ , that still covers  $X$
- Compactness can be characterized by the property that every open cover has a finite subcover

## Compact spaces

- Examples of compact spaces:
  - Any finite topological space
  - The closed interval  $[0, 1]$  with the subspace topology
  - The unit circle  $S^1$  with the subspace topology
- Examples of non-compact spaces:
  - The real line  $\mathbb{R}$  with the standard topology
  - The open interval  $(0, 1)$  with the subspace topology

## Compact subspaces

- A subset  $K$  of a topological space  $(X, \tau)$  is compact if it is compact as a subspace with the subspace topology
- Closed subsets of compact spaces are compact (Closed subset property)
- Compact subsets of Hausdorff spaces are closed (Closed in Hausdorff property)

## Continuity and compactness

- The continuous image of a compact space is compact (Continuous image property)
- If  $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$  is a continuous function and  $K \subseteq X$  is compact, then  $f(K)$  is compact in  $Y$
- This property is useful for proving the compactness of spaces constructed using continuous functions

## Tychonoff's theorem for product spaces

- Tychonoff's theorem states that the product of any collection of compact spaces is compact with respect to the product topology
- This theorem is a powerful tool for constructing compact spaces and proving the compactness of certain function spaces
- Example of an application of Tychonoff's theorem:
- The Hilbert cube  $[0, 1]^{\mathbb{N}}$  is compact with respect to the product topology

## Separation axioms

- Separation axioms are properties that describe how well points and closed sets can be separated by open sets in a topological space
- The separation axioms form a hierarchy, with each axiom implying the previous ones

## $T_0, T_1, T_2$ (Hausdorff) spaces

- A topological space  $(X, \tau)$  is  $T_0$  (Kolmogorov) if for any two distinct points  $x, y \in X$ , there exists an open set containing one of the points but not the other
- A topological space  $(X, \tau)$  is  $T_1$  (Fréchet) if for any two distinct points  $x, y \in X$ , there exist open sets  $U, V$  such that  $x \in U$  and  $y \in V$  and  $U \cap V = \emptyset$

## 1.2 Smooth manifolds

Smooth manifolds are the backbone of differential geometry, extending the concept of curves and surfaces to higher dimensions. They provide a framework for applying calculus and analysis to complex geometric spaces, enabling the study of intricate mathematical structures.

In this chapter, we explore the definition and properties of smooth manifolds, including their topological foundations and smooth structures. We'll examine various examples, from simple Euclidean spaces to more complex objects like spheres and matrix Lie groups, to build intuition about these fundamental mathematical entities.



## Definition of smooth manifolds

- Smooth manifolds are central objects of study in differential geometry, generalizing curves and surfaces to higher dimensions
- A smooth manifold is a topological space equipped with a smooth structure, allowing for calculus and analysis to be performed on the manifold
- Smooth manifolds provide a framework for studying geometric and topological properties of spaces using tools from calculus and linear algebra

## Topological spaces as manifolds

- A topological space is a set equipped with a collection of open sets satisfying certain axioms
- For a topological space to be a manifold, it must be locally Euclidean, meaning each point has a neighborhood homeomorphic to an open subset of Euclidean space
- Manifolds are required to be Hausdorff and second-countable, ensuring they have well-behaved topological properties

## Smooth structures on manifolds

- A smooth structure on a manifold is a collection of charts that are smoothly compatible with each other
- Charts are homeomorphisms from open subsets of the manifold to open subsets of Euclidean space
- The smooth structure allows for the definition of smooth functions and the use of calculus on the manifold

## Charts and atlases

- A chart on a manifold is a pair  $(U, \varphi)$ , where  $U$  is an open subset of the manifold and  $\varphi : U \rightarrow V$  is a homeomorphism onto an open subset  $V$  of Euclidean space
- An atlas is a collection of charts that cover the entire manifold
- The charts in an atlas must be smoothly compatible, meaning the transition maps between overlapping charts are smooth functions

## Transition maps between charts

- Given two overlapping charts  $(U, \varphi)$  and  $(V, \psi)$ , the transition map between them is the composition  $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$
- For a smooth manifold, the transition maps between charts must be smooth functions
- The smoothness of transition maps ensures that the notion of smoothness is well-defined and independent of the choice of charts

## Examples of smooth manifolds

- Smooth manifolds are abundant in mathematics and physics, with many important spaces naturally carrying a smooth structure
- Studying examples of smooth manifolds helps develop intuition and understanding of their properties and behavior

## Euclidean spaces as manifolds

- Euclidean spaces  $\mathbb{R}^n$  are the simplest examples of smooth manifolds
- The standard topology and smooth structure on  $\mathbb{R}^n$  are given by the identity chart, making it a smooth manifold of dimension  $n$
- Euclidean spaces serve as local models for general smooth manifolds through the use of charts

## Spheres and tori

- The  $n$ -sphere  $S^n$  is the set of points in  $\mathbb{R}^{n+1}$  at a fixed distance from the origin, inheriting a smooth structure as a submanifold of Euclidean space
- Tori, such as the 2-torus  $T^2$ , are obtained as products of circles, carrying a natural smooth structure
- Spheres and tori are important examples of compact smooth manifolds with non-trivial topology

## Matrix Lie groups

- Matrix Lie groups, such as the general linear group  $GL(n, \mathbb{R})$  and the special orthogonal group  $SO(n)$ , are smooth manifolds
- The smooth structure on matrix Lie groups is induced by the smooth structure on the space of matrices
- Matrix Lie groups play a central role in the study of symmetries and transformations in mathematics and physics

## Grassmann manifolds

- The Grassmann manifold  $Gr(k, n)$  is the set of  $k$ -dimensional linear subspaces of  $\mathbb{R}^n$ , carrying a natural smooth structure
- Grassmann manifolds are important in algebraic geometry and optimization, representing spaces of linear constraints or subspaces
- The projective space  $\mathbb{R}P^n$  is a special case of a Grassmann manifold, representing lines through the origin in  $\mathbb{R}^{n+1}$

## Smooth maps between manifolds

- Smooth maps between manifolds are the morphisms in the category of smooth manifolds, preserving the smooth structure
- Studying smooth maps allows for the comparison and relation of different manifolds, as well as the transfer of geometric and topological properties

## Definition of smooth maps

- A map  $f: M \rightarrow N$  between smooth manifolds is smooth if for every point  $p \in M$ , there exist charts  $(U, \varphi)$  around  $p$  and  $(V, \psi)$  around  $f(p)$  such that the composition  $\psi \circ f \circ \varphi^{-1}$  is a smooth function between open subsets of Euclidean spaces
- Smoothness of maps is a local property, depending only on the behavior in small neighborhoods of points

- Smooth maps are continuous and infinitely differentiable, allowing for the use of calculus techniques

## Diffeomorphisms and embeddings

- A diffeomorphism is a smooth map with a smooth inverse, providing an isomorphism between smooth manifolds
- Diffeomorphic manifolds have the same smooth structure and can be considered equivalent from the perspective of differential geometry
- An embedding is a smooth map that is a diffeomorphism onto its image, realizing one manifold as a submanifold of another

## Immersion and submersions

- An immersion is a smooth map with everywhere injective differential, locally embedding the domain manifold into the codomain
- A submersion is a smooth map with everywhere surjective differential, locally projecting the domain manifold onto the codomain
- Immersions and submersions are important in the study of foliations and fiber bundles on manifolds

## Inverse function theorem

- The inverse function theorem states that a smooth map with everywhere invertible differential is locally a diffeomorphism
- The theorem provides a powerful tool for constructing local inverses and charts on manifolds
- Applications of the inverse function theorem include the implicit function theorem and the constant rank theorem

## Tangent spaces and vectors

- Tangent spaces and vectors are the fundamental linear approximations to a manifold at each point, encoding infinitesimal information about the smooth structure
- The tangent space at a point can be thought of as the space of velocity vectors of curves passing through that point

## Tangent spaces at points

- The tangent space  $T_p M$  at a point  $p$  on a manifold  $M$  is a vector space of the same dimension as  $M$
- Tangent spaces can be defined equivalently as equivalence classes of curves through  $p$  or as derivations on the algebra of smooth functions at  $p$
- The disjoint union of all tangent spaces forms the tangent bundle  $TM$  of the manifold

## Tangent vectors as derivations

- A tangent vector at a point  $p$  can be defined as a derivation on the algebra of germs of smooth functions at  $p$
- Derivations are linear maps satisfying the Leibniz rule, capturing the idea of directional derivatives

- The derivation perspective provides a coordinate-free approach to tangent vectors and is useful in algebraic geometry

## Tangent bundles of manifolds

- The tangent bundle  $TM$  of a manifold  $M$  is the disjoint union of all tangent spaces, carrying a natural smooth structure
- The tangent bundle is a vector bundle over  $M$ , with each fiber being the tangent space at the corresponding point
- Sections of the tangent bundle are vector fields on the manifold, assigning a tangent vector to each point

## Vector fields on manifolds

- A vector field on a manifold  $M$  is a smooth assignment of a tangent vector to each point of  $M$
- Vector fields can be viewed as derivations on the algebra of smooth functions on  $M$ , generalizing the notion of directional derivatives
- The space of vector fields on a manifold forms a Lie algebra under the Lie bracket operation, capturing the non-commutativity of flows

## Cotangent spaces and forms

- Cotangent spaces and forms are the dual notions to tangent spaces and vectors, providing a way to measure and integrate on manifolds
- Differential forms are antisymmetric multilinear functionals on tangent spaces, generalizing the concept of integration

## Dual spaces and covectors

- The dual space  $T_p^*M$  of the tangent space at a point  $p$  is called the cotangent space, consisting of linear functionals on tangent vectors
- Elements of the cotangent space are called covectors or 1-forms, generalizing the notion of differentials of functions
- The disjoint union of all cotangent spaces forms the cotangent bundle  $T^*M$  of the manifold

## Cotangent bundles of manifolds

- The cotangent bundle  $T^*M$  of a manifold  $M$  is the vector bundle whose fiber at each point is the cotangent space at that point
- Sections of the cotangent bundle are differential 1-forms on the manifold, assigning a covector to each point
- The cotangent bundle carries a natural symplectic structure, making it a fundamental object in symplectic geometry

## Differential forms on manifolds

- A differential  $k$ -form on a manifold  $M$  is a smooth assignment of an alternating multilinear functional on the tangent space at each point
- Differential forms provide a way to integrate over submanifolds of  $M$ , generalizing the notion of integration of functions
- The exterior derivative  $d$  maps  $k$ -forms to  $(k + 1)$ -forms, satisfying  $d^2 = 0$  and leading to the de Rham cohomology of the manifold

## Exterior algebra of forms

- The exterior algebra of forms on a manifold  $M$  is the graded algebra generated by differential forms under the wedge product
- The wedge product is an antisymmetric multiplication operation, satisfying graded commutativity and associativity
- The exterior algebra provides a natural setting for the study of differential forms and their properties, such as the Hodge star operator and the Lie derivative

## Submanifolds and embeddings

- Submanifolds are manifolds that are embedded or immersed inside other manifolds, inheriting a smooth structure from the ambient space
- The study of submanifolds is important in understanding the local and global geometry of manifolds, as well as in applications such as constrained optimization

## Definition of submanifolds

- A subset  $S$  of a manifold  $M$  is a submanifold if it is a manifold in its own right and the inclusion map  $i : S \rightarrow M$  is an immersion
- The dimension of a submanifold is always less than or equal to the dimension of the ambient manifold
- Examples of submanifolds include open subsets of manifolds, level sets of submersions, and graphs of smooth functions

## Embedded vs immersed submanifolds

- An embedded submanifold is a submanifold for which the inclusion map is a homeomorphism onto its image
- An immersed submanifold is a submanifold for which the inclusion map is only locally a homeomorphism onto its image
- Embedded submanifolds are more rigid and have nicer topological properties, while immersed submanifolds allow for self-intersections and more flexible geometry

## Normal bundles of submanifolds

- The normal bundle of a submanifold  $S$  in a Riemannian manifold  $M$  is the orthogonal complement of the tangent bundle of  $S$  in the restricted tangent bundle of  $M$

- The normal bundle encodes the extrinsic geometry of the submanifold, measuring how it sits inside the ambient manifold
- Sections of the normal bundle are normal vector fields, which are important in the study of the second fundamental form and curvature of submanifolds

## Tubular neighborhood theorem

- The tubular neighborhood theorem states that every embedded submanifold of a manifold has a neighborhood that is diffeomorphic to a neighborhood of the zero section in its normal bundle
- The theorem provides a local model for the geometry of the submanifold and its relationship to the ambient manifold
- Tubular neighborhoods are useful in the study of the topology of submanifolds, as well as in the construction of collars and isotopies

## Smooth manifolds with additional structure

- Smooth manifolds can be equipped with additional geometric structures, leading to rich theories and applications
- Examples of such structures include Riemannian metrics, symplectic forms, complex structures, and Lie group actions

## Riemannian manifolds and metrics

- A Riemannian manifold is a smooth manifold equipped with a Riemannian metric, which is a smooth assignment of an inner product to each tangent space
- Riemannian metrics allow for the measurement of lengths, angles, and volumes on the manifold, as well as the definition of geodesics and curvature
- The study of Riemannian manifolds is central to differential geometry and has applications in physics, such as general relativity and mechanics

## Symplectic manifolds and forms

- A symplectic manifold is a smooth manifold equipped with a symplectic form, which is a closed, non-degenerate 2-form
- Symplectic forms provide a natural setting for Hamiltonian mechanics and the study of conservative dynamical systems
- Symplectic geometry is a rich and active area of research, with connections to algebraic geometry, topology, and mathematical physics

## Complex manifolds and charts

- A complex manifold is a smooth manifold equipped with an atlas of charts taking values in open subsets of complex Euclidean space, with holomorphic transition maps
- Complex manifolds are the natural setting for the study of holomorphic functions and complex geometry
- Examples of complex manifolds include Riemann surfaces, complex projective spaces, and complex Lie groups

## Lie groups as smooth manifolds

- A Lie group is a smooth manifold that is also a group, with smooth group operations of multiplication and inversion
- Lie groups provide a natural framework for studying continuous symmetries and transformations in mathematics and physics
- Examples of Lie groups include matrix groups (e.g.,  $GL(n, \mathbb{R})$ ,  $O(n)$ ,  $U(n)$ ), as well as more abstract groups such as the Heisenberg group and the Poincaré group

### 1.3 Coordinate charts and atlases

Coordinate charts and atlases are essential tools in differential geometry for describing manifolds. They provide a way to represent complex geometric structures using familiar coordinate systems, allowing for local analysis and calculations.

These concepts form the foundation for studying smooth manifolds. By using coordinate charts and atlases, we can define smoothness, perform computations, and analyze the geometric properties of manifolds in a systematic way.

#### Coordinate charts

- Coordinate charts are a fundamental concept in differential geometry that allow us to describe a manifold locally using a set of coordinates
- They provide a way to parametrize a portion of a manifold using a homeomorphism between an open subset of the manifold and an open subset of Euclidean space
- Coordinate charts are essential for performing calculations and analyzing the geometric properties of manifolds

#### Definition of coordinate chart

- A coordinate chart on a topological manifold  $M$  is a pair  $(U, \varphi)$ , where  $U$  is an open subset of  $M$  and  $\varphi : U \rightarrow \varphi(U) \subset \mathbb{R}^n$  is a homeomorphism
- The map  $\varphi$  assigns coordinates to each point in  $U$ , providing a local coordinate system
- The open set  $U$  is called the chart domain, and the homeomorphism  $\varphi$  is called the chart map or coordinate map

#### Homeomorphisms in coordinate charts

- The chart map  $\varphi$  in a coordinate chart is required to be a homeomorphism, which means it is a continuous bijection with a continuous inverse
- Homeomorphisms preserve the topological properties of the manifold, such as continuity and connectedness
- The existence of a homeomorphism between the chart domain and an open subset of Euclidean space ensures that the local structure of the manifold is preserved under the coordinate representation

## Transition maps between charts

- When two coordinate charts  $(U, \varphi)$  and  $(V, \psi)$  overlap, i.e.,  $U \cap V \neq \emptyset$ , we can define a transition map between them
- The transition map is given by the composition  $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$ , which relates the coordinates of points in the overlap region
- Transition maps are essential for ensuring consistency and compatibility between different coordinate charts on a manifold

## Atlases

- An atlas is a collection of coordinate charts that cover the entire manifold
- Atlases provide a way to describe the global structure of a manifold by patching together local coordinate descriptions
- The concept of an atlas is central to the study of differentiable manifolds and is used to define smooth structures on manifolds

## Definition of atlas

- An atlas on a topological manifold  $M$  is a collection of coordinate charts  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$  such that: 1. The chart domains  $U_\alpha$  cover the entire manifold, i.e.,  $\bigcup_{\alpha \in A} U_\alpha = M$  2. Any two charts in the atlas are compatible, meaning that the transition maps between overlapping charts are smooth (infinitely differentiable)
- An atlas provides a complete coordinate description of the manifold, allowing for the study of its geometric and topological properties

## Compatibility of charts in atlas

- Compatibility of charts in an atlas ensures that the transition maps between overlapping charts are smooth
- If two charts  $(U, \varphi)$  and  $(V, \psi)$  overlap, the transition map  $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$  must be a smooth function
- Compatibility allows for the consistent definition of smooth functions, vector fields, and other geometric objects on the manifold
- Charts in an atlas are often referred to as "smoothly compatible" or " $C^\infty$ -compatible" to emphasize the smoothness requirement

## Maximal atlas

- A maximal atlas on a manifold  $M$  is an atlas that contains all possible charts that are compatible with the charts already in the atlas
- Given an atlas  $\mathcal{A}$  on  $M$ , the maximal atlas containing  $\mathcal{A}$  is the union of all charts that are compatible with every chart in  $\mathcal{A}$
- The maximal atlas is unique and provides the most comprehensive coordinate description of the manifold



- The existence of a maximal atlas is guaranteed by the axiom of choice and is used in the definition of smooth manifolds

## Smooth structures

- A smooth structure on a manifold is a maximal atlas that defines the notion of smoothness for functions and maps on the manifold
- Smooth structures allow for the study of differential geometric properties, such as tangent spaces, vector fields, and differential forms
- The choice of a smooth structure on a manifold is not unique, and different smooth structures can give rise to non-diffeomorphic smooth manifolds

## Smooth atlas

- A smooth atlas on a manifold  $M$  is an atlas in which all the transition maps between overlapping charts are smooth (infinitely differentiable)
- A smooth atlas defines a smooth structure on the manifold, allowing for the consistent definition of smooth functions and maps
- The existence of a smooth atlas is a prerequisite for studying the differential geometric properties of a manifold

## Smooth manifolds

- A smooth manifold is a pair  $(M, \mathcal{A})$ , where  $M$  is a topological manifold and  $\mathcal{A}$  is a smooth atlas on  $M$
- Smooth manifolds are the primary objects of study in differential geometry and provide a framework for analyzing geometric properties using calculus and linear algebra
- Examples of smooth manifolds include Euclidean spaces, spheres, tori, and Lie groups

## Smooth functions and maps

- A function  $f: M \rightarrow \mathbb{R}$  on a smooth manifold  $M$  is called smooth if for every chart  $(U, \varphi)$  in the smooth atlas, the composition  $f \circ \varphi^{-1}: \varphi(U) \rightarrow \mathbb{R}$  is a smooth function on the open subset  $\varphi(U) \subset \mathbb{R}^n$
- A map  $F: M \rightarrow N$  between two smooth manifolds  $M$  and  $N$  is called smooth if for every pair of charts  $(U, \varphi)$  on  $M$  and  $(V, \psi)$  on  $N$  such that  $F(U) \subset V$ , the composition  $\psi \circ F \circ \varphi^{-1}: \varphi(U) \rightarrow \psi(V)$  is a smooth function
- Smooth functions and maps are the building blocks for studying the differential geometric properties of manifolds, such as tangent spaces, vector fields, and differential forms

## Coordinate representations

- Coordinate representations allow us to express geometric objects and properties of a manifold in terms of local coordinates
- By using coordinate charts, we can represent points, curves, functions, and other geometric objects as tuples of real numbers or functions of real variables
- Coordinate representations simplify calculations and provide a way to apply analytical methods to the study of manifolds

## Coordinate representation of points

- Given a coordinate chart  $(U, \varphi)$  on a manifold  $M$  and a point  $p \in U$ , the coordinate representation of  $p$  is the tuple  $\varphi(p) = (x^1(p), \dots, x^n(p)) \in \mathbb{R}^n$
- The components  $x^i(p)$  are called the coordinates of  $p$  with respect to the chart  $(U, \varphi)$
- Coordinate representations of points allow us to identify points on the manifold with tuples of real numbers, enabling the use of analytical methods

## Coordinate representation of curves

- A curve on a manifold  $M$  is a smooth map  $\gamma : I \rightarrow M$ , where  $I \subset \mathbb{R}$  is an open interval
- Given a coordinate chart  $(U, \varphi)$  on  $M$ , the coordinate representation of a curve  $\gamma$  in  $U$  is the composition  $\varphi \circ \gamma : I \rightarrow \varphi(U) \subset \mathbb{R}^n$
- The coordinate representation of a curve is a vector-valued function of a real variable, allowing for the study of its properties using calculus

## Coordinate representation of functions

- A function  $f : M \rightarrow \mathbb{R}$  on a manifold  $M$  can be expressed in local coordinates using a coordinate chart  $(U, \varphi)$
- The coordinate representation of  $f$  in  $U$  is the composition  $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}$ , which is a function of  $n$  real variables, where  $n$  is the dimension of the manifold
- Coordinate representations of functions allow for the application of analytical methods, such as differentiation and integration, to the study of functions on manifolds

## Computations in coordinates

- Computations in coordinates involve expressing geometric objects and operations in terms of local coordinate representations
- By using coordinate charts, we can perform calculations involving vector fields, differential forms, and other geometric quantities using the tools of linear algebra and calculus
- Computations in coordinates are essential for solving problems and deriving properties of manifolds and their associated geometric structures

## Vector fields in coordinates

- A vector field on a manifold  $M$  is a smooth assignment of a tangent vector to each point of the manifold
- In local coordinates, a vector field  $X$  can be expressed as a linear combination of the coordinate basis vectors:  $X = X^i \frac{\partial}{\partial x^i}$
- The components  $X^i$  are functions of the local coordinates and determine the behavior of the vector field in the given coordinate chart
- Coordinate representations of vector fields allow for the study of their properties, such as divergence, curl, and Lie derivatives

## Differential forms in coordinates

- A differential form on a manifold  $M$  is a smooth assignment of an alternating multilinear map on the tangent spaces of  $M$
- In local coordinates, a differential  $k$ -form  $\omega$  can be expressed as a linear combination of the coordinate basis forms:  $\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$
- The components  $\omega_{i_1 \dots i_k}$  are functions of the local coordinates and determine the behavior of the differential form in the given coordinate chart
- Coordinate representations of differential forms allow for the study of their properties, such as exterior derivatives, wedge products, and integration

## Christoffel symbols in coordinates

- Christoffel symbols are the components of the Levi-Civita connection, which is a fundamental object in Riemannian geometry
- In local coordinates, the Christoffel symbols  $\Gamma_{ij}^k$  are defined in terms of the metric tensor  $g_{ij}$  and its partial derivatives:  $\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$
- Christoffel symbols play a crucial role in the computation of geodesics, parallel transport, and curvature tensors in Riemannian manifolds
- Coordinate representations of Christoffel symbols allow for the study of the geometric properties of Riemannian manifolds using the tools of tensor calculus

## Coordinate transformations

- Coordinate transformations describe how geometric objects and properties change when transitioning between different coordinate charts on a manifold
- Understanding coordinate transformations is essential for ensuring the consistency and invariance of geometric concepts across different coordinate representations
- Coordinate transformations involve the Jacobian matrix, change of variables formulas, and the transformation laws for tensors and other geometric objects

## Jacobian matrix

- The Jacobian matrix of a coordinate transformation  $\varphi : U \rightarrow V$  between two coordinate charts  $(U, x^i)$  and  $(V, y^j)$  is the matrix of partial derivatives:  $J_{ij}^\varphi = \frac{\partial y^j}{\partial x^i}$
- The Jacobian matrix encodes the local behavior of the coordinate transformation and is used to transform vectors, differential forms, and other geometric objects between coordinate charts
- The determinant of the Jacobian matrix, called the Jacobian determinant, measures the local change in volume under the coordinate transformation

## Change of variables

- Change of variables formulas describe how integrals transform under a coordinate transformation
- Given a coordinate transformation  $\varphi : U \rightarrow V$  and a function  $f : V \rightarrow \mathbb{R}$ , the change of variables formula for integrals states:  $\int_V f(y) dy = \int_U f(\varphi(x)) |\det J^\varphi(x)| dx$

- Change of variables formulas are essential for computing integrals over manifolds and for deriving conservation laws and other integral identities

## Transformation of metric tensor

- The metric tensor  $\mathcal{G}$  on a Riemannian manifold is a symmetric, positive-definite tensor field that defines the inner product on tangent spaces
- Under a coordinate transformation  $\varphi : U \rightarrow V$ , the components of the metric tensor transform according to the rule:  $\mathcal{G}'_{ij}(y) = \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} \mathcal{G}_{kl}(x)$
- The transformation law for the metric tensor ensures that the length of curves, angles between vectors, and other geometric quantities are preserved under coordinate transformations
- Understanding the transformation properties of the metric tensor is crucial for studying the intrinsic geometry of Riemannian manifolds

## Atlas constructions

- Constructing atlases on manifolds is an important task in differential geometry, as it provides a way to define smooth structures and study the geometric properties of the manifold
- There are several standard techniques for constructing atlases, including stereographic projection, the exponential map, and coordinate patches on spheres
- These construction methods are often used to define smooth structures on common manifolds, such as spheres, projective spaces, and Lie groups

## Stereographic projection

- Stereographic projection is a method for constructing a coordinate chart on the sphere  $S^n$  by projecting points from the sphere onto a tangent plane
- The stereographic projection map  $\varphi : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  is defined by projecting a point  $P$  on the sphere from the north pole  $N$  onto the equatorial plane
- Stereographic projection provides a smooth coordinate chart on the sphere that covers all but one point (the north pole)
- The stereographic projection atlas consists of two charts: the stereographic projection from the north pole and the stereographic projection from the south pole, which together cover the entire sphere

## Exponential map

- The exponential map is a method for constructing coordinate charts on a Riemannian manifold using geodesics
- Given a point  $P$  on a Riemannian manifold  $M$ , the exponential map  $\exp_P : T_P M \rightarrow M$  maps a tangent vector  $V$  at  $P$  to the point on the manifold obtained by following the geodesic starting at  $P$  with initial velocity  $V$  for a unit time
- The exponential map provides a local diffeomorphism between a neighborhood of the origin in the tangent space  $T_P M$  and a neighborhood of  $P$  on the manifold
- Exponential maps are used to construct local coordinate charts, called normal coordinates, which are particularly useful for studying the geometry of the manifold near a given point

## Coordinate patches on spheres

- Coordinate patches on spheres are a way to construct atlases on spheres of arbitrary dimension using a combination of stereographic projections and rotations
- The standard coordinate patches on the unit sphere  $S^n$  are defined using stereographic projections from the north and south poles, along with rotations that move the poles to other points on the sphere
- For example, on the 2-sphere  $S^2$ , the standard coordinate patches consist of six charts: the stereographic projections from the north and south poles, and the stereographic projections from the four points obtained by rotating the poles by 90 degrees around the coordinate axes
- These coordinate patches provide a smooth atlas on the sphere and are often used to study the geometry and topology of spherical manifolds

## 1.4 Transition maps and compatibility

Transition maps are essential in differential geometry, allowing us to describe manifolds using multiple coordinate charts. They define the smooth structure of a manifold and ensure compatibility between different coordinate systems.

These maps relate overlapping coordinate charts, playing a crucial role in defining smooth functions and geometric objects. Understanding transition maps is key to grasping how manifolds are described and analyzed in differential geometry.

### Transition maps

- Transition maps are a fundamental concept in differential geometry that allow for the description of a manifold using multiple coordinate charts
- They play a crucial role in defining the smooth structure of a manifold and ensuring compatibility between different coordinate systems
- Transition maps are closely related to the notion of coordinate charts, which are local homeomorphisms from open subsets of the manifold to open subsets of Euclidean space

### Definition of transition maps

- A transition map is a function that relates two overlapping coordinate charts on a manifold
- Given two coordinate charts  $(U, \varphi)$  and  $(V, \psi)$  with overlapping domains, the transition map  $\varphi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \varphi(U \cap V)$  maps coordinates from one chart to the other
- Transition maps are typically denoted as  $\varphi_{ij} : \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j)$ , where  $U_i$  and  $U_j$  are the domains of the coordinate charts  $\varphi_i$  and  $\varphi_j$ , respectively

### Role in differential geometry

- Transition maps are essential for defining the differentiable structure of a manifold
- They ensure that the manifold has a well-defined notion of smoothness and that calculus can be performed consistently across different coordinate charts
- Transition maps allow for the definition of smooth functions, vector fields, and other geometric objects on the manifold

## Relationship to coordinate charts

- Coordinate charts are local homeomorphisms that map open subsets of the manifold to open subsets of Euclidean space
- Transition maps relate the coordinates of overlapping coordinate charts, enabling a consistent description of the manifold
- The compatibility of transition maps ensures that the coordinate charts form a coherent atlas for the manifold

## Properties of transition maps

- Transition maps possess several important properties that are crucial for the well-definedness and consistency of the manifold structure
- These properties include smoothness requirements, invertibility, and the behavior of the Jacobian matrix and its determinant
- Satisfying these properties ensures that the transition maps preserve the differentiable structure of the manifold

## Smoothness requirements

- Transition maps are required to be smooth functions, typically of class  $C^\infty$  (infinitely differentiable)
- Smoothness ensures that the transition maps preserve the differentiable structure of the manifold and allow for the consistent definition of smooth functions and other geometric objects
- The smoothness of transition maps is essential for performing calculus on the manifold and studying its geometric properties

## Invertibility and inverse functions

- Transition maps are required to be invertible, meaning that for each transition map  $\varphi_{ij}$ , there exists an inverse function  $\varphi_{ji} = \varphi_{ij}^{-1}$
- The invertibility of transition maps ensures that the coordinate transformations between overlapping charts are bijective and well-defined
- The inverse functions of transition maps are also required to be smooth, satisfying the same smoothness requirements as the original transition maps

## Jacobian matrix and its determinant

- The Jacobian matrix of a transition map  $\varphi_{ij}$  is the matrix of partial derivatives of the component functions of  $\varphi_{ij}$
- The Jacobian matrix encodes important information about the local behavior of the transition map, such as how it stretches or rotates vectors
- The determinant of the Jacobian matrix, known as the Jacobian determinant, must be non-zero at every point in the domain of the transition map
- A non-zero Jacobian determinant ensures that the transition map is locally invertible and preserves the orientation of the manifold

## Compatibility of transition maps

- Compatibility is a crucial property that ensures the consistency and well-definedness of the manifold structure
- Transition maps are said to be compatible if they satisfy certain conditions when composed with each other
- Compatibility guarantees that the coordinate charts form a coherent atlas for the manifold and that the differentiable structure is well-defined

## Definition of compatibility

- Transition maps  $\varphi_{ij}$  and  $\varphi_{jk}$  are said to be compatible if their composition  $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$  is also a transition map
- Compatibility ensures that the coordinate transformations between overlapping charts are consistent and do not depend on the choice of intermediate charts
- The compatibility condition can be expressed as  $\varphi_{ik}(x) = (\varphi_{ij} \circ \varphi_{jk})(x)$  for all  $x$  in the domain of  $\varphi_{ik}$

## Importance for well-defined manifolds

- Compatibility of transition maps is essential for the well-definedness of the manifold structure
- It ensures that the coordinate charts form a coherent atlas and that the differentiable structure is consistently defined across the manifold
- Without compatibility, the notion of smoothness and the ability to perform calculus on the manifold would be ambiguous and depend on the choice of coordinate charts

## Cocycle condition for compatibility

- The cocycle condition is a concise way of expressing the compatibility of transition maps
- It states that for any three overlapping coordinate charts  $(U_i, \varphi_i)$ ,  $(U_j, \varphi_j)$ , and  $(U_k, \varphi_k)$ , the composition of transition maps satisfies  $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \text{identity}$  on the triple overlap  $U_i \cap U_j \cap U_k$
- The cocycle condition ensures that the composition of transition maps around any closed loop in the manifold is the identity map, guaranteeing the consistency of the coordinate transformations

## Transition functions and atlas

- Transition functions and atlases are closely related concepts in the study of manifolds
- An atlas is a collection of coordinate charts that cover the entire manifold, and transition functions are the maps that relate these charts
- The properties of transition functions and the compatibility of the charts in an atlas determine the differentiable structure of the manifold

## Definition of an atlas

- An atlas on a manifold  $M$  is a collection of coordinate charts  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$  that satisfy the following conditions: 1. The domains  $U_\alpha$  cover the entire manifold:  $\bigcup_{\alpha \in A} U_\alpha = M$  2. The transition functions between overlapping charts are smooth (typically  $C^\infty$ )

- An atlas provides a complete description of the manifold using local coordinate systems

## Maximal atlas and its uniqueness

- A maximal atlas is an atlas that contains all possible compatible coordinate charts
- Given an atlas, a maximal atlas can be constructed by adding all charts that are compatible with the existing ones
- The maximal atlas is unique, meaning that any two maximal atlases on a manifold are equivalent and define the same differentiable structure

## Constructing an atlas using transition maps

- An atlas can be constructed using a collection of coordinate charts and their associated transition maps
- Starting with a set of overlapping coordinate charts, one can verify the compatibility of the transition maps between them
- If the transition maps satisfy the compatibility conditions (smoothness and cocycle condition), the collection of charts forms a valid atlas for the manifold
- Additional compatible charts can be added to the atlas to obtain a more complete description of the manifold

## Applications of transition maps

- Transition maps have numerous applications in differential geometry and related fields
- They play a crucial role in various geometric constructions and computations on manifolds
- Some important applications of transition maps include change of coordinates, pullback and pushforward of tensors, and induced maps on tangent and cotangent spaces

## Change of coordinates

- Transition maps allow for the change of coordinates between different charts on a manifold
- Given a point  $P$  on the manifold and two overlapping charts  $(U, \varphi)$  and  $(V, \psi)$  containing  $P$ , the transition map  $\varphi \circ \psi^{-1}$  relates the coordinates of  $P$  in the two charts
- Change of coordinates is essential for expressing geometric objects (functions, vectors, tensors) in different coordinate systems and studying their properties

## Pullback and pushforward of tensors

- Transition maps enable the pullback and pushforward of tensors between different coordinate charts
- The pullback of a tensor field from one chart to another is obtained by composing the tensor field with the inverse of the transition map
- The pushforward of a tensor field is obtained by composing the tensor field with the transition map itself
- Pullback and pushforward operations allow for the consistent definition and manipulation of tensor fields on the manifold



## Induced maps on tangent and cotangent spaces

- Transition maps induce linear maps between the tangent and cotangent spaces of the manifold at different points
- The differential of a transition map at a point  $P$  is a linear map between the tangent spaces at  $P$  in the two overlapping charts
- The dual of the differential of a transition map defines a linear map between the cotangent spaces at  $P$
- These induced maps are crucial for defining and studying vector fields, differential forms, and other geometric structures on the manifold

## Examples of transition maps

- Concrete examples of transition maps help illustrate their properties and behavior on specific manifolds
- Some common examples include transition maps on the sphere, torus, and in polar coordinates
- These examples demonstrate how transition maps relate different coordinate charts and ensure the compatibility of the manifold structure

## Transition maps on the sphere

- Consider the unit sphere  $S^2$  in  $\mathbb{R}^3$  and two stereographic projection charts: the north pole chart  $(U_N, \varphi_N)$  and the south pole chart  $(U_S, \varphi_S)$
- The transition map between these charts,  $\varphi_{NS} : \varphi_S(U_N \cap U_S) \rightarrow \varphi_N(U_N \cap U_S)$ , is given by 
$$\varphi_{NS}(x, y) = \left( \frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$$
- This transition map is smooth, invertible, and satisfies the compatibility conditions, ensuring the well-definedness of the differentiable structure on the sphere

## Transition maps on the torus

- Consider the torus  $T^2$  obtained by identifying opposite sides of a square
- Using the standard parametrization of the torus with angles  $\theta$  and  $\phi$ , the transition maps between the charts covering the torus are given by translations:  $\varphi_{ij}(\theta, \phi) = (\theta + 2\pi, \phi)$  and  $\varphi_{ij}(\theta, \phi) = (\theta, \phi + 2\pi)$
- These transition maps are smooth, invertible, and compatible, reflecting the periodic nature of the torus

## Transition maps in polar coordinates

- In the plane  $\mathbb{R}^2$ , consider the polar coordinate chart  $(U, \varphi)$  and the Cartesian coordinate chart  $(V, \psi)$
- The transition map from polar to Cartesian coordinates is given by  $\varphi \circ \psi^{-1}(r, \theta) = (r \cos \theta, r \sin \theta)$
- The inverse transition map from Cartesian to polar coordinates is given by 
$$\psi \circ \varphi^{-1}(x, y) = \left( \sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right) \right)$$
- These transition maps are smooth (except at the origin) and invertible, demonstrating the compatibility of the polar and Cartesian coordinate systems

## Transition maps and differentiable structures

- Transition maps play a fundamental role in defining and studying the differentiable structures on manifolds
- The compatibility of transition maps ensures that the notion of smoothness is well-defined and independent of the choice of coordinate charts
- Differentiable structures are essential for performing calculus on manifolds and investigating their geometric properties

## Equivalence of differentiable structures

- Two differentiable structures on a manifold are said to be equivalent if their corresponding maximal atlases are compatible
- Equivalence of differentiable structures means that the notion of smoothness and the results of calculus are the same for both structures
- Equivalent differentiable structures can be thought of as different ways of describing the same underlying smooth manifold

## Compatibility of differentiable structures

- Differentiable structures on a manifold are compatible if their transition maps are smooth with respect to both structures
- Compatibility of differentiable structures ensures that the notion of smoothness is consistent across the manifold, regardless of the choice of atlas
- Compatible differentiable structures can be combined to form a larger differentiable structure on the manifold

## Role in defining smooth manifolds

- Transition maps and their compatibility are crucial for defining smooth manifolds
- A smooth manifold is a topological manifold equipped with a maximal atlas whose transition maps are smooth
- The smoothness of transition maps ensures that the differentiable structure is well-defined and that calculus can be performed consistently on the manifold
- The study of smooth manifolds and their properties relies heavily on the properties and behavior of transition maps

## 1.5 Tangent spaces and tangent bundles

Tangent spaces and tangent bundles are key concepts in differential geometry. They provide a way to study the local behavior of manifolds by linearizing them at specific points, enabling calculus and geometric analysis.

These tools are essential for understanding the structure of manifolds. Tangent spaces represent infinitesimal displacements, while tangent bundles combine all tangent spaces into a single object, offering a global perspective on the manifold's geometry.

## Definition of tangent spaces

- Tangent spaces are a fundamental concept in differential geometry that allows us to study the local behavior of a manifold at a specific point
- Tangent spaces provide a way to linearize the manifold near a point, enabling us to perform calculus and analyze the geometry of the manifold

## Tangent vectors as derivations

- Tangent vectors can be defined as derivations on the algebra of smooth functions at a point on the manifold
- A derivation is a linear map that satisfies the Leibniz rule:  $D(fg) = D(f)g(p) + f(p)D(g)$  for smooth functions  $f$  and  $g$  at a point  $p$
- Tangent vectors as derivations capture the idea of directional derivatives along curves passing through the point

## Tangent vectors as equivalence classes

- Tangent vectors can also be defined as equivalence classes of curves passing through a point on the manifold
- Two curves are considered equivalent if they have the same velocity vector at the point of interest
- The equivalence class of curves defines a tangent vector at that point
- This definition provides a geometric interpretation of tangent vectors as infinitesimal displacements along curves

## Tangent space at a point

- The tangent space at a point  $p$  on a manifold  $M$ , denoted as  $T_pM$ , is the collection of all tangent vectors at  $p$
- The tangent space is a vector space with the same dimension as the manifold
- Elements of the tangent space are called tangent vectors
- The tangent space provides a local linear approximation of the manifold near the point  $p$

## Properties of tangent spaces

- Tangent spaces possess several important properties that make them useful for studying the geometry and dynamics of manifolds
- Understanding these properties is crucial for working with tangent spaces in various applications

## Vector space structure

- The tangent space  $T_pM$  at a point  $p$  on a manifold  $M$  has the structure of a vector space
- Tangent vectors can be added and scaled, satisfying the axioms of a vector space
- The zero vector in the tangent space represents the zero derivation or the equivalence class of constant curves

- The vector space structure allows for linear algebraic operations on tangent vectors

## Dimension of tangent spaces

- The dimension of the tangent space  $T_p M$  is equal to the dimension of the manifold  $M$
- For an  $n$ -dimensional manifold, the tangent space at each point is an  $n$ -dimensional vector space
- This property reflects the local linearity of the manifold and the number of independent directions in which curves can pass through a point

## Basis for tangent spaces

- A basis for the tangent space  $T_p M$  is a set of linearly independent tangent vectors that span the entire space
- In local coordinates  $(x^1, \dots, x^n)$  around the point  $P$ , the partial derivatives  $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$  evaluated at  $P$  form a natural basis for  $T_p M$
- The choice of basis allows for the representation of tangent vectors using their components with respect to the basis
- Different choices of coordinates or frames can lead to different bases for the tangent space

## Tangent bundles

- The tangent bundle is a construction that combines all the tangent spaces of a manifold into a single object
- It provides a global perspective on the collection of tangent vectors across the entire manifold

## Definition of tangent bundles

- The tangent bundle of a manifold  $M$ , denoted as  $TM$ , is the disjoint union of all the tangent spaces at each point of  $M$
- Formally,  $TM = \bigsqcup_{p \in M} T_p M$ , where  $\bigsqcup$  denotes the disjoint union
- An element of the tangent bundle is a pair  $(p, v)$ , where  $P$  is a point on the manifold and  $v$  is a tangent vector at  $P$
- The tangent bundle provides a way to study the totality of tangent vectors across the manifold

## Tangent bundle as a manifold

- The tangent bundle  $TM$  itself has the structure of a smooth manifold
- The dimension of  $TM$  is twice the dimension of the base manifold  $M$
- The smooth structure on  $TM$  is induced by the smooth structure on  $M$  and the vector space structure of the tangent spaces
- The tangent bundle manifold allows for the application of differential geometric techniques to study tangent vectors globally

## Projection map

- The tangent bundle comes equipped with a natural projection map  $\pi : TM \rightarrow M$
- The projection map sends each element  $(p, v)$  of the tangent bundle to its base point  $p$  on the manifold
- The projection map is a smooth surjective map that captures the relationship between the tangent bundle and the base manifold
- Fibers of the projection map,  $\pi^{-1}(p)$ , correspond to the tangent spaces  $T_p M$  at each point  $p$

## Local trivializations

- Locally, the tangent bundle can be trivialized, meaning it can be identified with the product of an open set in the manifold and a vector space
- A local trivialization of  $TM$  over an open set  $U \subset M$  is a diffeomorphism  $\phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ , where  $n$  is the dimension of  $M$
- The local trivialization allows for the representation of tangent vectors using their components in a local coordinate system
- Transition maps between different local trivializations are smooth and capture the change of coordinates on the tangent bundle

## Smooth maps and tangent spaces

- Smooth maps between manifolds induce linear maps between their tangent spaces
- These induced maps, called the differential or pushforward, allow for the study of how tangent vectors are transformed under smooth mappings

## Differential of a smooth map

- Let  $f : M \rightarrow N$  be a smooth map between manifolds  $M$  and  $N$
- The differential of  $f$  at a point  $p \in M$  is a linear map  $df_p : T_p M \rightarrow T_{f(p)} N$
- The differential maps tangent vectors from the tangent space of  $M$  at  $p$  to the tangent space of  $N$  at  $f(p)$
- In local coordinates, the differential can be represented by the Jacobian matrix of partial derivatives of  $f$

## Pushforward of tangent vectors

- The differential of a smooth map  $f : M \rightarrow N$  at a point  $p$  is also called the pushforward of tangent vectors
- Given a tangent vector  $v \in T_p M$ , the pushforward of  $v$  under  $f$  is denoted as  $f_*(v)$  or  $df_p(v)$
- The pushforward maps tangent vectors from  $T_p M$  to  $T_{f(p)} N$  in a way that preserves the algebraic structure of the tangent spaces
- The pushforward is a linear map that captures how tangent vectors are transformed under the smooth map  $f$

## Tangent map between tangent spaces

- The differential of a smooth map  $f: M \rightarrow N$  induces a map between the tangent bundles, called the tangent map
- The tangent map, denoted as  $Tf: TM \rightarrow TN$ , maps elements of the tangent bundle of  $M$  to elements of the tangent bundle of  $N$
- For each point  $p \in M$ , the tangent map restricts to the differential  $df_p: T_p M \rightarrow T_{f(p)} N$
- The tangent map is a smooth map between the tangent bundle manifolds that encodes the local linear behavior of  $f$

## Tangent spaces and curves

- Tangent spaces and tangent vectors have a natural connection with curves on a manifold
- Curves provide a way to visualize and interpret tangent vectors as velocity vectors or infinitesimal displacements

## Tangent vectors as velocity vectors

- Given a smooth curve  $\gamma: I \rightarrow M$  on a manifold  $M$ , where  $I$  is an interval in  $\mathbb{R}$ , the velocity vector of  $\gamma$  at a point  $\gamma(t)$  is a tangent vector in  $T_{\gamma(t)} M$
- The velocity vector  $\gamma'(t)$  represents the instantaneous direction and speed of the curve at the point  $\gamma(t)$
- The velocity vector can be obtained by differentiating the coordinate functions of the curve with respect to the parameter  $t$
- Tangent vectors can be thought of as velocity vectors of curves passing through a point

## Curves and tangent vectors

- Every tangent vector at a point  $p \in M$  can be realized as the velocity vector of some curve passing through  $p$
- Given a tangent vector  $v \in T_p M$ , there exists a smooth curve  $\gamma: I \rightarrow M$  such that  $\gamma(0) = p$  and  $\gamma'(0) = v$
- The curve  $\gamma$  can be constructed using the exponential map or by solving a system of differential equations
- This correspondence between curves and tangent vectors provides a geometric interpretation of tangent vectors

## Parallel transport along curves

- Parallel transport is a way to move tangent vectors along curves while preserving their direction and magnitude
- Given a curve  $\gamma: I \rightarrow M$  and a tangent vector  $v \in T_{\gamma(t_0)} M$  at a point  $\gamma(t_0)$ , parallel transport defines a way to move  $v$  along the curve to obtain a tangent vector at each point  $\gamma(t)$
- Parallel transport requires the notion of a connection on the manifold, which specifies how to transport vectors along curves

- Parallel transport allows for the comparison of tangent vectors at different points along a curve and is important in the study of geodesics and curvature

## Riemannian metrics and tangent spaces

- Riemannian metrics provide additional structure on tangent spaces by introducing an inner product
- The inner product allows for the measurement of lengths, angles, and distances on the manifold

## Inner product on tangent spaces

- A Riemannian metric  $g$  on a manifold  $M$  assigns to each point  $p \in M$  an inner product  $g_p$  on the tangent space  $T_p M$
- The inner product  $g_p$  is a symmetric, positive-definite, bilinear form that maps pairs of tangent vectors to real numbers
- The inner product allows for the computation of lengths, angles, and orthogonality of tangent vectors
- In local coordinates, the Riemannian metric is represented by a symmetric, positive-definite matrix of smooth functions

## Length of tangent vectors

- The Riemannian metric allows for the computation of the length or norm of a tangent vector
- Given a tangent vector  $v \in T_p M$ , its length is defined as  $\|v\| = \sqrt{g_p(v, v)}$
- The length of a tangent vector measures the magnitude or size of the vector with respect to the Riemannian metric
- The length induces a notion of distance on the manifold by integrating the lengths of tangent vectors along curves

## Angles between tangent vectors

- The Riemannian metric allows for the definition of angles between tangent vectors
- Given two tangent vectors  $v, w \in T_p M$ , the angle  $\theta$  between them is given by  $\cos \theta = \frac{g_p(v, w)}{\|v\| \|w\|}$
- The angle measures the relative orientation of the tangent vectors with respect to the Riemannian metric
- Orthogonal or perpendicular tangent vectors have an angle of  $\frac{\pi}{2}$  between them

## Connections and tangent bundles

- Connections provide a way to differentiate vector fields and transport vectors along curves on a manifold
- Connections are defined on the tangent bundle and are closely related to the study of parallel transport and curvature

## Covariant derivatives on tangent bundles

- A connection on a manifold  $M$  is a map that assigns to each vector field  $X$  on  $M$  a covariant derivative operator  $\nabla_X$

- The covariant derivative  $\nabla_X Y$  of a vector field  $Y$  with respect to a vector field  $X$  is another vector field that measures the rate of change of  $Y$  along the direction of  $X$
- The covariant derivative satisfies certain properties, such as linearity and the Leibniz rule, which make it compatible with the smooth structure of the manifold
- In local coordinates, the connection is specified by a set of smooth functions called Christoffel symbols

## Parallel transport using connections

- Connections provide a way to define parallel transport of vectors along curves on a manifold
- Given a curve  $\gamma : I \rightarrow M$  and a vector  $v \in T_{\gamma(t_0)}M$  at a point  $\gamma(t_0)$ , parallel transport along  $\gamma$  defines a vector field  $V(t)$  along the curve such that  $V(t_0) = v$  and  $\nabla_{\gamma'(t)} V(t) = 0$
- Parallel transport preserves the length and angle of vectors with respect to the connection
- The parallel-transported vector field  $V(t)$  is obtained by solving a system of differential equations involving the connection coefficients

## Levi-Civita connection

- The Levi-Civita connection is a special connection that is compatible with a given Riemannian metric on a manifold
- It is the unique connection that is symmetric (torsion-free) and compatible with the metric (metric-compatible)
- The Christoffel symbols of the Levi-Civita connection can be expressed in terms of the partial derivatives of the metric components
- The Levi-Civita connection is widely used in Riemannian geometry and is essential for the study of geodesics, curvature, and parallel transport on Riemannian manifolds

## Applications of tangent spaces

- Tangent spaces and their associated concepts find applications in various fields beyond pure mathematics
- They provide a framework for studying dynamical systems, optimization problems, and control systems

## Tangent spaces in physics

- In physics, tangent spaces are used to describe the state space of a system and its evolution over time
- The tangent vectors represent velocities or momenta of particles in the system
- The equations of motion can be formulated using the language of tangent spaces and vector fields
- Parallel transport and curvature play a crucial role in the study of gravitational fields and general relativity



## Tangent spaces in optimization

- Tangent spaces are used in optimization to study the local behavior of objective functions and constraints
- The gradient of a function at a point can be viewed as a tangent vector that points in the direction of steepest ascent
- Optimization algorithms, such as gradient descent, utilize the tangent space structure to iteratively update the solution and converge to a local optimum
- The Hessian matrix, which captures second-order information, can be interpreted as a quadratic form on the tangent space

## Tangent spaces in control theory

- In control theory, tangent spaces are used to describe the state space and control inputs of a dynamical system
- The tangent vectors represent infinitesimal changes in the system's state or control variables
- The controllability and observability of a system can be studied using the properties of tangent spaces and vector fields
- The Lie bracket of vector fields, which measures their non-commutativity, plays a crucial role in the study of nonlinear control systems and the design of control laws

## 1.6 Vector fields and flows

Vector fields are fundamental in differential geometry, assigning tangent vectors to each point on a manifold. They describe infinitesimal behavior of curves and flows, crucial for analyzing geometry and topology. Understanding vector fields unlocks insights into manifold structure and dynamics.

Smooth vector fields form a vector space, with properties like linearity and scalar multiplication. They can be represented in local coordinates, enabling explicit calculations. Vector fields interact with differential forms through operations like interior product and Lie derivative, revealing deep connections in differential geometry.

### Definition of vector fields

- Vector fields are a fundamental object of study in differential geometry, assigning a tangent vector to each point on a manifold
- They provide a way to describe the infinitesimal behavior of curves and flows on the manifold
- Understanding vector fields is crucial for analyzing the geometry and topology of manifolds

## Tangent vectors on manifolds

- At each point  $P$  on a manifold  $M$ , the tangent space  $T_P M$  is a vector space consisting of tangent vectors to curves passing through  $P$
- Tangent vectors can be thought of as velocity vectors of curves at the point  $P$
- The collection of all tangent spaces forms the tangent bundle  $TM$ , a vector bundle over the manifold  $M$

## Smooth vector fields

- A vector field  $X$  on a manifold  $M$  is a smooth assignment of a tangent vector  $X_p \in T_p M$  to each point  $p \in M$
- Smoothness means that the components of  $X$  in any local coordinate chart vary smoothly as functions of the coordinates
- The set of all smooth vector fields on  $M$  is denoted by  $\mathfrak{X}(M)$  and forms a vector space over  $\mathbb{R}$

## Local coordinate representations

- In a local coordinate chart  $(U, (x^1, \dots, x^n))$ , a vector field  $X$  can be expressed as a linear combination of the coordinate basis vectors:

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$$

- The coefficients  $X^i$  are smooth functions on  $U$  called the components of  $X$  in the given coordinate chart
- The local coordinate representation allows for explicit calculations and analysis of vector fields

## Properties of vector fields

- Vector fields possess various algebraic and analytical properties that make them a rich and versatile tool in differential geometry
- These properties enable the study of the geometry and topology of manifolds through the behavior of vector fields
- Understanding these properties is essential for working with vector fields in applications

## Linearity and scalar multiplication

- The set of smooth vector fields  $\mathfrak{X}(M)$  forms a vector space over  $\mathbb{R}$
- For vector fields  $X, Y \in \mathfrak{X}(M)$  and scalars  $a, b \in \mathbb{R}$ , the linear combination  $aX + bY$  is defined pointwise:  $(aX + bY)_p = aX_p + bY_p$
- Scalar multiplication of a vector field  $X$  by a smooth function  $f \in C^\infty(M)$  is defined pointwise:  $(fX)_p = f(p)X_p$

## Support and compact support

- The support of a vector field  $X$  is the closure of the set of points where  $X$  is non-zero:  
 $\text{supp}(X) = \overline{\{p \in M : X_p \neq 0\}}$
- A vector field  $X$  is said to have compact support if its support is a compact subset of  $M$
- Vector fields with compact support are particularly useful in the study of local properties and in the construction of partitions of unity

## Divergence and curl

- The divergence of a vector field  $X$  on a Riemannian manifold  $(M, g)$  is a scalar function that measures the infinitesimal rate of change of volume under the flow of  $X$ :  $\operatorname{div} X = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} (\sqrt{|g|} X^i)$
- The curl of a vector field  $X$  on a 3-dimensional Riemannian manifold is a vector field that measures the infinitesimal rotation of  $X$ :  $\operatorname{curl} X = (*dX^\flat)^\sharp$ , where  $\flat$  and  $\sharp$  denote the musical isomorphisms and  $*$  is the Hodge star operator
- Divergence and curl provide important information about the local behavior of vector fields and are related to conservation laws and the topology of the manifold

## Vector fields and differential forms

- Vector fields and differential forms are dual objects on a manifold, with each providing complementary information about the geometry and topology
- The interplay between vector fields and differential forms is a fundamental aspect of modern differential geometry
- This duality is expressed through various operations, such as the interior product, contraction, and Lie derivative

## Dual relationship with 1-forms

- The dual space to the tangent space  $T_p M$  at each point  $p \in M$  is the cotangent space  $T_p^* M$ , consisting of linear functionals on  $T_p M$
- A differential 1-form  $\omega$  on  $M$  is a smooth assignment of a cotangent vector  $\omega_p \in T_p^* M$  to each point  $p \in M$
- The pairing between a vector field  $X$  and a 1-form  $\omega$  is given by the evaluation:  $\omega(X) = \omega_p(X_p)$  at each point  $p \in M$

## Interior product and contraction

- The interior product (or contraction) of a vector field  $X$  with a differential  $k$ -form  $\omega$  is a  $(k-1)$ -form denoted by  $i_X \omega$  or  $X \lrcorner \omega$
- In local coordinates,  $(i_X \omega)_{i_1 \dots i_{k-1}} = X^j \omega_{j i_1 \dots i_{k-1}}$
- The interior product satisfies various properties, such as linearity, the Leibniz rule, and compatibility with the exterior derivative

## Lie derivative of differential forms

- The Lie derivative  $\mathcal{L}_X \omega$  of a differential  $k$ -form  $\omega$  along a vector field  $X$  measures the change of  $\omega$  under the flow of  $X$
- It is defined as  $\mathcal{L}_X \omega = \lim_{t \rightarrow 0} \frac{1}{t} (\varphi_t^* \omega - \omega)$ , where  $\varphi_t$  is the flow of  $X$
- The Lie derivative satisfies the Leibniz rule, commutes with the exterior derivative, and is related to the interior product by Cartan's magic formula:  $\mathcal{L}_X = d \circ i_X + i_X \circ d$

## Integral curves and flows

- Integral curves and flows are fundamental concepts in the study of vector fields, describing the motion of points under the influence of a vector field
- They provide a way to visualize and analyze the global behavior of vector fields on a manifold
- The existence and uniqueness of integral curves and flows are guaranteed by the fundamental theorem of ordinary differential equations

### Definition of integral curves

- An integral curve of a vector field  $X$  on a manifold  $M$  is a smooth curve  $\gamma : I \rightarrow M$  (where  $I \subseteq \mathbb{R}$  is an interval) such that  $\gamma'(t) = X_{\gamma(t)}$  for all  $t \in I$
- Integral curves are parametrized curves whose velocity vector at each point coincides with the value of the vector field at that point
- They can be thought of as the trajectories of particles moving under the influence of the vector field

### Existence and uniqueness theorem

- The fundamental theorem of ordinary differential equations guarantees the existence and uniqueness of integral curves for a given vector field and initial condition
- For a smooth vector field  $X$  on  $M$  and a point  $p \in M$ , there exists a unique maximal integral curve  $\gamma : I \rightarrow M$  such that  $\gamma(0) = p$
- The theorem ensures that the flow of a vector field is well-defined and determines the global behavior of the vector field

### Local and global flows

- The flow of a vector field  $X$  on  $M$  is a smooth map  $\varphi : D \rightarrow M$ , where  $D \subseteq \mathbb{R} \times M$  is an open subset containing  $\{0\} \times M$ , such that for each  $p \in M$ , the curve  $t \mapsto \varphi(t, p)$  is the unique maximal integral curve of  $X$  starting at  $p$
- The flow is defined locally around each point, and the domain  $D$  may not be the entire  $\mathbb{R} \times M$  due to the possibility of incomplete integral curves
- When the vector field is complete (i.e., all integral curves are defined for all  $t \in \mathbb{R}$ ), the flow is global and defined on the entire  $\mathbb{R} \times M$

### One-parameter groups of diffeomorphisms

- For each  $t \in \mathbb{R}$ , the map  $\varphi_t : M \rightarrow M$  defined by  $\varphi_t(p) = \varphi(t, p)$  is a diffeomorphism of  $M$
- The collection  $\{\varphi_t\}_{t \in \mathbb{R}}$  forms a one-parameter group of diffeomorphisms, satisfying  $\varphi_0 = \text{id}_M$  and  $\varphi_{t+s} = \varphi_t \circ \varphi_s$  for all  $t, s \in \mathbb{R}$
- The one-parameter group of diffeomorphisms encodes the symmetries and dynamics of the vector field

## Lie brackets and Lie derivatives

- Lie brackets and Lie derivatives are fundamental operations on vector fields that capture their infinitesimal behavior and relationships
- They play a crucial role in the study of symmetries, integrability, and the geometry of manifolds
- Understanding Lie brackets and Lie derivatives is essential for working with vector fields and their applications

## Lie bracket of vector fields

- The Lie bracket of two vector fields  $X, Y \in \mathfrak{X}(M)$  is another vector field  $[X, Y] \in \mathfrak{X}(M)$  that measures the failure of  $X$  and  $Y$  to commute
- In local coordinates,  $[X, Y]^i = X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}$
- The Lie bracket satisfies antisymmetry ( $[X, Y] = -[Y, X]$ ) and the Jacobi identity ( $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ )

## Jacobi identity and Lie algebra

- The Jacobi identity is a crucial property of the Lie bracket, ensuring that the space of vector fields  $\mathfrak{X}(M)$  forms a Lie algebra
- A Lie algebra is a vector space equipped with a bilinear operation (the Lie bracket) satisfying antisymmetry and the Jacobi identity
- The Lie algebra structure of vector fields encodes the infinitesimal symmetries and deformations of the manifold

## Lie derivative of vector fields

- The Lie derivative  $\mathcal{L}_X Y$  of a vector field  $Y$  along another vector field  $X$  is a vector field that measures the change of  $Y$  under the flow of  $X$
- It is defined as  $\mathcal{L}_X Y = \lim_{t \rightarrow 0} \frac{1}{t} (\varphi_t^* Y - Y)$ , where  $\varphi_t$  is the flow of  $X$
- The Lie derivative satisfies various properties, such as linearity, the Leibniz rule, and compatibility with the Lie bracket

## Relation to Lie brackets and flows

- The Lie bracket of two vector fields  $X$  and  $Y$  can be expressed in terms of their Lie derivatives:  $[X, Y] = \mathcal{L}_X Y - \mathcal{L}_Y X$
- The flow of a vector field  $X$  is related to its Lie derivative by the formula  $\frac{d}{dt} \varphi_t^* Y = \varphi_t^* (\mathcal{L}_X Y)$ , where  $\varphi_t$  is the flow of  $X$
- These relations highlight the deep connections between Lie brackets, Lie derivatives, and the geometry of vector fields and flows

## Invariant submanifolds and foliations

- Invariant submanifolds and foliations are geometric structures that capture the symmetries and decompositions of a manifold with respect to a given vector field or distribution

- They provide a way to study the global behavior of vector fields and their integral curves
- The Frobenius theorem is a fundamental result that characterizes the integrability of distributions and the existence of invariant submanifolds and foliations

## Invariant submanifolds under flows

- A submanifold  $N \subseteq M$  is said to be invariant under the flow of a vector field  $X$  if for every  $p \in N$  and  $t \in \mathbb{R}$  such that  $\varphi_t(p)$  is defined, we have  $\varphi_t(p) \in N$
- Invariant submanifolds are preserved by the flow of the vector field and contain entire integral curves
- They play a crucial role in the study of dynamical systems and the qualitative behavior of vector fields

## Frobenius theorem and integrability

- A distribution  $\Delta$  on a manifold  $M$  is a smooth assignment of a subspace  $\Delta_p \subseteq T_p M$  to each point  $p \in M$
- A distribution is called involutive if for any two vector fields  $X, Y$  tangent to  $\Delta$ , their Lie bracket  $[X, Y]$  is also tangent to  $\Delta$
- The Frobenius theorem states that a distribution  $\Delta$  is integrable (i.e., there exists a foliation of  $M$  tangent to  $\Delta$ ) if and only if it is involutive

## Codimension-one foliations

- A codimension-one foliation of a manifold  $M$  is a decomposition of  $M$  into disjoint connected submanifolds (called leaves) of codimension one
- Codimension-one foliations are particularly important in the study of contact structures and Reeb vector fields
- They can be defined by integrable codimension-one distributions or by the kernels of non-vanishing 1-forms

## Reeb vector fields and contact structures

- A contact structure on a manifold  $M$  of odd dimension  $2n + 1$  is a maximally non-integrable codimension-one distribution  $\xi$
- A contact form  $\alpha$  is a 1-form such that  $\alpha \wedge (d\alpha)^n \neq 0$  everywhere, and its kernel defines a contact structure
- The Reeb vector field  $R_\alpha$  associated with a contact form  $\alpha$  is the unique vector field satisfying  $\alpha(R_\alpha) = 1$  and  $i_{R_\alpha} d\alpha = 0$
- Reeb vector fields and contact structures play a central role in contact geometry and have applications in mechanics and thermodynamics

## Applications and examples

- Vector fields and their associated concepts have numerous applications in various areas of mathematics and physics
- They provide a powerful framework for studying the geometry, topology, and dynamics of manifolds

- Some notable applications include Hamiltonian mechanics, Riemannian geometry, and the study of symmetries and conservation laws

## Hamiltonian vector fields in symplectic geometry

- A symplectic manifold  $(M, \omega)$  is an even-dimensional manifold  $M$  equipped with a closed, non-degenerate 2-form  $\omega$
- For a smooth function  $H \in C^\infty(M)$ , the Hamiltonian vector field  $X_H$  is defined by the equation  $i_{X_H}\omega = dH$
- Hamiltonian vector fields describe the dynamics of conservative mechanical systems and are a fundamental object of study in symplectic geometry

## Geodesic flows on Riemannian manifolds

- On a Riemannian manifold  $(M, g)$ , the geodesic flow is a vector field on the tangent bundle  $TM$  that describes the motion of particles along geodesics
- The geodesic flow is defined by the Hamiltonian vector field of the energy function  $E(v) = \frac{1}{2}g(v, v)$  on  $TM$  with respect to the canonical symplectic structure
- Geodesic flows encode the geometry of the Riemannian manifold and have applications in mechanics and optics

## Killing vector fields and isometries

- A Killing vector field on a Riemannian manifold  $(M, g)$  is a vector field  $X$  that preserves the metric, i.e.,  $\mathcal{L}_X g = 0$
- Killing vector fields are the infinitesimal generators of isometries

# Unit 2 – Curves and surfaces

## 2.1 Parametrized curves

Parametrized curves are the building blocks of differential geometry, describing paths in space using functions of a parameter. They allow us to analyze curves mathematically, capturing their shape, length, and behavior through properties like regularity, arc length, and curvature.

Understanding parametrized curves is crucial for studying more complex geometric objects. This topic introduces key concepts like the Frenet frame, which provides a local coordinate system for curves, and special types of curves with unique properties used in various applications.

### Definition of parametrized curves

- Parametrized curves are a fundamental concept in differential geometry that describe curves using a parametric equation
- Parametric equations represent a curve as a function of an independent variable, typically denoted as  $t$ , which maps to points  $(x(t), y(t))$  in the plane or  $(x(t), y(t), z(t))$  in space

### Parametrization of curves

- Parametrization is the process of representing a curve using parametric equations
- A curve  $C$  can be parametrized by a function  $\gamma : I \rightarrow \mathbb{R}^n$ , where  $I$  is an interval in  $\mathbb{R}$  and  $n$  is the dimension of the ambient space (typically 2 or 3)
- The function  $\gamma(t) = (x(t), y(t))$  or  $\gamma(t) = (x(t), y(t), z(t))$  assigns a unique point on the curve to each value of the parameter  $t$

### Smooth vs non-smooth curves

- Smooth curves are parametrized curves whose component functions (e.g.,  $x(t)$ ,  $y(t)$ ,  $z(t)$ ) are continuously differentiable
- Non-smooth curves have points where the derivative is not defined or is discontinuous (cusps, corners)
- Smoothness is an important property for analyzing the geometric properties of curves, such as tangent vectors and curvature

### Properties of parametrized curves

- Parametrized curves have several important properties that allow for their geometric analysis and manipulation
- These properties include regularity, reparametrization, and equivalence classes of parametrizations

### Regularity of curves

- A parametrized curve  $\gamma : I \rightarrow \mathbb{R}^n$  is regular if its derivative  $\gamma'(t)$  is non-zero for all  $t \in I$
- Regularity ensures that the curve does not have any singular points or self-intersections
- Regular curves have well-defined tangent vectors at every point



## Reparametrization of curves

- Reparametrization is the process of changing the parameter of a curve without altering its geometric shape
- Given a curve  $\gamma(t)$  and a bijective, continuously differentiable function  $\phi : J \rightarrow I$ , the reparametrized curve is  $\tilde{\gamma}(s) = \gamma(\phi(s))$ , where  $s \in J$
- Reparametrization is useful for simplifying calculations or achieving desired properties (unit speed)

## Equivalence classes of parametrizations

- Two parametrizations  $\gamma_1 : I_1 \rightarrow \mathbb{R}^n$  and  $\gamma_2 : I_2 \rightarrow \mathbb{R}^n$  are equivalent if there exists a bijective, continuously differentiable function  $\phi : I_2 \rightarrow I_1$  such that  $\gamma_2 = \gamma_1 \circ \phi$
- Equivalent parametrizations describe the same geometric curve
- Equivalence classes of parametrizations allow for the study of curves independent of their specific parametrization

## Arc length of parametrized curves

- Arc length is a fundamental concept in the study of parametrized curves that measures the distance along the curve between two points
- Arc length is essential for defining arc length parametrization and curvature

## Definition of arc length

- The arc length of a parametrized curve  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  is given by the integral:  $L(\gamma) = \int_a^b \|\gamma'(t)\| dt$
- $\|\gamma'(t)\|$  represents the magnitude of the tangent vector at each point along the curve
- Arc length is independent of the parametrization and is an intrinsic property of the curve

## Arc length parametrization

- An arc length parametrization is a reparametrization of a curve such that the parameter represents the arc length along the curve from a fixed starting point
- For a curve  $\gamma(t)$ , the arc length parametrization  $\tilde{\gamma}(s)$  satisfies:  $s = \int_a^t \|\gamma'(u)\| du$
- Arc length parametrizations have unit speed, meaning  $\|\tilde{\gamma}'(s)\| = 1$  for all  $s$

## Existence of arc length parametrization

- Every regular parametrized curve admits an arc length parametrization
- The existence of an arc length parametrization is guaranteed by the fundamental theorem of calculus and the regularity of the curve
- Arc length parametrizations are unique up to a constant shift in the parameter

## Curvature of parametrized curves

- Curvature is a measure of how much a curve deviates from a straight line at a given point

- Curvature is an intrinsic property of the curve and is independent of the parametrization

## Definition of curvature

- The curvature of a regular parametrized curve  $\gamma(t)$  at a point  $t$  is given by:  $\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$
- $\gamma'(t)$  and  $\gamma''(t)$  represent the first and second derivatives of the curve at the point  $t$
- The cross product  $\gamma'(t) \times \gamma''(t)$  measures the deviation of the curve from a straight line

## Signed vs unsigned curvature

- Unsigned curvature is the absolute value of the curvature,  $|\kappa(t)|$ , and does not distinguish between clockwise and counterclockwise bending
- Signed curvature assigns a positive or negative value to the curvature depending on the orientation of the curve (clockwise or counterclockwise)
- Signed curvature is useful for analyzing the behavior of the curve in relation to its ambient space

## Curvature in terms of arc length

- For a curve  $\gamma(s)$  parametrized by arc length, the curvature can be expressed as:  $\kappa(s) = \|\gamma''(s)\|$
- This simplification is possible because  $\|\gamma'(s)\| = 1$  for arc length parametrizations
- Expressing curvature in terms of arc length simplifies calculations and provides a geometric interpretation

## Frenet frame of parametrized curves

- The Frenet frame is a moving orthonormal frame attached to each point of a regular parametrized curve
- The Frenet frame consists of the tangent, normal, and binormal vectors, which provide a local coordinate system for describing the curve's geometry

## Tangent vector of curves

- The tangent vector  $\mathbf{T}(t)$  of a regular parametrized curve  $\gamma(t)$  is the unit vector pointing in the direction of the curve's velocity:  $\mathbf{T}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$
- The tangent vector is always tangent to the curve and indicates the instantaneous direction of motion

## Normal vector of curves

- The normal vector  $\mathbf{N}(t)$  is the unit vector perpendicular to the tangent vector and points in the direction of the curve's acceleration:  $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$
- The normal vector lies in the osculating plane, which is the plane spanned by the tangent and acceleration vectors

## Binormal vector of curves

- The binormal vector  $\mathbf{B}(t)$  is the unit vector perpendicular to both the tangent and normal vectors, completing the right-handed orthonormal frame:  $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

- The binormal vector is normal to the osculating plane and indicates the direction in which the curve is twisting

## Frenet-Serret formulas

- The Frenet-Serret formulas describe the rates of change of the tangent, normal, and binormal vectors along the curve:

$$\begin{aligned}\mathbf{T}'(s) &= \kappa(s)\mathbf{N}(s) \\ \mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s) \\ \mathbf{B}'(s) &= -\tau(s)\mathbf{N}(s)\end{aligned}$$

- $\kappa(s)$  is the curvature, and  $\tau(s)$  is the torsion, which measures the rate of change of the osculating plane
- The Frenet-Serret formulas provide a complete description of the local geometry of the curve

## Special parametrized curves

- Certain classes of parametrized curves have distinctive properties that make them useful for various applications
- These special curves include plane curves, space curves, simple curves, closed curves, and curves of constant curvature

## Plane curves vs space curves

- Plane curves are parametrized curves that lie entirely in a single plane (e.g.,  $\gamma(t) = (x(t), y(t))$ )
- Space curves are parametrized curves that do not lie in a single plane and have non-zero torsion (e.g.,  $\gamma(t) = (x(t), y(t), z(t))$ )
- Plane curves have simpler geometric properties compared to space curves, as they do not exhibit twisting behavior

## Simple vs closed curves

- Simple curves are parametrized curves that do not self-intersect, meaning  $\gamma(t_1) \neq \gamma(t_2)$  for any  $t_1 \neq t_2$  in the domain
- Closed curves are parametrized curves whose start and end points coincide, i.e.,  $\gamma(a) = \gamma(b)$  for the domain  $[a, b]$
- Simple closed curves are both simple and closed, forming a loop without self-intersections (Jordan curves)

## Curves of constant curvature

- Curves of constant curvature have a curvature function  $\kappa(t)$  that is constant along the entire curve
- Examples of curves with constant curvature include circles ( $\kappa > 0$ ), straight lines ( $\kappa = 0$ ), and hyperbolic curves ( $\kappa < 0$ )
- Curves of constant curvature have special geometric properties and are often used in modeling and analysis

## Applications of parametrized curves

- Parametrized curves have numerous applications in various fields, including physics, engineering, computer graphics, and data analysis
- These applications leverage the geometric properties and flexibility of parametrized curves to model, analyze, and visualize complex phenomena

## Modeling of physical phenomena

- Parametrized curves are used to model the trajectories of particles, celestial bodies, and other objects in motion
- Examples include the path of a projectile under the influence of gravity, the orbit of a satellite around a planet, and the motion of a charged particle in an electromagnetic field
- Parametrized curves enable the study of the geometric properties of these trajectories, such as curvature and torsion, which provide insights into the underlying physical processes

## Curve fitting in data analysis

- Parametrized curves are employed in data analysis to fit curves to experimental or observational data points
- Curve fitting techniques, such as least squares and spline interpolation, use parametrized curves to approximate the underlying trends and relationships in the data
- Fitted curves can be used for data visualization, prediction, and extracting meaningful features from the data

## Parametrized curves in computer graphics

- Parametrized curves are fundamental in computer graphics for representing and manipulating curves and surfaces
- Bézier curves, B-splines, and NURBS (Non-Uniform Rational B-Splines) are popular parametric curve representations used in computer-aided design (CAD), animation, and 3D modeling
- Parametrized curves enable the creation of smooth, flexible, and easily controllable shapes, which are essential for generating visually appealing and geometrically accurate graphics

## 2.2 Arc length and reparametrization

Arc length and reparameterization are key concepts in differential geometry. They allow us to measure curves and change how we describe them without altering their shape. This topic builds our understanding of intrinsic curve properties, independent of specific parameterizations.

Arc length lets us quantify curve lengths, while reparameterization gives us flexibility in describing curves. Together, they provide powerful tools for analyzing curve geometry and solving problems in physics, engineering, and other fields that involve curved paths and surfaces.

### Definition of arc length

- Arc length is a fundamental concept in differential geometry that measures the length of a curve segment in a metric space

- Intuitively, arc length can be thought of as the distance traveled along a curve if one were to "walk" along it
- Arc length is defined using the metric of the space and is independent of the parameterization of the curve

## Arc length formula

- The arc length of a curve  $\gamma : [a, b] \rightarrow M$  in a metric space  $(M, d)$  is given by the formula:  

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt$$
where  $g$  is the metric tensor and  $\gamma'(t)$  is the tangent vector to the curve at  $t$
- In Euclidean space  $\mathbb{R}^n$ , the arc length formula simplifies to:  $L(\gamma) = \int_a^b \|\gamma'(t)\| dt$  where  $\|\cdot\|$  denotes the Euclidean norm
- The arc length formula involves an integral, which can be interpreted as a limit of Riemann sums approximating the length of the curve by summing the lengths of small linear segments

## Arc length parameterization

- A curve  $\gamma : [a, b] \rightarrow M$  is said to be parameterized by arc length (or unit speed) if  $\|\gamma'(t)\| = 1$  for all  $t \in [a, b]$
- In other words, the tangent vector to the curve has unit length at every point along the curve
- Arc length parameterization is a natural and intrinsic way to parameterize a curve, as it depends only on the geometry of the curve and not on any particular coordinate system
- Given a curve  $\gamma : [a, b] \rightarrow M$ , one can always reparameterize it by arc length by defining a new parameter  $s(t) = \int_a^t \|\gamma'(u)\| du$  and considering the curve  $\tilde{\gamma}(s) = \gamma(t(s))$ , where  $t(s)$  is the inverse function of  $s(t)$

## Unit speed curves

- A curve parameterized by arc length is also called a unit speed curve
- Unit speed curves have several nice properties that make them useful in differential geometry:
- The tangent vector field  $\tilde{\gamma}'(s)$  along a unit speed curve has constant length 1
- The arc length parameter  $s$  corresponds to the actual distance traveled along the curve
- Geodesics (locally length-minimizing curves) are often parameterized by arc length
- Examples of unit speed curves include:
  - Great circles on a sphere (with appropriate parameterization)
  - Straight lines in Euclidean space (with linear parameterization)

## Properties of arc length

- Arc length is a fundamental concept in differential geometry and has several important properties that make it useful for studying curves and surfaces
- These properties are intrinsic to the geometry of the curve and do not depend on any particular parameterization or coordinate system

- Understanding the properties of arc length is crucial for developing a deeper understanding of the geometry of curves and surfaces

## Invariance under reparameterization

- One of the most important properties of arc length is its invariance under reparameterization
- This means that the arc length of a curve does not change if we parameterize the curve differently, as long as the new parameterization is still regular (i.e., has non-vanishing tangent vectors)
- More precisely, if  $\gamma : [a, b] \rightarrow M$  and  $\tilde{\gamma} : [c, d] \rightarrow M$  are two regular parameterizations of the same curve, then  $L(\gamma) = L(\tilde{\gamma})$
- This property reflects the fact that arc length is an intrinsic property of the curve itself, not of any particular parameterization

## Additivity of arc lengths

- Another important property of arc length is its additivity
- If a curve  $\gamma : [a, b] \rightarrow M$  is split into two subcurves  $\gamma_1 : [a, c] \rightarrow M$  and  $\gamma_2 : [c, b] \rightarrow M$ , where  $a < c < b$ , then the arc length of  $\gamma$  is equal to the sum of the arc lengths of  $\gamma_1$  and  $\gamma_2$ :  

$$L(\gamma) = L(\gamma_1) + L(\gamma_2)$$
- This property allows us to compute the arc length of a curve by breaking it up into smaller, more manageable pieces
- It also reflects the idea that arc length measures the total distance traveled along the curve, and this distance can be accumulated over different segments of the curve

## Arc length and line integrals

- Arc length can be expressed as a special case of a line integral
- Given a curve  $\gamma : [a, b] \rightarrow M$  and a continuous function  $f : M \rightarrow \mathbb{R}$ , the line integral of  $f$  along  $\gamma$  is defined as: 
$$\int_{\gamma} f ds = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt$$
- If we take  $f \equiv 1$ , then the line integral reduces to the arc length formula:  

$$L(\gamma) = \int_{\gamma} 1 ds = \int_a^b \|\gamma'(t)\| dt$$
- This connection between arc length and line integrals is important in many applications, such as computing the work done by a force along a curved path

## Reparameterization of curves

- Reparameterization is the process of changing the parameter used to describe a curve while keeping the image (shape) of the curve unchanged
- It is an important tool in differential geometry for simplifying calculations and highlighting different properties of curves
- Reparameterization can be used to make a curve unit speed, to change its orientation, or to focus on a particular portion of the curve

## Definition of reparameterization

- A reparameterization of a curve  $\gamma : [a, b] \rightarrow M$  is a new curve  $\tilde{\gamma} : [c, d] \rightarrow M$  of the form  $\tilde{\gamma} = \gamma \circ \varphi$ , where  $\varphi : [c, d] \rightarrow [a, b]$  is a diffeomorphism (smooth bijective map with smooth inverse)
- The map  $\varphi$  is called the reparameterization function and it determines how the parameter of the curve is changed
- Common examples of reparameterization functions include:
  - Linear functions:  $\varphi(t) = \alpha t + \beta$ , which change the speed and shift the parameter interval
  - Inverse functions:  $\varphi(t) = t^{-1}$ , which reverse the orientation of the curve
  - Trigonometric functions:  $\varphi(t) = \tan(t)$ , which can be used to focus on a particular portion of the curve

## Regular and singular points

- A point  $\gamma(t_0)$  on a curve  $\gamma : [a, b] \rightarrow M$  is called regular if the tangent vector  $\gamma'(t_0)$  is non-zero
- If  $\gamma'(t_0) = 0$ , then  $\gamma(t_0)$  is called a singular point or a stationary point of the curve
- Regular points are important because they allow us to define the tangent line to the curve, which is a key concept in differential geometry
- Singular points, on the other hand, can indicate important features of the curve, such as cusps or self-intersections
- When reparameterizing a curve, it is important to ensure that regular points remain regular under the reparameterization, so that the geometry of the curve is preserved

## Reparameterization and orientation

- The orientation of a curve refers to the direction in which the curve is traversed, as determined by the direction of its tangent vectors
- A curve can be parameterized in two opposite orientations, corresponding to traversing the curve in opposite directions
- Reparameterization can be used to change the orientation of a curve by composing it with a reparameterization function that reverses the order of the parameter interval
- For example, if  $\gamma : [a, b] \rightarrow M$  is a curve and  $\varphi(t) = a + b - t$  is a reparameterization function, then  $\tilde{\gamma} = \gamma \circ \varphi$  is a reparameterization of  $\gamma$  with the opposite orientation
- Changing the orientation of a curve can be useful for various purposes, such as aligning the orientation of multiple curves or ensuring consistency with a chosen convention

## Arc length parameterization

- Arc length parameterization is a special type of reparameterization in which the parameter of the curve corresponds to the arc length along the curve
- In other words, a curve  $\gamma : [a, b] \rightarrow M$  is parameterized by arc length if for any  $t_1, t_2 \in [a, b]$ , the arc length of the portion of the curve between  $\gamma(t_1)$  and  $\gamma(t_2)$  is equal to  $|t_2 - t_1|$
- Arc length parameterization is a natural and intrinsic way to parameterize a curve, as it depends only on the geometry of the curve and not on any particular coordinate system

## Existence and uniqueness

- Given a regular curve  $\gamma : [a, b] \rightarrow M$ , there always exists a reparameterization of  $\gamma$  that is parameterized by arc length
- Moreover, this arc length parameterization is unique up to a shift in the parameter (i.e., up to composition with a translation function  $\varphi(t) = t + c$ )
- The existence of arc length parameterization follows from the fact that the arc length function  $s(t) = \int_a^t \|\gamma'(u)\| du$  is always strictly increasing and continuous for a regular curve, and therefore has a continuous inverse function  $t(s)$
- The uniqueness (up to a shift) follows from the fact that any two arc length parameterizations of the same curve must differ by a constant, since they both measure the same arc length along the curve

## Construction of arc length parameterization

- To construct an arc length parameterization of a given curve  $\gamma : [a, b] \rightarrow M$ , we follow these steps: 1. Compute the arc length function  $s(t) = \int_a^t \|\gamma'(u)\| du$  2. Find the inverse function  $t(s)$  of  $s(t)$ , defined on an interval  $[0, L]$  where  $L = s(b)$  is the total arc length of the curve 3. Define the arc length parameterized curve  $\tilde{\gamma} : [0, L] \rightarrow M$  by  $\tilde{\gamma}(s) = \gamma(t(s))$
- In practice, finding the inverse function  $t(s)$  can be difficult or impossible to do analytically, so numerical methods or approximations may be necessary
- However, the construction is still useful for theoretical purposes and for understanding the properties of arc length parameterized curves

## Properties of arc length parameterized curves

- Arc length parameterized curves have several useful properties that make them important in differential geometry:
- The tangent vector  $\tilde{\gamma}'(s)$  has unit length for all  $s$ , i.e.,  $\|\tilde{\gamma}'(s)\| = 1$
- The arc length parameter  $s$  corresponds to the actual distance traveled along the curve from the starting point
- The integral of a function  $f$  along the curve simplifies to  $\int_{\tilde{\gamma}} f ds = \int_0^L f(\tilde{\gamma}(s)) ds$
- Geodesics (locally length-minimizing curves) are often parameterized by arc length
- These properties make arc length parameterization a natural and convenient choice for many problems in differential geometry, such as studying the curvature and torsion of curves, or finding the shortest path between two points on a surface

## Applications of arc length and reparameterization

- Arc length and reparameterization are fundamental concepts in differential geometry with a wide range of applications in mathematics, physics, and engineering
- These applications often involve studying the intrinsic geometry of curves and surfaces, or finding optimal paths and trajectories subject to certain constraints
- Some of the key applications of arc length and reparameterization include:



## Computation of curvature

- Curvature is a measure of how much a curve deviates from being a straight line at a given point
- For a regular curve  $\gamma : [a, b] \rightarrow \mathbb{R}^3$  in Euclidean space, the curvature at a point  $\gamma(t)$  is defined as:  
$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$
- This formula involves the first and second derivatives of the curve, which can be complicated to compute in an arbitrary parameterization
- However, if the curve is parameterized by arc length, then  $\|\gamma'(s)\| = 1$  and the formula simplifies to:  
$$\kappa(s) = \|\gamma''(s)\|$$
- This makes it much easier to compute and study the curvature of the curve, which has important applications in geometry, physics, and engineering (e.g., in the design of roads, railways, and mechanical parts)

## Frenet-Serret formulas

- The Frenet-Serret formulas are a set of equations that describe the kinematic properties of a regular curve in Euclidean space, such as its curvature and torsion
- They involve the Frenet frame, which is a moving orthonormal frame along the curve consisting of the tangent, normal, and binormal vectors
- For a regular curve  $\gamma : [a, b] \rightarrow \mathbb{R}^3$  parameterized by arc length, the Frenet-Serret formulas are:

$$T'(s) = \kappa(s)N(s)$$

$$N'(s) = -\kappa(s)T(s) + \tau(s)B(s)$$

$$B'(s) = -\tau(s)N(s)$$

where  $T, N, B$  are the tangent, normal, and binormal vectors, and  $\kappa, \tau$  are the curvature and torsion functions

- These formulas provide a powerful tool for studying the intrinsic geometry of curves and have applications in various fields, such as physics (e.g., in the study of particle trajectories and magnetic fields) and computer graphics (e.g., in the design of 3D curves and surfaces)

## Geodesics and shortest paths

- Geodesics are curves that locally minimize the distance between two points on a surface or a manifold
- They are the natural generalization of straight lines in Euclidean space and play a fundamental role in differential geometry and its applications
- Geodesics are often parameterized by arc length, which makes it easier to study their properties and compute their lengths
- In many applications, such as in robotics, computer vision, and geophysics, finding the shortest path between two points on a surface is an important problem that involves computing geodesics
- Reparameterization can also be used to simplify the equations of geodesics and make them easier to solve numerically, by choosing a parameterization that is adapted to the geometry of the surface or manifold

## Relationship between arc length and energy

- Arc length and energy are two closely related concepts in differential geometry and its applications
- The energy of a curve is a measure of how much the curve "stretches" or "bends" as it moves through space, and it can be used to characterize the shape and geometry of the curve
- In many problems, such as in the study of elasticity or the design of optimal trajectories, finding curves that minimize energy is an important goal

## Definition of energy of a curve

- For a regular curve  $\gamma : [a, b] \rightarrow M$  in a Riemannian manifold  $(M, g)$ , the energy of  $\gamma$  is defined as:  
$$E(\gamma) = \frac{1}{2} \int_a^b g_{\gamma(t)}(\gamma'(t), \gamma'(t)) dt$$
- This formula involves the Riemannian metric  $g$  and the velocity vector  $\gamma'(t)$  of the curve, and it measures the total "kinetic energy" of the curve as it moves through the manifold
- In Euclidean space, the energy formula simplifies to:  $E(\gamma) = \frac{1}{2} \int_a^b \|\gamma'(t)\|^2 dt$
- The energy of a curve is always non-negative and vanishes only for constant curves (i.e., curves whose velocity is zero)

## Energy minimizing curves

- A curve  $\gamma : [a, b] \rightarrow M$  is said to be energy minimizing if it minimizes the energy functional  $E(\gamma)$  among all curves with the same endpoints  $\gamma(a)$  and  $\gamma(b)$

## 2.3 Frenet-Serret formulas

The Frenet-Serret formulas are key tools in Metric Differential Geometry. They describe the movement of a particle along a curve in 3D space, using tangent, normal, and binormal vectors to create a moving coordinate system.

These formulas help us understand a curve's shape by measuring its curvature and torsion. They're used in physics, engineering, and computer graphics to analyze and create curves with specific properties in three-dimensional space.

## Definition of Frenet-Serret formulas

- Set of equations describing the kinematic properties of a particle moving along a continuous, differentiable curve in three-dimensional Euclidean space
- Formulas relate the tangent, normal, and binormal unit vectors to each other and to the curvature and torsion of the curve
- Fundamental tool in the study of curves and their geometric properties within the context of Metric Differential Geometry

## Derivation of Frenet-Serret formulas

- Derived by considering the motion of a particle along a curve and analyzing the changes in its velocity and acceleration vectors
- Involves the use of vector calculus and the concept of arc length parameterization

## Tangent vector

- Unit vector denoted as  $\vec{T}(s)$  pointing in the direction of the curve at each point
- Defined as the derivative of the position vector  $\vec{r}(s)$  with respect to the arc length parameter  $S$ :  
$$\vec{T}(s) = \frac{d\vec{r}}{ds}$$
- Always points in the direction of the curve's velocity

## Normal vector

- Unit vector denoted as  $\vec{N}(s)$  perpendicular to the tangent vector and pointing towards the center of curvature
- Defined as the derivative of the tangent vector with respect to the arc length parameter, divided by its magnitude: 
$$\vec{N}(s) = \frac{\frac{d\vec{T}}{ds}}{\left\| \frac{d\vec{T}}{ds} \right\|}$$
- Measures how the curve deviates from a straight line

## Binormal vector

- Unit vector denoted as  $\vec{B}(s)$  perpendicular to both the tangent and normal vectors, forming a right-handed orthonormal basis
- Defined as the cross product of the tangent and normal vectors: 
$$\vec{B}(s) = \vec{T}(s) \times \vec{N}(s)$$
- Measures the torsion or twisting of the curve out of the plane defined by the tangent and normal vectors

## Geometric interpretation of Frenet-Serret formulas

- Frenet-Serret frame provides a moving coordinate system along the curve, with the tangent, normal, and binormal vectors as the basis vectors
- Formulas describe how these basis vectors change as the particle moves along the curve

## Tangent vector direction

- Always points in the direction of the curve's velocity
- Indicates the instantaneous direction of motion at each point on the curve (e.g., the direction a car is moving along a winding road)

## Normal vector direction

- Points towards the center of curvature of the curve
- Indicates the direction in which the curve is turning at each point (e.g., the direction a car's steering wheel points when navigating a turn)

## Binormal vector direction

- Perpendicular to the plane containing the tangent and normal vectors
- Indicates the direction of the curve's torsion or twisting (e.g., the direction a roller coaster track twists as it spirals upward)

## Curvature in Frenet-Serret formulas

- Measures how much a curve deviates from a straight line at each point
- Quantifies the rate at which the tangent vector changes direction along the curve

### Definition of curvature

- Denoted as  $\kappa(s)$  and defined as the magnitude of the rate of change of the tangent vector with respect to the arc length parameter:  $\kappa(s) = \left\| \frac{d\vec{T}}{ds} \right\|$
- Higher curvature values indicate sharper turns or bends in the curve (e.g., a tight hairpin turn on a mountain road)

### Relationship between curvature and tangent vector

- Curvature is directly related to the rate of change of the tangent vector
- As the curvature increases, the tangent vector changes direction more rapidly (e.g., the tangent vector rotates quickly when moving along a highly curved section of a spiral)

## Torsion in Frenet-Serret formulas

- Measures how much a curve twists or deviates from a plane at each point
- Quantifies the rate at which the binormal vector changes direction along the curve

### Definition of torsion

- Denoted as  $\tau(s)$  and defined as the rate of change of the binormal vector with respect to the arc length parameter:  $\tau(s) = -\vec{N}(s) \cdot \frac{d\vec{B}}{ds}$
- Positive torsion indicates a right-handed twist, while negative torsion indicates a left-handed twist (e.g., the twisting of a DNA double helix)

### Relationship between torsion and binormal vector

- Torsion is directly related to the rate of change of the binormal vector
- As the torsion increases, the binormal vector rotates more rapidly around the tangent vector (e.g., the binormal vector rotates quickly when moving along a highly twisted section of a helix)

## Applications of Frenet-Serret formulas

- Frenet-Serret formulas have numerous applications in various fields, including physics, engineering, and computer graphics
- Enable the study and analysis of curves and their properties in three-dimensional space

## Curve reconstruction from curvature and torsion

- Given the curvature and torsion functions of a curve, the Frenet-Serret formulas can be used to reconstruct the original curve up to a rigid motion
- Allows for the creation of curves with desired properties or the analysis of existing curves (e.g., designing a roller coaster track with specific curvature and torsion profiles)

## Characterization of space curves

- Frenet-Serret formulas provide a way to classify and compare space curves based on their curvature and torsion
- Curves with the same curvature and torsion functions are congruent up to a rigid motion (e.g., identifying similar protein structures based on their backbone curves)

## Frenet-Serret formulas in higher dimensions

- Frenet-Serret formulas can be generalized to curves in  $n$ -dimensional Euclidean space
- Allows for the study of higher-dimensional curves and their geometric properties

## Generalization to $n$ -dimensional space

- In  $n$ -dimensional space, the Frenet-Serret frame consists of  $n$  orthonormal vectors: the tangent vector and  $(n-1)$  normal vectors
- Generalized curvatures, denoted as  $K_1(s), K_2(s), \dots, K_{n-1}(s)$ , measure the rate of change of each normal vector

## Frenet-Serret frame in higher dimensions

- The generalized Frenet-Serret formulas describe the relationships between the orthonormal vectors and the generalized curvatures
- Provide a complete description of the curve's geometry in  $n$ -dimensional space (e.g., analyzing the motion of a particle in a high-dimensional configuration space)

## Limitations of Frenet-Serret formulas

- Frenet-Serret formulas have certain limitations and may not be well-defined for all types of curves
- Understanding these limitations is essential for the proper application of the formulas

## Non-differentiable points on curves

- Frenet-Serret formulas require the curve to be continuously differentiable
- At non-differentiable points, such as cusps or corners, the formulas may not be well-defined (e.g., the tangent vector may not exist at a sharp corner)

## Vanishing curvature or torsion

- If the curvature or torsion vanishes at a point, the corresponding normal or binormal vector may not be well-defined
- Requires special treatment or alternative formulations to handle these cases (e.g., using a different parameterization or considering one-sided limits)

## Relationship to other concepts in differential geometry

- Frenet-Serret formulas are closely related to other important concepts in differential geometry

- Understanding these connections provides a deeper insight into the geometric properties of curves and surfaces

## Connection to parallel transport

- Parallel transport is the process of moving a vector along a curve while maintaining its angle with respect to the curve's tangent vector
- Frenet-Serret formulas describe how the tangent, normal, and binormal vectors change under parallel transport (e.g., the normal vector remains perpendicular to the tangent vector)

## Relationship to geodesics

- Geodesics are curves that represent the shortest path between two points on a surface
- In the context of Frenet-Serret formulas, geodesics are curves with zero geodesic curvature (i.e., the normal vector coincides with the surface normal)
- Frenet-Serret formulas can be used to study the properties of geodesics on surfaces (e.g., analyzing the curvature and torsion of a geodesic on a sphere)

## 2.4 Parametrized surfaces

Parametrized surfaces are a key concept in differential geometry, allowing us to study two-dimensional surfaces in three-dimensional space. They provide a mathematical framework for describing and analyzing the shape, curvature, and other properties of surfaces.

This topic covers the definition of parametrized surfaces, local parametrizations, Riemannian metrics, isometries, Gaussian curvature, geodesics, minimal surfaces, and the Gauss-Bonnet theorem. These concepts form the foundation for understanding the geometry of curved spaces.

### Definition of parametrized surfaces

- A parametrized surface is a smooth mapping  $X : U \rightarrow \mathbb{R}^3$  from an open set  $U \subset \mathbb{R}^2$  to three-dimensional Euclidean space  $\mathbb{R}^3$
- Parametrized surfaces provide a way to describe and study two-dimensional surfaces embedded in three-dimensional space, which is a fundamental concept in differential geometry
- The mapping  $X$  assigns to each point  $(u, v) \in U$  a point  $X(u, v) = (x(u, v), y(u, v), z(u, v))$  on the surface in  $\mathbb{R}^3$

### Local parametrizations

#### Coordinate patches

- A coordinate patch is a pair  $(U, X)$ , where  $U \subset \mathbb{R}^2$  is an open set and  $X : U \rightarrow \mathbb{R}^3$  is a smooth mapping
- Coordinate patches provide a local description of a surface, allowing for the study of local properties such as tangent planes, curvature, and geodesics
- The mapping  $X$  is often required to be a homeomorphism onto its image, ensuring that the patch is a faithful representation of the surface locally

## Transition maps

- Transition maps, also known as change of coordinate maps, describe how different coordinate patches of a surface are related to each other
- Given two overlapping coordinate patches  $(U, X)$  and  $(V, Y)$ , the transition map  $\varphi : X(U \cap V) \rightarrow Y(U \cap V)$  is defined by  $\varphi = Y \circ X^{-1}$
- Transition maps are required to be smooth (differentiable) to ensure that the surface is well-defined and has consistent properties across different patches

## Atlas for a surface

- An atlas for a surface is a collection of coordinate patches  $\{(U_i, X_i)\}_{i \in I}$  that cover the entire surface, such that any two overlapping patches are related by a smooth transition map
- The existence of an atlas ensures that the surface is a smooth manifold, allowing for the application of differential geometric techniques
- The study of atlases and their properties is central to the field of differential topology, which is closely related to differential geometry

## Riemannian metrics on surfaces

### First fundamental form

- The first fundamental form, denoted as  $I$ , is a quadratic form that measures the infinitesimal distance between points on a surface
- In local coordinates  $(u, v)$ , the first fundamental form is given by  $I = Edu^2 + 2Fdudv + Gdv^2$ , where  $E, F, G$  are the coefficients of the first fundamental form
- The coefficients  $E, F, G$  are determined by the inner products of the partial derivatives of the parametrization:  $E = \langle X_u, X_u \rangle$ ,  $F = \langle X_u, X_v \rangle$ ,  $G = \langle X_v, X_v \rangle$

### Induced metric from $\mathbb{R}^3$

- The induced metric on a surface is the Riemannian metric obtained by restricting the Euclidean metric of the ambient space  $\mathbb{R}^3$  to the surface
- Given a parametrized surface  $X : U \rightarrow \mathbb{R}^3$ , the induced metric is the first fundamental form determined by the coefficients  $E, F, G$  as described above
- The induced metric allows for the measurement of lengths, angles, and areas on the surface, as well as the computation of geometric quantities such as curvature and geodesics

## Isometries of surfaces

### Isometric surfaces

- Two surfaces  $S_1$  and  $S_2$  are said to be isometric if there exists a diffeomorphism  $f : S_1 \rightarrow S_2$  that preserves the Riemannian metric, i.e., the pullback of the metric on  $S_2$  by  $f$  equals the metric on  $S_1$
- Isometric surfaces have the same intrinsic geometry, meaning that they have the same Gaussian curvature, geodesics, and other metric-dependent properties

- Examples of isometric surfaces include a plane and a cylinder, or a catenoid and a helicoid

## Local isometries

- A local isometry between two surfaces  $S_1$  and  $S_2$  is a smooth mapping  $f: S_1 \rightarrow S_2$  that preserves the Riemannian metric in a neighborhood of each point
- Local isometries preserve the intrinsic geometry of the surfaces locally, but may not be globally one-to-one or onto
- The study of local isometries is important in understanding the local behavior of surfaces and their relationships to each other

## Gaussian curvature of surfaces

### Definition of Gaussian curvature

- The Gaussian curvature  $K$  of a surface at a point  $P$  is the product of the principal curvatures  $K_1$  and  $K_2$  at that point, i.e.,  $K = K_1 K_2$
- Principal curvatures are the eigenvalues of the shape operator (or Weingarten map), which describes how the surface bends in different directions at a given point
- Gaussian curvature is an intrinsic property of the surface, meaning that it depends only on the first fundamental form and not on the embedding of the surface in  $\mathbb{R}^3$

### Gaussian curvature in local coordinates

- In local coordinates  $(u, v)$ , the Gaussian curvature can be expressed in terms of the coefficients of the first and second fundamental forms
- The second fundamental form, denoted as  $II$ , measures how the surface bends in  $\mathbb{R}^3$  and is given by  $II = Ldu^2 + 2Mdudv + Ndv^2$ , where  $L, M, N$  are the coefficients of the second fundamental form
- The Gaussian curvature is then given by  $K = \frac{LN - M^2}{EG - F^2}$ , where  $E, F, G$  are the coefficients of the first fundamental form

## Theorema Egregium

- The Theorema Egregium, or "Remarkable Theorem," states that the Gaussian curvature of a surface is invariant under local isometries
- This means that if two surfaces are locally isometric, they must have the same Gaussian curvature at corresponding points
- The Theorema Egregium highlights the importance of Gaussian curvature as an intrinsic property of surfaces and has far-reaching consequences in the study of geometry and topology

## Geodesics on surfaces

### Definition of geodesics

- A geodesic on a surface is a curve that locally minimizes the distance between any two nearby points on the curve
- Geodesics are the generalization of straight lines in Euclidean space to curved surfaces



- On a parametrized surface, geodesics are characterized by a system of second-order differential equations called the geodesic equations

## Geodesic equations

- The geodesic equations are a system of second-order differential equations that describe the path of a geodesic on a surface in local coordinates
- In terms of the Christoffel symbols  $\Gamma_{ij}^k$ , which are determined by the coefficients of the first fundamental form, the geodesic equations are given by  $\ddot{u}^k + \Gamma_{ij}^k \dot{u}^i \dot{u}^j = 0$ , where  $u^k$  are the coordinates of the geodesic and dots represent derivatives with respect to the curve parameter
- Solving the geodesic equations allows for the computation of geodesics on a given surface and the study of their properties

## Clairaut's relation

- Clairaut's relation is a first integral of the geodesic equations for surfaces of revolution, providing a conserved quantity along geodesics on such surfaces
- For a surface of revolution with parametrization  $X(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$ , Clairaut's relation states that  $f(u)^2 \dot{v} = c$ , where  $c$  is a constant determined by the initial conditions of the geodesic
- Clairaut's relation simplifies the study of geodesics on surfaces of revolution and allows for the classification of different types of geodesic behavior

## Minimal surfaces

### Definition of minimal surfaces

- A minimal surface is a surface that locally minimizes its area among all surfaces with the same boundary
- Equivalently, a surface is minimal if its mean curvature, given by  $H = \frac{\kappa_1 + \kappa_2}{2}$ , vanishes identically
- Examples of minimal surfaces include the plane, the catenoid, and the helicoid

## Isothermal coordinates

- Isothermal coordinates are a special parametrization of a surface in which the coefficients of the first fundamental form satisfy  $E = G$  and  $F = 0$
- In isothermal coordinates, the first fundamental form takes the form  $I = \lambda^2(du^2 + dv^2)$ , where  $\lambda$  is a positive function called the conformal factor
- The existence of isothermal coordinates is guaranteed by the uniformization theorem, and they are particularly useful in the study of minimal surfaces and conformal geometry

## Weierstrass-Enneper parameterization

- The Weierstrass-Enneper parameterization is a method for constructing minimal surfaces using complex analysis
- Given two holomorphic functions  $\phi(z)$  and  $\psi(z)$  satisfying certain conditions, the Weierstrass-Enneper parameterization defines a minimal surface

$X(u, v) = \operatorname{Re} \int_{z_0}^z (\phi(z), i\phi(z), \psi(z)) dz$ , where  $z = u + iv$  is a complex variable

- This parameterization provides a powerful tool for generating and studying minimal surfaces, and has led to the discovery of many interesting examples

## Gauss-Bonnet theorem

### Statement of Gauss-Bonnet theorem

- The Gauss-Bonnet theorem relates the total Gaussian curvature of a closed surface to its Euler characteristic, a topological invariant
- For a closed surface  $S$ , the theorem states that  $\int_S K dA + \int_{\partial S} \kappa_g ds = 2\pi\chi(S)$ , where  $K$  is the Gaussian curvature,  $dA$  is the area element,  $\kappa_g$  is the geodesic curvature of the boundary (if any),  $ds$  is the line element along the boundary, and  $\chi(S)$  is the Euler characteristic of the surface
- The Gauss-Bonnet theorem is a deep result that connects the intrinsic geometry of a surface to its topology, and has numerous applications in mathematics and physics

### Applications of Gauss-Bonnet

- The Gauss-Bonnet theorem can be used to compute the Euler characteristic of a surface, which provides information about its topology (genus, number of holes, etc.)
- In physics, the theorem plays a crucial role in the study of general relativity and the geometry of spacetime, as it relates curvature to topological properties
- The Gauss-Bonnet theorem also has applications in the study of geometric flows, such as the Ricci flow, which has been used to prove the Poincaré conjecture and other important results in geometry and topology

### Classification of surfaces

- The Gauss-Bonnet theorem, combined with other results in differential geometry and topology, allows for the classification of closed surfaces based on their Euler characteristic
- Surfaces with Euler characteristic  $\chi = 2$  are topologically equivalent to a sphere, those with  $\chi = 0$  are equivalent to a torus (or a connected sum of tori), and those with  $\chi < 0$  are equivalent to a connected sum of projective planes
- This classification provides a powerful framework for understanding the global structure of surfaces and has far-reaching implications in various areas of mathematics, including algebraic geometry, complex analysis, and topology

## 2.5 First and second fundamental forms

The first and second fundamental forms are essential tools in differential geometry for studying surfaces. They provide a way to measure intrinsic and extrinsic properties, respectively. The first fundamental form deals with lengths, angles, and areas, while the second fundamental form captures curvature and shape.

These forms allow us to analyze surfaces without relying on their embedding in space. By using them, we can classify surface points, study minimal and developable surfaces, and understand geodesics. This topic connects intrinsic and extrinsic geometry, forming a foundation for deeper surface analysis.

## Definition of first fundamental form

- The first fundamental form is a quadratic form that measures intrinsic properties of a surface, such as length, angle, and area
- It is defined using the metric tensor, which is a symmetric, positive definite matrix that varies smoothly over the surface
- The first fundamental form is independent of the embedding of the surface in a higher-dimensional space and only depends on the intrinsic geometry of the surface

## Metric tensor in local coordinates

- In local coordinates  $(u, v)$  on a surface, the metric tensor  $g$  is given by  $g = Edu^2 + 2Fdudv + Gdv^2$
- The coefficients  $E$ ,  $F$ , and  $G$  are smooth functions of  $(u, v)$  and are called the first fundamental coefficients
- The metric tensor determines the inner product of tangent vectors and the length of curves on the surface
- For example, if  $(u, v)$  are orthogonal coordinates (such as latitude and longitude on a sphere), then  $F = 0$  and the metric tensor simplifies to  $g = Edu^2 + Gdv^2$

## Length of curves using first fundamental form

- The length of a curve  $\gamma(t) = (u(t), v(t))$  on a surface can be computed using the first fundamental form as  $L(\gamma) = \int \sqrt{E(u', v')^2 + 2F(u', v')(v', u') + G(v', u')^2} dt$
- Here,  $(u', v')$  denotes the derivative of  $(u, v)$  with respect to  $t$
- This formula generalizes the Pythagorean theorem to curved surfaces and reduces to the usual formula for length in Euclidean space when the surface is a plane

## Area of regions using first fundamental form

- The area of a region  $R$  on a surface can be computed using the first fundamental form as  $A(R) = \iint_R \sqrt{EG - F^2} dudv$
- This formula generalizes the formula for area in Euclidean space to curved surfaces and reduces to the usual formula when the surface is a plane
- The quantity  $\sqrt{EG - F^2}$  is called the area element and measures the local stretching or compression of area on the surface

## Conformal mappings and first fundamental form

- A mapping between two surfaces is called conformal if it preserves angles between curves
- Conformal mappings preserve the shape of infinitesimal figures on the surface but may change their size
- A mapping is conformal if and only if it preserves the first fundamental form up to a scalar factor, i.e.,  $\tilde{g} = \lambda g$  for some positive function  $\lambda$

- For example, the stereographic projection from a sphere to a plane is a conformal mapping that preserves angles but distorts areas

## Geometric meaning of first fundamental form

- The first fundamental form encodes the intrinsic geometry of a surface, which is the geometry that can be measured by an inhabitant of the surface without reference to the ambient space
- It allows us to define geometric quantities such as length, angle, and area on the surface using only intrinsic measurements

## Measuring angles on surfaces

- The angle between two curves on a surface can be computed using the first fundamental form and the inner product of their tangent vectors
- If  $\gamma_1(t)$  and  $\gamma_2(t)$  are two curves on a surface with tangent vectors  $v_1$  and  $v_2$ , then the angle  $\theta$  between them is given by  $\cos \theta = \frac{g(v_1, v_2)}{\sqrt{g(v_1, v_1)g(v_2, v_2)}}$
- This formula generalizes the formula for the angle between vectors in Euclidean space to curved surfaces

## Isometries and first fundamental form

- An isometry is a mapping between two surfaces that preserves the intrinsic geometry, i.e., it preserves lengths, angles, and areas
- Isometries preserve the first fundamental form, i.e., if  $f$  is an isometry between surfaces  $M$  and  $N$ , then  $f^* g_N = g_M$ , where  $f^*$  denotes the pullback of the metric tensor by  $f$
- For example, rotations and translations of a plane are isometries that preserve the Euclidean metric

## Gaussian curvature and first fundamental form

- The Gaussian curvature of a surface at a point can be computed using the first fundamental form and its derivatives
- The Gaussian curvature  $K$  is given by  $K = \frac{LN - M^2}{EG - F^2}$ , where  $L$ ,  $M$ , and  $N$  are the second fundamental coefficients
- The sign of the Gaussian curvature determines the local shape of the surface:  $K > 0$  for spherical points,  $K = 0$  for flat points, and  $K < 0$  for hyperbolic points
- The Gaussian curvature is an intrinsic quantity that depends only on the first fundamental form and not on the embedding of the surface in space

## Definition of second fundamental form

- The second fundamental form is a quadratic form that measures extrinsic properties of a surface, such as curvature and torsion
- It is defined using the shape operator, which is a self-adjoint linear operator that maps tangent vectors to tangent vectors
- The second fundamental form depends on the embedding of the surface in a higher-dimensional space and measures how the surface bends and twists in space

## Shape operator and second fundamental form

- The shape operator  $S$  is a self-adjoint linear operator that maps each tangent vector  $V$  to the derivative of the unit normal vector  $N$  in the direction of  $V$ , i.e.,  $SV = -dN_V$
- In local coordinates  $(u, v)$ , the shape operator is given by a symmetric matrix

$$S = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

, where  $L$ ,  $M$ , and  $N$  are smooth functions of  $(u, v)$  called the second fundamental coefficients

- The second fundamental form  $II$  is the quadratic form associated to the shape operator, i.e.,  $II(V, W) = g(SV, W)$  for tangent vectors  $V$  and  $W$

## Principal curvatures and second fundamental form

- The principal curvatures of a surface at a point are the eigenvalues of the shape operator at that point
- The principal curvatures  $K_1$  and  $K_2$  are the solutions of the characteristic equation  $\det(S - \kappa I) = 0$ , which can be written in terms of the second fundamental coefficients as  $\kappa^2 - (L + N)\kappa + (LN - M^2) = 0$
- The principal curvatures measure the maximum and minimum curvatures of the surface at a point and determine the local shape of the surface

## Mean curvature and second fundamental form

- The mean curvature of a surface at a point is the average of the principal curvatures at that point, i.e.,  $H = \frac{K_1 + K_2}{2}$
- The mean curvature can be computed using the second fundamental coefficients as  $H = \frac{L + N}{2\sqrt{EG - F^2}}$
- The mean curvature measures the average curvature of the surface at a point and determines the local bending of the surface

## Gaussian curvature and second fundamental form

- The Gaussian curvature of a surface at a point is the product of the principal curvatures at that point, i.e.,  $K = K_1 K_2$
- The Gaussian curvature can be computed using the second fundamental coefficients as  $K = \frac{LN - M^2}{EG - F^2}$
- The Gaussian curvature is an intrinsic quantity that can be computed using only the first fundamental form, but it can also be expressed in terms of the second fundamental form

## Geometric meaning of second fundamental form

- The second fundamental form encodes the extrinsic geometry of a surface, which is the geometry that depends on the embedding of the surface in a higher-dimensional space
- It allows us to define geometric quantities such as curvature and torsion that measure how the surface bends and twists in space

## Normal curvature and second fundamental form

- The normal curvature of a surface at a point in a given direction is the curvature of the curve obtained by intersecting the surface with the plane spanned by the unit normal vector and the given direction
- The normal curvature  $\kappa_n$  in the direction of a unit tangent vector  $V$  can be computed using the second fundamental form as  $\kappa_n = II(V, V) = L v_1^2 + 2M v_1 v_2 + N v_2^2$ , where  $V = (v_1, v_2)$  in local coordinates
- The maximum and minimum values of the normal curvature at a point are the principal curvatures, and the corresponding directions are the principal directions

## Principal directions and second fundamental form

- The principal directions of a surface at a point are the eigenvectors of the shape operator at that point
- The principal directions are orthogonal and correspond to the directions of maximum and minimum normal curvature
- The principal directions can be computed by solving the eigenvector equation  $SV = KV$ , which can be written in terms of the second fundamental coefficients as  $(L - \kappa)v_1 + Mv_2 = 0$  and  $Mv_1 + (N - \kappa)v_2 = 0$

## Umbilical points and second fundamental form

- An umbilical point is a point on a surface where the principal curvatures are equal, i.e.,  $\kappa_1 = \kappa_2$
- At an umbilical point, the shape operator is a scalar multiple of the identity, i.e.,  $S = KI$ , and the second fundamental form is a scalar multiple of the first fundamental form, i.e.,  $II = \kappa I$
- Umbilical points are the points where the surface is locally spherical, and the principal directions are not uniquely defined
- For example, every point on a sphere is an umbilical point, while a torus has no umbilical points

## Relationship between first and second fundamental forms

- The first and second fundamental forms are closely related and together determine the local geometry of a surface
- The Gaussian curvature can be expressed in terms of both fundamental forms, which leads to important relationships between intrinsic and extrinsic geometry

## Gaussian curvature in terms of fundamental forms

- The Gaussian curvature can be computed using only the first fundamental form as  $K = \frac{R_{1212}}{EG - F^2}$ , where  $R_{1212}$  is the Riemann curvature tensor of the surface
- The Gaussian curvature can also be computed using both fundamental forms as  $K = \frac{LN - M^2}{EG - F^2}$ , which shows that it depends on the embedding of the surface in space
- The equality of these two expressions for Gaussian curvature is a fundamental result in differential geometry and relates intrinsic and extrinsic curvature

## Codazzi-Mainardi equations

- The Codazzi-Mainardi equations are a system of partial differential equations that relate the first and second fundamental forms
- In local coordinates  $(u, v)$ , the Codazzi-Mainardi equations are given by  $L_v - M_u = \Gamma_{12}^1 L + \Gamma_{12}^2 M - \Gamma_{11}^1 M - \Gamma_{11}^2 N$  and  $M_v - N_u = \Gamma_{22}^1 L + \Gamma_{22}^2 M - \Gamma_{12}^1 M - \Gamma_{12}^2 N$ , where  $\Gamma_{ij}^k$  are the Christoffel symbols of the surface
- The Codazzi-Mainardi equations are necessary and sufficient conditions for a pair of quadratic forms to be the first and second fundamental forms of a surface

## Fundamental theorem of surface theory

- The fundamental theorem of surface theory states that a surface is uniquely determined up to isometry by its first and second fundamental forms
- More precisely, if two surfaces have the same first and second fundamental forms, then there exists an isometry between them that preserves the fundamental forms
- The fundamental theorem of surface theory is a powerful result that allows us to study surfaces intrinsically by specifying their fundamental forms, without referring to their embedding in space

## Applications of fundamental forms

- The first and second fundamental forms have numerous applications in differential geometry, physics, and engineering
- They can be used to classify points on surfaces, study special classes of surfaces, and analyze the behavior of geodesics

## Classification of points on surfaces

- The first and second fundamental forms can be used to classify points on surfaces into different types based on their local geometry
- Elliptic points are points where the Gaussian curvature is positive, i.e.,  $K > 0$ , and the surface is locally spherical
- Hyperbolic points are points where the Gaussian curvature is negative, i.e.,  $K < 0$ , and the surface is locally saddle-shaped
- Parabolic points are points where the Gaussian curvature is zero, i.e.,  $K = 0$ , and one of the principal curvatures is zero
- Planar points are points where both principal curvatures are zero, i.e.,  $K_1 = K_2 = 0$ , and the surface is locally flat

## Minimal surfaces and fundamental forms

- A minimal surface is a surface that locally minimizes its area among all surfaces with the same boundary
- Minimal surfaces have zero mean curvature, i.e.,  $H = 0$ , which means that their principal curvatures are equal and opposite, i.e.,  $K_1 = -K_2$

- The first and second fundamental forms of a minimal surface satisfy the equation  $EN - 2FM + GL = 0$ , which is a necessary and sufficient condition for a surface to be minimal
- Examples of minimal surfaces include the catenoid, the helicoid, and the Enneper surface

## Developable surfaces and fundamental forms

- A developable surface is a surface that can be flattened onto a plane without stretching or tearing
- Developable surfaces have zero Gaussian curvature, i.e.,  $K = 0$ , which means that one of their principal curvatures is zero at every point
- The first and second fundamental forms of a developable surface satisfy the equation  $LN - M^2 = 0$ , which is a necessary and sufficient condition for a surface to be developable
- Examples of developable surfaces include cylinders, cones, and tangent developables of curves

## Geodesics and fundamental forms

- A geodesic is a curve on a surface that locally minimizes the distance between its endpoints
- Geodesics are the straightest possible curves on a surface and are determined by the first fundamental form
- The geodesic equations are a system of ordinary differential equations that describe the motion of a particle on a surface under the influence of the metric
- In local coordinates  $(u, v)$ , the geodesic equations are given by  $u'' + \Gamma_{11}^1(u')^2 + 2\Gamma_{12}^1u'v' + \Gamma_{22}^1(v')^2 = 0$  and  $v'' + \Gamma_{11}^2(u')^2 + 2\Gamma_{12}^2u'v' + \Gamma_{22}^2(v')^2 = 0$ , where  $\Gamma_{ij}^k$  are the Christoffel symbols of the surface
- The behavior of geodesics on a surface is closely related to its curvature, and the deviation of geodesics from straight lines is measured by the geodesic curvature, which can be expressed in terms of the first and second fundamental forms

## 2.6 Gaussian and mean curvatures

Gaussian and mean curvatures are key concepts in differential geometry that describe how surfaces bend and curve. They provide crucial insights into a surface's shape, helping us understand its local and global properties.

These curvatures play a vital role in various fields, from pure mathematics to physics and computer graphics. By studying their properties and relationships, we gain a deeper understanding of surface geometry and its applications in the real world.

### Definition of Gaussian curvature

- Gaussian curvature is a fundamental concept in differential geometry that measures the intrinsic curvature of a surface at a given point
- It quantifies how the surface bends in different directions and provides insight into the local geometry of the surface



## Intrinsic definition

- The intrinsic definition of Gaussian curvature relies solely on measurements within the surface itself, without referring to the ambient space in which the surface is embedded
- It can be defined as the product of the principal curvatures at a point, which are the maximum and minimum curvatures of curves passing through that point
- Alternatively, it can be expressed in terms of the Riemannian metric and its derivatives using the Theorema Egregium

## Extrinsic definition

- The extrinsic definition of Gaussian curvature considers the surface as embedded in a higher-dimensional space (usually Euclidean space)
- It is defined as the determinant of the shape operator (also known as the second fundamental form) at a given point
- The shape operator measures how the surface normal changes as one moves along the surface

## Gauss map

- The Gauss map is a continuous map from a surface to the unit sphere that assigns each point on the surface to its unit normal vector
- It provides a way to study the geometry of the surface by analyzing how the normal vector field varies across the surface
- The Jacobian determinant of the Gauss map at a point is equal to the Gaussian curvature at that point

## Relationship between intrinsic and extrinsic definitions

- The Theorema Egregium, proved by Gauss, states that the Gaussian curvature of a surface can be computed solely from the first fundamental form (intrinsic properties) without referring to the embedding of the surface in a higher-dimensional space
- This remarkable result establishes the equivalence between the intrinsic and extrinsic definitions of Gaussian curvature
- It implies that Gaussian curvature is an intrinsic property of the surface and remains invariant under isometric deformations

## Properties of Gaussian curvature

- Gaussian curvature possesses several important properties that shed light on its geometric significance and behavior under various transformations
- Understanding these properties is crucial for analyzing and classifying surfaces based on their curvature characteristics

## Invariance under isometries

- Gaussian curvature is an intrinsic property of a surface, meaning it remains unchanged under isometric transformations (distance-preserving mappings)

- Isometries include rigid motions such as translations, rotations, and reflections
- This invariance allows for the study of surfaces up to isometry and the classification of surfaces based on their Gaussian curvature

## Local nature of Gaussian curvature

- Gaussian curvature is a local property, meaning it depends only on the local geometry of the surface in a neighborhood of a point
- It can vary from point to point on the surface, allowing for the existence of regions with different curvature characteristics (positive, negative, or zero curvature)
- The local nature of Gaussian curvature enables the analysis of surface features and the identification of special points (umbilical points, saddle points, etc.)

## Relation to principal curvatures

- The Gaussian curvature at a point is equal to the product of the principal curvatures at that point
- Principal curvatures ( $k_1$  and  $k_2$ ) are the maximum and minimum values of the normal curvatures of curves passing through the point
- The sign of the Gaussian curvature determines the local shape of the surface:
  - Positive curvature ( $k_1 k_2 > 0$ ): elliptic point, locally resembling a sphere or ellipsoid
  - Negative curvature ( $k_1 k_2 < 0$ ): hyperbolic point, locally resembling a saddle
  - Zero curvature ( $k_1 k_2 = 0$ ): parabolic point, locally resembling a cylinder or plane

## Gaussian curvature of surfaces of revolution

- Surfaces of revolution are generated by rotating a curve (profile curve) around an axis
- The Gaussian curvature of a surface of revolution can be computed using a simple formula involving the profile curve and its derivatives
- For a profile curve  $\gamma(u) = (f(u), 0, g(u))$  rotated around the Z-axis, the Gaussian curvature at a point  $(u, v)$  is given by:

$$K(u, v) = \frac{f''(u)g'(u) - f'(u)g''(u)}{(f'(u)^2 + g'(u)^2)^2}$$

- This formula allows for the efficient computation and analysis of the Gaussian curvature of surfaces of revolution

## Gauss-Bonnet theorem

- The Gauss-Bonnet theorem is a fundamental result in differential geometry that relates the Gaussian curvature of a surface to its topology
- It establishes a deep connection between the local geometry of a surface and its global topological properties

## Statement of the theorem

- For a compact, oriented, two-dimensional Riemannian manifold  $M$  with boundary  $\partial M$ , the Gauss-Bonnet theorem states:

$$\int_M K dA + \int_{\partial M} k_g ds = 2\pi\chi(M)$$

where: -  $K$  is the Gaussian curvature of  $M$  -  $dA$  is the area element of  $M$  -  $k_g$  is the geodesic curvature of  $\partial M$  -  $ds$  is the line element of  $\partial M$  -  $\chi(M)$  is the Euler characteristic of  $M$

## Proof of the theorem

- The proof of the Gauss-Bonnet theorem involves several key steps: 1. Triangulating the surface  $M$  into a finite number of geodesic triangles 2. Expressing the integral of Gaussian curvature over each triangle in terms of the interior angles using the Gauss-Bonnet formula for geodesic triangles 3. Summing the contributions from all triangles and applying the angle-sum formula for the Euler characteristic 4. Taking the limit as the triangulation becomes finer and using the additivity of integration to obtain the final result

## Applications of the theorem

- The Gauss-Bonnet theorem has numerous applications in geometry, topology, and physics, including:
- Classification of compact surfaces based on their Euler characteristic
- Computation of total curvature for closed surfaces
- Study of geodesics and shortest paths on surfaces
- Analysis of defects and singularities in physical systems (dislocations, vortices, etc.)

## Topological implications

- The Gauss-Bonnet theorem reveals a deep connection between the Gaussian curvature and the topology of a surface
- It implies that the total Gaussian curvature of a closed surface is a topological invariant, determined solely by its Euler characteristic
- Surfaces with the same Euler characteristic have the same total Gaussian curvature, regardless of their specific geometry
- The theorem provides a powerful tool for understanding the global structure of surfaces and their topological properties

## Definition of mean curvature

- Mean curvature is another important concept in differential geometry that measures the average curvature of a surface at a given point
- It quantifies the overall bending of the surface and provides information about its extrinsic geometry

## Extrinsic definition

- The extrinsic definition of mean curvature considers the surface as embedded in a higher-dimensional space (usually Euclidean space)

- It is defined as the arithmetic mean of the principal curvatures at a point:

$$H = \frac{k_1 + k_2}{2}$$

where  $k_1$  and  $k_2$  are the principal curvatures

- Mean curvature measures the average rate of change of the surface normal in the principal directions

## Relation to principal curvatures

- The mean curvature is directly related to the principal curvatures of the surface
- It is the average of the maximum and minimum curvatures at a point
- The sign of the mean curvature provides information about the local shape of the surface:
  - Positive mean curvature: the surface is locally convex (bulging outward)
  - Negative mean curvature: the surface is locally concave (bulging inward)
  - Zero mean curvature: the surface is locally minimal (has equal principal curvatures with opposite signs)

## Mean curvature of surfaces of revolution

- For surfaces of revolution, the mean curvature can be computed using a formula involving the profile curve and its derivatives
- Given a profile curve  $\gamma(u) = (f(u), 0, g(u))$  rotated around the  $Z$ -axis, the mean curvature at a point  $(u, v)$  is:

$$H(u, v) = \frac{f''(u)g'(u) - f'(u)g''(u)}{2(f'(u)^2 + g'(u)^2)^{3/2}} + \frac{g'(u)}{2f(u)\sqrt{f'(u)^2 + g'(u)^2}}$$

- This formula simplifies the computation of mean curvature for surfaces of revolution and enables their analysis and classification

## Properties of mean curvature

- Mean curvature exhibits several interesting properties that highlight its geometric significance and behavior under certain transformations
- Understanding these properties is essential for studying surfaces and their interaction with physical phenomena

## Invariance under conformal mappings

- Mean curvature is invariant under conformal mappings, which are angle-preserving transformations that locally scale distances
- Conformal mappings preserve the mean curvature of a surface at each point
- This invariance property is particularly useful in the study of minimal surfaces and conformal geometry

## Variational characterization

- Mean curvature has a variational characterization in terms of the area functional

- It arises as the Euler-Lagrange equation for the problem of minimizing the surface area subject to certain constraints
- Surfaces with constant mean curvature are critical points of the area functional and satisfy the minimal surface equation

## Relation to minimal surfaces

- Minimal surfaces are surfaces with zero mean curvature at every point
- They locally minimize the surface area and have equal principal curvatures with opposite signs
- Examples of minimal surfaces include the catenoid, helicoid, and Enneper's surface
- The study of minimal surfaces is a rich area of research in differential geometry, with connections to various fields such as physics, materials science, and architecture

## Mean curvature flow

- Mean curvature flow is a geometric evolution equation that describes the deformation of a surface over time
- The surface evolves in the direction of its mean curvature vector, which is proportional to the Laplacian of the surface position
- Mean curvature flow has applications in image processing, surface smoothing, and the study of geometric singularities
- It provides a powerful tool for understanding the behavior of surfaces under curvature-driven deformations

## Relationship between Gaussian and mean curvatures

- Gaussian curvature and mean curvature are two fundamental measures of surface curvature that provide complementary information about the geometry of a surface
- Understanding their relationship and the special cases that arise from certain curvature conditions is crucial for the classification and analysis of surfaces

## Surfaces with constant Gaussian curvature

- Surfaces with constant Gaussian curvature have uniform intrinsic curvature throughout
- They can be classified into three categories based on the sign of the Gaussian curvature:
  - Positive constant Gaussian curvature: spheres and ellipsoids
  - Zero Gaussian curvature: planes, cylinders, and cones
  - Negative constant Gaussian curvature: hyperbolic surfaces (pseudosphere)
- Surfaces with constant Gaussian curvature have special geometric properties and are of interest in various fields, including geometry, physics, and computer graphics

## Surfaces with constant mean curvature

- Surfaces with constant mean curvature have uniform average curvature at every point

- They include minimal surfaces (zero mean curvature) and non-minimal surfaces with non-zero constant mean curvature
- Examples of surfaces with constant mean curvature are spheres, cylinders, and Delaunay surfaces (unduloids, nodoids, catenoids)
- These surfaces arise in physical phenomena such as soap films, bubbles, and capillary surfaces

## Minimal surfaces and their curvatures

- Minimal surfaces have zero mean curvature at every point, implying that their principal curvatures are equal in magnitude but opposite in sign
- The Gaussian curvature of a minimal surface is non-positive (zero or negative) at every point
- Minimal surfaces locally minimize the surface area and have fascinating geometric and topological properties
- Examples of minimal surfaces include the catenoid, helicoid, Enneper's surface, and the Scherk surface

## Gaussian vs mean curvature in surface classification

- The signs of Gaussian and mean curvatures provide a way to classify surface points into different categories:
- Elliptic points: positive Gaussian curvature, positive or negative mean curvature
- Hyperbolic points: negative Gaussian curvature, positive or negative mean curvature
- Parabolic points: zero Gaussian curvature, non-zero mean curvature
- Planar points: zero Gaussian curvature, zero mean curvature
- This classification scheme helps in understanding the local geometry of surfaces and identifying special points and regions of interest

## Computational aspects

- Computing Gaussian and mean curvatures is essential for various applications in computer graphics, computer vision, and numerical analysis
- Several computational methods and tools are available for estimating and visualizing curvatures on discrete surfaces or from surface parametrizations

## Calculating Gaussian curvature from parametrizations

- Given a parametric surface  $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ , the Gaussian curvature can be computed using the first and second fundamental forms
- The coefficients of the first fundamental form  $(E, F, G)$  and the second fundamental form  $(L, M, N)$  are calculated from the partial derivatives of  $\mathbf{r}$
- The Gaussian curvature is then given by:

$$K = \frac{LN - M^2}{EG - F^2}$$

- This formula allows for the computation of Gaussian curvature at any point on a parametric surface

## Calculating mean curvature from parametrizations

- The mean curvature of a parametric surface can also be computed using the coefficients of the first and second fundamental forms
- The formula for mean curvature in terms of these coefficients is:

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)}$$

- By evaluating this expression at different parameter values, the mean curvature can be calculated at any point on the surface

## Numerical methods for estimating curvatures

- For discrete surfaces represented by meshes or point clouds, numerical methods are employed to estimate Gaussian and mean curvatures
- Some common approaches include:
  - Finite difference methods: approximating derivatives using neighboring vertices or faces
  - Integral methods: estimating curvatures by integrating over local neighborhoods
  - Fitting methods: fitting local surface patches (polynomials, quadrics) to estimate curvatures
- These numerical methods provide approximate curvature values and are widely used in geometry processing and analysis tasks

## Software tools for visualizing curvatures

- Various software tools and libraries are available for visualizing Gaussian and mean curvatures on surfaces
- Some popular options include:
  - ParaView: an open-source, multi-platform data analysis and visualization application
  - MeshLab: an open-source system for processing and editing 3D triangular meshes
  - MATLAB: a numerical computing environment with built-in functions for surface curvature computation and visualization
  - Python libraries: NumPy, SciPy, and Matplotlib for numerical computations and plotting
- These tools enable the visual exploration of curvature distributions, the identification of surface features, and the analysis of geometric properties

# Unit 3 – Riemannian metrics

## 3.1 Riemannian manifolds

Riemannian manifolds combine smooth structures with geometric properties, allowing us to generalize concepts from Euclidean geometry to curved spaces. They provide a framework for studying the interplay between topology, geometry, and analysis in diverse mathematical and physical contexts.

This chapter explores the fundamental aspects of Riemannian manifolds, including metrics, connections, curvature, and geodesics. We'll examine how these concepts shape our understanding of curved spaces and their applications in various fields of mathematics and physics.

### Definition of Riemannian manifolds

- Riemannian manifolds are fundamental objects in Metric Differential Geometry that combine the smooth structure of differentiable manifolds with the geometric properties induced by a Riemannian metric
- The study of Riemannian manifolds allows for the generalization of many concepts from Euclidean geometry to curved spaces, enabling a deeper understanding of the interplay between topology, geometry, and analysis

### Smooth manifolds with Riemannian metrics

- A Riemannian manifold is a smooth manifold  $M$  equipped with a Riemannian metric  $\mathcal{G}$ , which is a smoothly varying family of inner products on the tangent spaces  $T_p M$  at each point  $p \in M$
- The Riemannian metric  $\mathcal{G}$  allows for the measurement of angles, lengths, and volumes on the manifold, providing a notion of distance and geometry that is compatible with the smooth structure
- Examples of smooth manifolds include differentiable curves, surfaces (2-sphere, torus), and higher-dimensional spaces (n-spheres, projective spaces)

### Tangent spaces and inner products

- At each point  $p$  of a smooth manifold  $M$ , the tangent space  $T_p M$  is a vector space that captures the infinitesimal behavior of smooth curves passing through  $p$
- The Riemannian metric  $\mathcal{G}$  assigns an inner product  $\mathcal{G}_p$  to each tangent space  $T_p M$ , allowing for the computation of angles and lengths of tangent vectors
- The inner product  $\mathcal{G}_p$  is a symmetric, positive definite bilinear form that varies smoothly with respect to the point  $p$  on the manifold

### Riemannian metrics

- Riemannian metrics are the fundamental objects that endow smooth manifolds with geometric properties, enabling the measurement of distances, angles, and curvature
- The choice of a Riemannian metric on a manifold determines its geometric structure and influences the behavior of geodesics, isometries, and other geometric quantities



## Positive definite symmetric bilinear forms

- A Riemannian metric  $\mathcal{G}$  on a smooth manifold  $M$  assigns to each point  $p \in M$  a positive definite symmetric bilinear form  $g_p$  on the tangent space  $T_p M$
- Positive definiteness ensures that the metric assigns a positive length to all non-zero tangent vectors, while symmetry implies that  $g_p(v, w) = g_p(w, v)$  for all tangent vectors  $v, w \in T_p M$
- The bilinearity of  $g_p$  means that it is linear in each argument separately, satisfying properties such as  $g_p(av + bw, u) = ag_p(v, u) + bg_p(w, u)$  for scalars  $a, b$  and tangent vectors  $u, v, w$

## Induced norms and distances

- The Riemannian metric  $\mathcal{G}$  induces a norm on each tangent space  $T_p M$  given by  $\|v\|_p = \sqrt{g_p(v, v)}$  for  $v \in T_p M$ , measuring the length of tangent vectors
- The induced norm allows for the definition of the length of a smooth curve  $\gamma : [a, b] \rightarrow M$  as  $L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)} dt$ , where  $\gamma'(t)$  is the tangent vector to the curve at  $\gamma(t)$
- The Riemannian distance between two points  $p, q \in M$  is defined as the infimum of the lengths of all smooth curves connecting  $p$  and  $q$ , providing a notion of shortest paths on the manifold

## Length of curves on Riemannian manifolds

- The Riemannian metric allows for the computation of the length of a smooth curve  $\gamma : [a, b] \rightarrow M$  using the formula  $L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt$
- Minimizing the length functional among all smooth curves connecting two points leads to the concept of geodesics, which are the shortest paths between points on the manifold
- The length of curves plays a crucial role in the study of geometric properties such as the diameter of a manifold, the injectivity radius, and the behavior of geodesics

## Levi-Civita connection

- The Levi-Civita connection is a fundamental object in Riemannian geometry that provides a way to differentiate vector fields and study parallel transport and geodesics on a Riemannian manifold
- It is a unique connection that is both metric-compatible and torsion-free, ensuring compatibility with the Riemannian metric and symmetry of the connection coefficients

## Uniqueness of Levi-Civita connection

- The Levi-Civita connection is uniquely determined by the Riemannian metric  $\mathcal{G}$  on a smooth manifold  $M$
- The uniqueness follows from the requirements of metric compatibility, which means that the connection preserves the metric under parallel transport, and the torsion-free property, which ensures symmetry of the connection coefficients
- The existence and uniqueness of the Levi-Civita connection is a fundamental result in Riemannian geometry, known as the Fundamental Theorem of Riemannian Geometry

## Christoffel symbols and metric compatibility

- The Christoffel symbols  $\Gamma_{ij}^k$  are the components of the Levi-Civita connection with respect to a chosen coordinate system on the Riemannian manifold
- They are determined by the Riemannian metric  $g$  through the formula  $\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ , where  $g^{kl}$  are the components of the inverse metric tensor
- The metric compatibility of the Levi-Civita connection is expressed by the condition  $\nabla_X g = 0$  for any vector field  $X$ , ensuring that the metric is preserved under parallel transport

## Parallel transport and geodesics

- Parallel transport is a way to move tangent vectors along curves on a Riemannian manifold while preserving their angle and length with respect to the Levi-Civita connection
- A vector field  $V$  along a curve  $\gamma$  is said to be parallel if  $\nabla_{\gamma'(t)} V = 0$  for all  $t$ , where  $\nabla$  is the Levi-Civita connection and  $\gamma'(t)$  is the tangent vector to the curve at  $\gamma(t)$
- Geodesics are curves on a Riemannian manifold that parallel transport their own tangent vector, characterized by the equation  $\nabla_{\gamma'(t)} \gamma'(t) = 0$ , and they represent the shortest paths between points on the manifold

## Curvature of Riemannian manifolds

- The curvature of a Riemannian manifold measures the deviation of the manifold from being flat or Euclidean, and it encodes important geometric and topological properties
- There are various notions of curvature, including the Riemann curvature tensor, sectional curvature, Ricci curvature, and scalar curvature, each capturing different aspects of the manifold's geometry

## Riemann curvature tensor

- The Riemann curvature tensor  $R$  is a (1,3)-tensor that measures the non-commutativity of the Levi-Civita connection and the deviation of parallel transport from being path-independent
- In local coordinates, the components of the Riemann curvature tensor are given by  $R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{im}^l \Gamma_{jk}^m - \Gamma_{jm}^l \Gamma_{ik}^m$ , where  $\Gamma_{ij}^k$  are the Christoffel symbols of the Levi-Civita connection
- The Riemann curvature tensor satisfies symmetries such as  $R_{ijkl} = -R_{jikl} = -R_{ijlk}$  and the Bianchi identity  $\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0$

## Sectional curvature and Ricci curvature

- Sectional curvature is a scalar-valued function that assigns a real number to each 2-dimensional subspace (or plane) of the tangent space at a point on the manifold
- For a pair of linearly independent tangent vectors  $u, v$ , the sectional curvature  $K(u, v)$  is defined as  $K(u, v) = \frac{R(u, v, u, v)}{|u \wedge v|^2}$ , where  $R$  is the Riemann curvature tensor and  $|u \wedge v|$  is the area of the parallelogram spanned by  $u$  and  $v$
- Ricci curvature is a contraction of the Riemann curvature tensor, obtained by summing over two of its indices, and it provides a measure of the average sectional curvature at a point

- In local coordinates, the Ricci curvature tensor is given by  $Ric_{jk} = R^i_{j k i}$ , and it is a symmetric (0,2)-tensor that captures the trace of the Riemann curvature tensor

## Scalar curvature and Einstein manifolds

- Scalar curvature is a further contraction of the Ricci curvature tensor, obtained by taking its trace with respect to the metric
- In local coordinates, the scalar curvature is given by  $S = g^{jk} Ric_{jk}$ , where  $g^{jk}$  are the components of the inverse metric tensor
- Einstein manifolds are Riemannian manifolds whose Ricci curvature tensor is proportional to the metric tensor, satisfying the equation  $Ric = \lambda g$  for some constant  $\lambda$
- Examples of Einstein manifolds include Euclidean spaces, spheres with the round metric, and hyperbolic spaces with the hyperbolic metric

## Geodesics on Riemannian manifolds

- Geodesics are the natural generalization of straight lines in Euclidean space to curved Riemannian manifolds, representing the shortest paths between points
- The study of geodesics is central to Riemannian geometry, as they encode important geometric and topological properties of the manifold

## Geodesic equations and exponential map

- Geodesics on a Riemannian manifold are characterized by the geodesic equations, which are a system of second-order ordinary differential equations
- In local coordinates, the geodesic equations are given by  $\ddot{\gamma}^k + \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0$ , where  $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$  is a curve on the manifold and  $\Gamma^k_{ij}$  are the Christoffel symbols of the Levi-Civita connection
- The exponential map  $\exp_p : T_p M \rightarrow M$  at a point  $p$  on the manifold maps a tangent vector  $v \in T_p M$  to the point  $\exp_p(v)$  obtained by following the unique geodesic starting at  $p$  with initial velocity  $v$  for a unit time

## Minimizing properties of geodesics

- Geodesics are locally distance-minimizing curves, meaning that for sufficiently small segments, they provide the shortest path between their endpoints
- The minimizing property of geodesics is a consequence of the fact that they are critical points of the energy functional  $E(\gamma) = \frac{1}{2} \int_a^b g_{\gamma(t)}(\gamma'(t), \gamma'(t)) dt$  among all curves connecting two points
- However, geodesics may fail to be globally minimizing, as there can exist multiple geodesics connecting two points on the manifold (conjugate points)

## Geodesic completeness and Hopf-Rinow theorem

- A Riemannian manifold is said to be geodesically complete if every geodesic can be extended indefinitely in both directions

- Geodesic completeness is a global property that ensures the existence of geodesics between any two points on the manifold
- The Hopf-Rinow theorem states that for a connected Riemannian manifold, the following conditions are equivalent: 1. The manifold is geodesically complete 2. The exponential map  $\exp_p$  is defined on the entire tangent space  $T_p M$  for every point  $p$  on the manifold 3. Every closed and bounded subset of the manifold is compact

## Isometries and Riemannian submersions

- Isometries and Riemannian submersions are important tools in the study of Riemannian manifolds, as they allow for the comparison and relation of different geometric structures
- Isometries preserve the Riemannian metric and geometric properties, while Riemannian submersions provide a way to map between Riemannian manifolds while preserving certain geometric features

## Definition and properties of isometries

- An isometry between two Riemannian manifolds  $(M, g)$  and  $(N, h)$  is a diffeomorphism  $f: M \rightarrow N$  that preserves the Riemannian metrics, i.e.,  $f^* h = g$ , where  $f^*$  denotes the pullback of the metric
- Isometries preserve distances, angles, and geodesics between the manifolds, providing a notion of equivalence in Riemannian geometry
- The set of all isometries of a Riemannian manifold forms a Lie group under composition, called the isometry group of the manifold

## Riemannian submersions and O'Neill's formulas

- A Riemannian submersion is a smooth surjective map  $\pi: (M, g) \rightarrow (N, h)$  between Riemannian manifolds that satisfies the following conditions: 1. For each  $p \in M$ , the differential  $d\pi_p: T_p M \rightarrow T_{\pi(p)} N$  is surjective 2. The restriction of  $d\pi_p$  to the orthogonal complement of its kernel (called the horizontal space) is an isometry
- Riemannian submersions allow for the construction of new Riemannian manifolds from existing ones, such as quotient spaces and warped products
- O'Neill's formulas relate the curvature of the total space  $M$  to the curvature of the base space  $N$  and the geometry of the fibers, providing a way to compute curvature in terms of the submersion

## Riemannian coverings and lifting properties

- A Riemannian covering is a special case of a Riemannian submersion where the map  $\pi: (M, g) \rightarrow (N, h)$  is a covering map, meaning that every point in  $N$  has an evenly covered neighborhood
- Riemannian coverings have the property that geodesics, isometries, and other geometric objects can be lifted from the base space  $N$  to the total space  $M$
- The lifting properties of Riemannian coverings allow for the study of the geometry and topology of the base space in terms of the often simpler total space

## Riemannian submanifolds

- Riemannian submanifolds are subsets of a Riemannian manifold that inherit a Riemannian metric from the ambient space, allowing for the study of their intrinsic and extrinsic geometry

- The geometry of Riemannian submanifolds is characterized by the induced metric, the second fundamental form, and the relationship between the curvature of the submanifold and the ambient manifold

## Induced Riemannian metrics on submanifolds

- Let  $(M, g)$  be a Riemannian manifold and  $N \subset M$  a smooth submanifold. The induced Riemannian metric on  $N$  is the pullback of the metric  $g$  to the tangent spaces of  $N$ , given by  $h = i^*g$ , where  $i : N \hookrightarrow M$  is the inclusion map
- The induced metric makes  $(N, h)$  a Riemannian manifold in its own right, allowing for the study of its intrinsic geometric properties such as geodesics, curvature, and isometries
- Examples of Riemannian submanifolds include hypersurfaces (codimension-1 submanifolds), geodesic submanifolds (submanifolds containing the geodesics of the ambient space)

## 3.2 Metric tensor

The metric tensor is a fundamental concept in differential geometry that defines geometric properties of manifolds. It allows us to measure distances, angles, and curvature on curved spaces, providing a mathematical framework for understanding the shape of our universe.

This powerful tool finds applications in physics, particularly in general relativity and string theory. By describing the geometry of spacetime, the metric tensor helps us model gravitational phenomena and explore the nature of reality at its most fundamental level.

### Definition of metric tensor

- The metric tensor is a fundamental object in differential geometry that defines the geometric properties of a manifold, such as distances, angles, and curvature
- It is a symmetric, positive definite, and coordinate-free tensor field on a manifold

### Symmetric covariant tensor

- The metric tensor is a symmetric covariant tensor of rank 2, denoted as  $g_{ij}$  or  $ds^2 = g_{ij}dx^i dx^j$
- Symmetry means that  $g_{ij} = g_{ji}$  for all  $i$  and  $j$
- Covariance ensures that the metric tensor transforms correctly under coordinate changes

### Positive definiteness

- The metric tensor is positive definite, meaning that for any non-zero tangent vector  $v$ ,  $g(v, v) > 0$
- Positive definiteness ensures that the metric tensor defines a genuine inner product on the tangent space at each point of the manifold
- This property is crucial for defining lengths, angles, and volumes on the manifold

### Coordinate-free definition

- The metric tensor can be defined in a coordinate-free manner as a smooth, symmetric, positive definite section of the tensor product of the cotangent bundle with itself

- This definition emphasizes the intrinsic geometric nature of the metric tensor, independent of any particular choice of coordinates
- The coordinate-free definition allows for a more abstract and general treatment of the metric tensor and its properties

## Components of metric tensor

- The metric tensor can be expressed in terms of its components in a given coordinate system, which are functions of the coordinates

## Covariant components

- The covariant components of the metric tensor are denoted as  $g_{ij}$ , where  $i$  and  $j$  are indices running from 1 to the dimension of the manifold
- Covariant components transform according to the tensor transformation law under coordinate changes
- In a coordinate basis, the line element is given by  $ds^2 = g_{ij}dx^i dx^j$

## Contravariant components

- The contravariant components of the metric tensor are denoted as  $g^{ij}$  and are obtained by inverting the matrix of covariant components
- Contravariant components transform according to the inverse tensor transformation law under coordinate changes
- The contravariant metric tensor is used for raising indices of other tensors

## Transformation laws

- Under a coordinate transformation  $x^i \rightarrow x'^i$ , the covariant components of the metric tensor transform as  $g'_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} g_{kl}$
- The contravariant components transform as  $g'^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} g^{kl}$
- These transformation laws ensure that the metric tensor remains a tensor under coordinate changes

## Metric tensor in local coordinates

- In local coordinates, the metric tensor can be represented by its components, which are functions of the coordinates

## Metric coefficients

- The components of the metric tensor in a given coordinate system are called metric coefficients
- Metric coefficients are often denoted as  $g_{ij}(x)$ , where  $x$  represents the coordinates
- The metric coefficients determine the geometric properties of the manifold in the given coordinate system

## First fundamental form

- In the context of surfaces embedded in a higher-dimensional space, the metric tensor induces the first fundamental form
- The first fundamental form is a quadratic form that measures distances and angles on the surface
- It is given by  $ds^2 = g_{ij}du^i du^j$ , where  $u^i$  are the coordinates on the surface

## Induced metric on submanifolds

- When a submanifold is embedded in a higher-dimensional manifold with a metric tensor, the submanifold inherits an induced metric
- The induced metric is obtained by restricting the ambient metric tensor to the tangent spaces of the submanifold
- The induced metric allows for the study of the intrinsic geometry of the submanifold

## Geometric meaning of metric tensor

- The metric tensor encodes various geometric properties of a manifold, such as distances, angles, and volumes

## Measuring distances and angles

- The metric tensor allows for the computation of distances between points on the manifold using the line element  $ds^2 = g_{ij}dx^i dx^j$
- The angle between two tangent vectors  $u$  and  $v$  at a point is given by  $\cos \theta = \frac{g(u, v)}{\sqrt{g(u, u)}\sqrt{g(v, v)}}$
- The metric tensor provides a way to measure lengths of curves and angles between vectors on the manifold

## Volume element

- The metric tensor determines the volume element on the manifold, which is used for integration
- In local coordinates, the volume element is given by  $dV = \sqrt{|\det(g_{ij})|} dx^1 \wedge \cdots \wedge dx^n$
- The volume element allows for the computation of volumes, surface areas, and other measures on the manifold

## Raising and lowering indices

- The metric tensor can be used to raise or lower the indices of other tensors
- Raising an index is achieved by contracting with the contravariant metric tensor, e.g.,  $v^i = g^{ij}v_j$
- Lowering an index is achieved by contracting with the covariant metric tensor, e.g.,  $v_i = g_{ij}v^j$
- Raising and lowering indices allows for the conversion between covariant and contravariant components of tensors

## Metric tensor in special cases

- The metric tensor takes on specific forms in certain special cases, depending on the geometry of the manifold

### Euclidean metric

- In Euclidean space, the metric tensor is represented by the identity matrix, i.e.,  $g_{ij} = \delta_{ij}$
- The Euclidean metric is flat and corresponds to the familiar notions of distance and angle in Euclidean geometry
- The line element in Euclidean space is given by  $ds^2 = dx^2 + dy^2 + dz^2$  in Cartesian coordinates

### Minkowski metric

- In special relativity, the metric tensor is the Minkowski metric, which has a signature of  $(-, +, +, +)$
- The Minkowski metric describes the geometry of spacetime in flat spacetime
- The line element in Minkowski spacetime is given by  $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$  in Cartesian coordinates

## Riemannian vs pseudo-Riemannian metrics

- Riemannian metrics are positive definite metric tensors, meaning that  $g(v, v) > 0$  for all non-zero tangent vectors  $v$
- Pseudo-Riemannian metrics, such as the Minkowski metric, have indefinite signature and allow for the existence of null vectors (vectors with zero length)
- Riemannian metrics are used in the study of Riemannian geometry, while pseudo-Riemannian metrics are used in the study of spacetime geometry and general relativity

## Metric tensor and connections

- The metric tensor is closely related to the concept of a connection on a manifold, which describes how tangent vectors are transported along curves

### Levi-Civita connection

- The Levi-Civita connection is a unique, metric-compatible, and torsion-free connection associated with a given metric tensor
- Metric-compatibility means that the covariant derivative of the metric tensor vanishes, i.e.,  $\nabla_k g_{ij} = 0$
- Torsion-free means that the torsion tensor of the connection vanishes, i.e.,  $T^i_{jk} = 0$

## Christoffel symbols

- The Christoffel symbols are the components of the Levi-Civita connection in a given coordinate system
- They are defined in terms of the metric tensor and its derivatives as  $\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^l} - \frac{\partial g_{jl}}{\partial x^k} \right)$



- Christoffel symbols appear in the expressions for covariant derivatives and geodesic equations

## Parallel transport and geodesics

- Parallel transport is the process of moving a tangent vector along a curve while preserving its direction relative to the connection
- Geodesics are curves that parallel transport their own tangent vector, i.e., the covariant derivative of the tangent vector along the curve vanishes
- The geodesic equation, which describes the paths of freely falling particles, is given by 
$$\frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

## Curvature and the metric tensor

- The curvature of a manifold is determined by the metric tensor and its derivatives

## Riemann curvature tensor

- The Riemann curvature tensor is a tensor that measures the intrinsic curvature of a manifold
- It is defined in terms of the Christoffel symbols and their derivatives as 
$$R_{jkl}^i = \frac{\partial \Gamma_{jl}^i}{\partial x^k} - \frac{\partial \Gamma_{jk}^i}{\partial x^l} + \Gamma_{mk}^i \Gamma_{jl}^m - \Gamma_{ml}^i \Gamma_{jk}^m$$
- The Riemann tensor satisfies certain symmetry properties and the Bianchi identities

## Ricci tensor and scalar curvature

- The Ricci tensor is obtained by contracting the Riemann tensor, i.e.,  $R_{ij} = R_{ikj}^k$
- The scalar curvature is the contraction of the Ricci tensor with the metric tensor, i.e.,  $R = g^{ij} R_{ij}$
- The Ricci tensor and scalar curvature provide scalar measures of the curvature of a manifold

## Einstein tensor and field equations

- The Einstein tensor is a symmetric tensor defined as  $G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$
- The Einstein field equations of general relativity relate the Einstein tensor to the stress-energy tensor  $T_{ij}$ , which describes the distribution of matter and energy
- The Einstein field equations are given by  $G_{ij} = \frac{8\pi G}{c^4} T_{ij}$ , where  $G$  is Newton's gravitational constant and  $c$  is the speed of light

## Physical applications of metric tensor

- The metric tensor plays a crucial role in various areas of physics, particularly in the description of spacetime and gravitational phenomena

## Metric in general relativity

- In general relativity, the metric tensor describes the geometry of spacetime and how it is affected by the presence of matter and energy
- The Einstein field equations relate the curvature of spacetime, described by the metric tensor, to the distribution of matter and energy

- Solutions to the Einstein field equations, such as the Schwarzschild metric and the Kerr metric, describe the spacetime geometry around massive objects and rotating black holes

## Metric in string theory

- In string theory, the metric tensor appears in the action that describes the dynamics of strings propagating in a background spacetime
- The metric tensor determines the geometry of the target space in which the strings live and influences their interactions
- Different background geometries, such as Calabi-Yau manifolds or AdS spaces, lead to different physical properties of the string theory

## Metric in continuum mechanics

- In continuum mechanics, the metric tensor is used to describe the deformation and strain of continuous media, such as solids and fluids
- The metric tensor relates the reference configuration of a material to its deformed configuration and allows for the computation of strain measures
- The metric tensor plays a role in the formulation of constitutive equations that relate stress and strain in the material

## 3.3 Induced metrics on submanifolds

Induced metrics on submanifolds are a key concept in Metric Differential Geometry. They allow us to study the intrinsic geometry of subsets within larger spaces, connecting local properties to global structures.

By examining how metrics from ambient spaces transfer to submanifolds, we gain insights into curvature, geodesics, and other geometric features. This topic bridges intrinsic and extrinsic perspectives, revealing the interplay between a submanifold's internal structure and its embedding.

## Submanifolds in metric spaces

- Submanifolds are subsets of a metric space that locally resemble Euclidean space and inherit the metric structure from the ambient space
- The study of submanifolds is a central topic in Metric Differential Geometry as it allows for the analysis of geometric properties and structures within a larger space
- Submanifolds provide a framework for understanding the interplay between the intrinsic geometry of the submanifold and the extrinsic geometry induced by the ambient space

## Definition of submanifolds

- A subset  $M$  of a metric space  $X$  is called a submanifold if for every point  $p \in M$ , there exists a neighborhood  $U \subset X$  of  $p$  and a homeomorphism  $\phi : U \rightarrow V \subset \mathbb{R}^n$  such that  $\phi(U \cap M) = V \cap (\mathbb{R}^k \times \{0\})$  for some  $k \leq n$
- The dimension of the submanifold  $M$  is defined as the dimension  $k$  of the Euclidean space  $\mathbb{R}^k$  in the local parametrization
- Submanifolds can be characterized by their local charts and transition maps, which are compatible with the subspace topology and the metric structure

## Topological properties of submanifolds

- Submanifolds inherit the topological properties of the ambient space, such as separability, connectedness, and compactness
- The subspace topology on a submanifold coincides with the topology induced by the metric structure
- Submanifolds can be classified according to their topological properties, such as being closed, open, or properly embedded

## Smooth structures on submanifolds

- A submanifold  $M$  of a smooth manifold  $X$  is called a smooth submanifold if the inclusion map  $i : M \rightarrow X$  is a smooth immersion
- Smooth submanifolds inherit the smooth structure from the ambient manifold, allowing for the study of differential geometric properties
- The tangent space of a smooth submanifold can be identified with a subspace of the tangent space of the ambient manifold at each point

## Riemannian metrics on submanifolds

- Riemannian metrics on submanifolds describe the intrinsic geometry of the submanifold and its relationship to the ambient space
- The study of Riemannian metrics on submanifolds is crucial for understanding curvature, geodesics, and other geometric properties
- Submanifolds with induced Riemannian metrics provide a rich setting for exploring the interplay between intrinsic and extrinsic geometry

## Restriction of ambient metric

- Given a Riemannian manifold  $(X, g)$  and a submanifold  $M \subset X$ , the restriction of the metric  $g$  to the tangent spaces of  $M$  defines a Riemannian metric on the submanifold
- The restricted metric, denoted by  $g|_M$ , is compatible with the smooth structure on the submanifold and induces an inner product on each tangent space
- The restriction of the ambient metric allows for the study of the intrinsic geometry of the submanifold using tools from Riemannian geometry

## Intrinsic vs extrinsic geometry

- Intrinsic geometry refers to the properties of a submanifold that depend only on the induced Riemannian metric and not on the embedding in the ambient space (e.g., Gaussian curvature of a surface)
- Extrinsic geometry describes the properties of a submanifold that depend on its embedding in the ambient space and the relationship between the intrinsic and ambient metrics (e.g., mean curvature of a hypersurface)
- The interplay between intrinsic and extrinsic geometry is a central theme in the study of submanifolds and their geometric properties

## Induced Riemannian metrics

- An induced Riemannian metric on a submanifold  $M$  is obtained by restricting the ambient metric to the tangent spaces of  $M$  and extending it to a smooth tensor field
- The induced metric is uniquely determined by the embedding of the submanifold and is compatible with the submanifold topology and smooth structure
- Induced Riemannian metrics allow for the study of geometric properties, such as curvature and geodesics, using techniques from Riemannian geometry

## Induced metrics and isometric embeddings

- Induced metrics and isometric embeddings are fundamental concepts in the study of submanifolds and their geometric properties
- The relationship between induced metrics and isometric embeddings provides a framework for understanding the intrinsic geometry of submanifolds and their relationship to the ambient space
- Characterizing induced metrics and isometric embeddings is essential for classifying submanifolds and studying their geometric properties

## Definition of induced metric

- Given a smooth map  $f: M \rightarrow N$  between Riemannian manifolds  $(M, g)$  and  $(N, h)$ , the induced metric on  $M$  is defined as  $f^*h$ , where  $(f^*h)(X, Y) = h(df(X), df(Y))$  for all vector fields  $X, Y$  on  $M$
- The induced metric is a symmetric, positive-definite  $(0, 2)$ -tensor field on  $M$  that is compatible with the smooth structure and topology of the submanifold
- The induced metric captures the intrinsic geometry of the submanifold as determined by the embedding map  $f$

## Isometric embeddings and immersions

- An isometric embedding is a smooth map  $f: (M, g) \rightarrow (N, h)$  between Riemannian manifolds such that  $f^*h = g$ , i.e., the induced metric on  $M$  coincides with the given metric  $g$
- An isometric immersion is a smooth map  $f: (M, g) \rightarrow (N, h)$  that is an immersion (i.e.,  $df$  is injective at each point) and satisfies  $f^*h = g$
- Isometric embeddings and immersions preserve the intrinsic geometry of the submanifold and provide a way to realize abstract Riemannian manifolds as submanifolds of other spaces

## Characterization of induced metrics

- A Riemannian metric  $g$  on a manifold  $M$  is said to be induced by an embedding if there exists a Riemannian manifold  $(N, h)$  and an embedding  $f: M \rightarrow N$  such that  $f^*h = g$
- The Nash Embedding Theorem states that every Riemannian manifold can be isometrically embedded into a Euclidean space of sufficiently high dimension
- Characterizing induced metrics helps in understanding the relationship between the intrinsic geometry of a submanifold and its extrinsic geometry in the ambient space

## Geometric properties of submanifolds

- The geometric properties of submanifolds, such as curvature and minimality, provide insights into the intrinsic and extrinsic geometry of the submanifold
- Studying the geometric properties of submanifolds is essential for understanding their shape, behavior, and relationship to the ambient space
- Metric Differential Geometry provides tools and techniques for analyzing the geometric properties of submanifolds and their variations

## First and second fundamental forms

- The first fundamental form of a submanifold is the induced Riemannian metric, which captures the intrinsic geometry of the submanifold
- The second fundamental form is a symmetric bilinear form that measures the extrinsic curvature of the submanifold and its deviation from being totally geodesic
- The first and second fundamental forms together describe the local geometry of the submanifold and its relationship to the ambient space

## Gaussian and mean curvature

- Gaussian curvature is an intrinsic measure of curvature for surfaces, defined as the product of the principal curvatures at each point
- Mean curvature is an extrinsic measure of curvature for hypersurfaces, defined as the average of the principal curvatures at each point
- Gaussian and mean curvature provide important information about the shape and behavior of submanifolds, such as the presence of umbilical points or the minimality of the submanifold

## Minimal submanifolds and variations

- A submanifold is called minimal if its mean curvature vanishes identically, i.e., it locally minimizes area among nearby submanifolds
- Minimal submanifolds are characterized by the vanishing of the first variation of the area functional, which leads to the minimal surface equation
- The study of minimal submanifolds and their variations is a rich area of research in Metric Differential Geometry, with applications in geometry, topology, and mathematical physics

## Examples and applications

- Concrete examples and applications of submanifolds help to illustrate the abstract concepts and techniques developed in Metric Differential Geometry
- Studying specific classes of submanifolds, such as surfaces in Euclidean space or hypersurfaces in Riemannian manifolds, provides insights into the general theory and its implications
- Applications of submanifold theory can be found in various areas of mathematics, physics, and engineering, such as the study of geometric flows, relativity theory, and computer graphics

## Surfaces in Euclidean space

- Surfaces in Euclidean space, such as spheres, cylinders, and tori, serve as basic examples of submanifolds and their geometric properties
- The induced Riemannian metric on a surface in Euclidean space is related to the first fundamental form, which can be expressed in terms of the local parametrization
- The Gaussian curvature of a surface in Euclidean space can be computed using the first and second fundamental forms, and it provides information about the intrinsic geometry of the surface

## Hypersurfaces in Riemannian manifolds

- Hypersurfaces are codimension-one submanifolds of Riemannian manifolds, such as hyperspheres in Euclidean space or hypersurfaces in Lorentzian manifolds
- The geometry of hypersurfaces is governed by the induced Riemannian metric and the second fundamental form, which captures the extrinsic curvature
- Hypersurfaces play a crucial role in the study of geometric flows, such as mean curvature flow, and in the formulation of the positive mass theorem in general relativity

## Geodesic submanifolds and totally geodesic submanifolds

- A submanifold is called geodesic if every geodesic in the submanifold with respect to the induced metric is also a geodesic in the ambient space
- A submanifold is called totally geodesic if every geodesic in the ambient space that is tangent to the submanifold at one point remains in the submanifold
- Geodesic and totally geodesic submanifolds are important examples of submanifolds with special geometric properties, and they arise naturally in the study of symmetric spaces and homogeneous manifolds

## Connections and parallel transport

- Connections and parallel transport are fundamental tools in Metric Differential Geometry for studying the intrinsic geometry of submanifolds and their relationship to the ambient space
- The Levi-Civita connection on a submanifold is induced by the Levi-Civita connection on the ambient space and is compatible with the induced Riemannian metric
- Parallel transport and geodesics on submanifolds are closely related to the intrinsic and extrinsic geometry of the submanifold, as described by the Gauss and Codazzi-Mainardi equations

## Levi-Civita connection on submanifolds

- The Levi-Civita connection on a Riemannian manifold is a unique torsion-free metric connection that is compatible with the Riemannian metric
- The Levi-Civita connection on a submanifold is induced by the Levi-Civita connection on the ambient space and satisfies the properties of metric compatibility and torsion-freeness
- The induced Levi-Civita connection on a submanifold allows for the study of parallel transport, geodesics, and curvature on the submanifold using techniques from Riemannian geometry

## Parallel transport and geodesics

- Parallel transport is a way of moving tangent vectors along curves in a manifold while preserving their angle and length with respect to the Riemannian metric
- Geodesics are curves that minimize the distance between points and are characterized by the property that their tangent vectors remain parallel along the curve with respect to the Levi-Civita connection
- The relationship between parallel transport and geodesics on submanifolds is governed by the Gauss and Codazzi-Mainardi equations, which relate the intrinsic and extrinsic curvature of the submanifold

## Gauss and Codazzi-Mainardi equations

- The Gauss equation relates the intrinsic curvature of a submanifold (as measured by the Riemann curvature tensor of the induced metric) to the extrinsic curvature (as measured by the second fundamental form) and the curvature of the ambient space
- The Codazzi-Mainardi equation is a compatibility condition between the first and second fundamental forms, expressing the symmetry of the second fundamental form with respect to the induced Levi-Civita connection
- The Gauss and Codazzi-Mainardi equations provide a fundamental link between the intrinsic and extrinsic geometry of submanifolds and are essential tools in the study of submanifolds and their geometric properties

## Submanifolds of special Riemannian manifolds

- The study of submanifolds of special Riemannian manifolds, such as spaces of constant curvature, symmetric spaces, and Kähler manifolds, reveals important connections between the intrinsic geometry of the submanifold and the geometric structure of the ambient space
- Submanifolds of special Riemannian manifolds often inherit additional geometric properties and symmetries from the ambient space, leading to interesting classes of submanifolds with rich geometric structures
- Understanding the behavior of submanifolds in special Riemannian manifolds is important for applications in various areas of mathematics and physics, such as the study of minimal surfaces, Einstein manifolds, and harmonic maps

## Submanifolds of spaces of constant curvature

- Spaces of constant curvature, such as Euclidean spaces, spheres, and hyperbolic spaces, are Riemannian manifolds with constant sectional curvature
- Submanifolds of spaces of constant curvature inherit special geometric properties, such as the existence of a parallel second fundamental form or the property of being totally umbilical
- The study of submanifolds in spaces of constant curvature is closely related to the theory of isometric immersions and the classification of space forms

## Submanifolds of symmetric spaces

- Symmetric spaces are Riemannian manifolds with a rich algebraic and geometric structure, characterized by the existence of symmetries at each point that preserve the Riemannian metric

- Submanifolds of symmetric spaces, such as Lie groups or Grassmannians, often inherit the symmetries of the ambient space and can be studied using techniques from representation theory and Lie theory
- The geometry of submanifolds in symmetric spaces is related to the theory of Jordan algebras, isoparametric submanifolds, and the study of homogeneous spaces

## Kähler submanifolds of Kähler manifolds

- Kähler manifolds are complex manifolds equipped with a compatible Riemannian metric and a complex structure that satisfies the Kähler condition
- Kähler submanifolds are complex submanifolds of Kähler manifolds that inherit the Kähler structure from the ambient space, i.e., the induced metric and complex structure satisfy the Kähler condition
- The study of Kähler submanifolds is important in complex differential geometry and has applications in the theory of minimal surfaces, Einstein manifolds, and the geometry of moduli spaces

## 3.4 Length and volume

Length and volume are fundamental concepts in metric differential geometry. They quantify the extent of curves and regions in manifolds, providing essential tools for measuring and comparing geometric objects. These concepts form the basis for understanding distances, shortest paths, and the intrinsic properties of curved spaces.

Arc length, geodesics, and Riemannian volume play crucial roles in this topic. They allow us to calculate distances along curves, find optimal paths between points, and measure the size of subsets in manifolds. These ideas connect local geometric properties to global topological features, as seen in the Gauss-Bonnet theorem and isoperimetric inequalities.

### Definition of length

- Length is a fundamental concept in metric differential geometry that quantifies the extent of curves in a manifold
- The definition of length relies on the notion of arc length, which measures the distance traveled along a curve
- Length is an intrinsic property of a curve that does not depend on the choice of coordinates or parameterization

### Arc length of curves

- Arc length is defined as the integral of the speed of a curve over the interval of parameterization
- For a curve  $\gamma : [a, b] \rightarrow M$  in a Riemannian manifold  $(M, g)$ , the arc length is given by 
$$L(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} dt$$
- Arc length measures the distance traveled along the curve, taking into account the metric structure of the manifold
- The arc length of a curve is independent of the parameterization, as long as the orientation is preserved



## Additivity property

- The arc length satisfies the additivity property, which states that the length of a concatenation of curves is the sum of their individual lengths
- Formally, if  $\gamma_1 : [a, b] \rightarrow M$  and  $\gamma_2 : [b, c] \rightarrow M$  are two curves such that  $\gamma_1(b) = \gamma_2(b)$ , then  $L(\gamma_1 * \gamma_2) = L(\gamma_1) + L(\gamma_2)$ , where  $\gamma_1 * \gamma_2$  denotes the concatenation of the curves
- The additivity property allows for the computation of lengths of piecewise smooth curves by breaking them into smaller segments

## Invariance under reparameterization

- The arc length of a curve is invariant under reparameterization, meaning that it does not change if the curve is reparameterized while preserving its orientation
- If  $\gamma : [a, b] \rightarrow M$  is a curve and  $\phi : [c, d] \rightarrow [a, b]$  is a smooth, monotonically increasing function, then the reparameterized curve  $\tilde{\gamma} = \gamma \circ \phi$  has the same arc length as  $\gamma$
- Invariance under reparameterization ensures that the length of a curve is an intrinsic property that depends only on the geometric shape of the curve, not on the specific parameterization chosen

## Calculating arc length

- Computing the arc length of a curve involves evaluating the integral that defines it, which can be done using various techniques depending on the representation of the curve
- The choice of parameterization and the dimension of the ambient space affect the complexity of the arc length calculation

## Line integrals

- In Euclidean space, the arc length of a curve can be expressed as a line integral of the Euclidean metric along the curve
- For a curve  $\gamma(t) = (x(t), y(t), z(t))$  in  $\mathbb{R}^3$ , the arc length is given by 
$$L(\gamma) = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} dt$$
- Line integrals provide a straightforward way to compute arc lengths in Euclidean space, as long as the curve is given in a suitable parameterization

## Parametric equations

- Parametric equations describe a curve by expressing each coordinate as a function of a parameter, typically denoted by  $t$
- In the case of a curve  $\gamma(t) = (x(t), y(t))$  in the plane, the arc length can be computed using the formula 
$$L(\gamma) = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$$
- Parametric equations allow for the calculation of arc lengths by substituting the coordinate functions into the integral and evaluating it over the parameter interval

## Examples in different dimensions

- In one dimension, the arc length of a curve  $\gamma(t) = f(t)$  on an interval  $[a, b]$  is simply the absolute value of the difference of the function values at the endpoints,  $L(\gamma) = |f(b) - f(a)|$

- For a curve  $\gamma(t) = (x(t), y(t))$  in the plane, the arc length is given by  $L(\gamma) = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$ , which can be evaluated using techniques from calculus
- In higher dimensions, such as for a curve  $\gamma(t) = (x_1(t), \dots, x_n(t))$  in  $\mathbb{R}^n$ , the arc length is computed using the formula

$$L(\gamma) = \int_a^b \sqrt{\sum_{i=1}^n (dx_i/dt)^2} dt$$

, which involves the Euclidean metric in  $n$  dimensions

## Geodesics

- Geodesics are curves that generalize the notion of straight lines to curved spaces, such as Riemannian manifolds
- They play a central role in metric differential geometry, as they represent the shortest paths between points and capture the intrinsic geometry of the manifold

## Definition and properties

- A geodesic is a curve  $\gamma : I \rightarrow M$  in a Riemannian manifold  $(M, g)$  that locally minimizes the distance between any two nearby points on the curve
- Geodesics are characterized by the property that their acceleration vector is always perpendicular to the tangent space of the manifold
- Geodesics have constant speed, meaning that the norm of their velocity vector remains constant along the curve
- Geodesics are locally length-minimizing, but they may not be globally length-minimizing if the manifold has a complex topology or if the points are far apart

## Geodesic equations

- The geodesic equations are a system of second-order differential equations that characterize geodesics in a Riemannian manifold
- In local coordinates  $(x^1, \dots, x^n)$ , the geodesic equations are given by  $\ddot{x}^k + \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0$ , where  $\Gamma_{ij}^k$  are the Christoffel symbols of the Levi-Civita connection
- The geodesic equations express the fact that the acceleration of a geodesic is always perpendicular to the tangent space
- Solving the geodesic equations with given initial conditions (a point and a tangent vector) yields the unique geodesic passing through that point with the specified initial velocity

## Geodesics on surfaces

- On a surface  $(S, g)$  embedded in  $\mathbb{R}^3$ , geodesics can be visualized as curves that minimize the distance between points while staying on the surface
- The geodesic equations on a surface simplify to a system of two second-order differential equations, as the surface has only two local coordinates
- Examples of geodesics on surfaces include great circles on a sphere (the shortest paths between points on the sphere's surface) and straight lines on a plane (which is a flat surface)

- The study of geodesics on surfaces provides insight into the intrinsic geometry of the surface and helps characterize its curvature properties

## Shortest paths

- In metric differential geometry, the concept of shortest paths is closely related to geodesics, as geodesics are often the curves that minimize the distance between points on a manifold
- The existence and uniqueness of shortest paths depend on the geometric properties of the manifold, such as its curvature and topology

## Geodesics as shortest paths

- Geodesics are locally length-minimizing curves, meaning that they minimize the distance between nearby points along the curve
- In many cases, geodesics are also globally length-minimizing, providing the shortest path between any two points on the manifold
- On complete Riemannian manifolds with non-positive sectional curvature (such as the Euclidean space or the hyperbolic space), geodesics are always globally length-minimizing
- However, on manifolds with positive curvature (such as the sphere), geodesics may not be globally length-minimizing, as there can be multiple geodesics connecting two points

## Existence and uniqueness

- The existence of shortest paths between points on a Riemannian manifold depends on the completeness of the manifold and the distance between the points
- On complete Riemannian manifolds (manifolds where geodesics can be extended indefinitely), the Hopf-Rinow theorem guarantees the existence of a shortest path between any two points
- The uniqueness of shortest paths is related to the presence of conjugate points along geodesics
- If there are no conjugate points along a geodesic, then it is the unique shortest path between its endpoints
- In the presence of conjugate points, there may be multiple geodesics of the same length connecting two points, leading to non-uniqueness of shortest paths

## Variational principles

- Shortest paths can be characterized using variational principles, which seek to minimize the length functional among all curves connecting two points
- The length functional  $L(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} dt$  assigns a length value to each curve  $\gamma$  connecting two points
- Shortest paths are the curves that minimize the length functional, and they can be found by solving the corresponding Euler-Lagrange equations
- The Euler-Lagrange equations for the length functional lead to the geodesic equations, confirming that geodesics are the stationary points of the length functional
- Variational principles provide a powerful tool for studying shortest paths and their properties, as well as for deriving the geodesic equations in a coordinate-free manner

## Riemannian volume

- Riemannian volume is a measure that quantifies the size of subsets of a Riemannian manifold, taking into account the metric structure
- It generalizes the concept of volume from Euclidean space to curved spaces, allowing for the integration of functions on manifolds

## Volume element

- The volume element on a Riemannian manifold  $(M, g)$  is a differential form that assigns a volume density to each tangent space
- In local coordinates  $(x^1, \dots, x^n)$ , the volume element is given by  $dV = \sqrt{|\det(g_{ij})|} dx^1 \wedge \dots \wedge dx^n$ , where  $g_{ij}$  are the components of the Riemannian metric tensor
- The volume element is invariant under isometries of the manifold, meaning that it does not change under coordinate transformations that preserve the metric
- The volume of a subset  $U \subset M$  is obtained by integrating the volume element over  $U$ ,  $Vol(U) = \int_U dV$

## Integration on manifolds

- Integration on Riemannian manifolds is performed using the volume element, which provides a natural measure for the manifold
- Given a function  $f: M \rightarrow \mathbb{R}$  on a Riemannian manifold  $(M, g)$ , its integral over a subset  $U \subset M$  is defined as  $\int_U f dV = \int_U f(x) \sqrt{|\det(g_{ij})|} dx^1 \dots dx^n$
- Integration on manifolds allows for the computation of volumes, averages, and other geometric quantities that depend on the metric structure
- The divergence theorem and Stokes' theorem generalize to Riemannian manifolds, relating the integral of a function over a region to the integral of its derivative over the boundary

## Examples of volume calculations

- On the Euclidean space  $\mathbb{R}^n$  with the standard metric, the volume element reduces to the usual Lebesgue measure  $dV = dx^1 \dots dx^n$ , and integration corresponds to the standard Riemann integral
- For a sphere  $S^n$  of radius  $r$  in  $\mathbb{R}^{n+1}$ , the volume element is given by  $dV = r^n \sin^{n-1}(\theta_1) \sin^{n-2}(\theta_2) \dots \sin(\theta_{n-1}) d\theta_1 \dots d\theta_n$ , where  $(\theta_1, \dots, \theta_n)$  are the spherical coordinates
- The volume of a ball  $B(p, r)$  of radius  $r$  centered at a point  $p$  in a Riemannian manifold  $(M, g)$  can be computed by integrating the volume element over the ball,  $Vol(B(p, r)) = \int_{B(p, r)} dV$
- In hyperbolic space  $\mathbb{H}^n$  with constant negative curvature, the volume of a ball grows exponentially with its radius, in contrast to the polynomial growth in Euclidean space

## Gauss-Bonnet theorem

- The Gauss-Bonnet theorem is a fundamental result in Riemannian geometry that relates the curvature of a compact, oriented, two-dimensional Riemannian manifold (surface) to its topology

- It establishes a deep connection between the local geometry of a surface, characterized by its Gaussian curvature, and its global topological properties, such as its genus

## Statement of theorem

- For a compact, oriented, two-dimensional Riemannian manifold  $(M, g)$  with boundary  $\partial M$ , the Gauss-Bonnet theorem states that  $\int_M K dA + \int_{\partial M} \kappa_g ds = 2\pi\chi(M)$
- Here,  $K$  is the Gaussian curvature of the surface,  $dA$  is the area element,  $\kappa_g$  is the geodesic curvature of the boundary,  $ds$  is the line element along the boundary, and  $\chi(M)$  is the Euler characteristic of the surface
- The Euler characteristic is a topological invariant that depends only on the genus  $g$  of the surface, given by  $\chi(M) = 2 - 2g$
- The theorem holds for surfaces with boundary, where the boundary term  $\int_{\partial M} \kappa_g ds$  accounts for the contribution of the boundary's geodesic curvature

## Curvature and topology

- The Gauss-Bonnet theorem reveals a deep connection between the curvature of a surface and its topology
- The Gaussian curvature  $K$  measures the local deviation of the surface from flatness, with positive curvature corresponding to spherical-like regions and negative curvature corresponding to saddle-like regions
- The Euler characteristic  $\chi(M)$  encodes the topological properties of the surface, such as the number of holes or handles
- The theorem implies that the total integral of the Gaussian curvature over the surface is completely determined by its Euler characteristic, which depends only on the surface's genus

## Applications and examples

- For a sphere  $S^2$ , the Gaussian curvature is constant and positive,  $K = 1/r^2$ , where  $r$  is the radius. The Euler characteristic of a sphere is  $\chi(S^2) = 2$ , and the Gauss-Bonnet theorem gives  $\int_{S^2} K dA = 4\pi$ , confirming the known surface area of a sphere
- For a torus  $T^2$ , the Euler characteristic is  $\chi(T^2) = 0$ , and the Gauss-Bonnet theorem implies that the total integral of the Gaussian curvature over the torus is zero,  $\int_{T^2} K dA = 0$
- The Gauss-Bonnet theorem can be used to classify compact surfaces based on their Euler characteristic and to study the relationship between curvature and topology in more general settings
- The theorem has important applications in physics, such as in the study of general relativity and the geometry of spacetime, where the curvature of spacetime is related to the presence of matter and energy

## Isoperimetric inequality

- The isoperimetric inequality is a geometric result that relates the area of a region to the length of its boundary, providing a lower bound for the area in terms of the boundary length
- It captures the idea that among all regions with a given boundary length, the disk (in the plane) or the ball (in higher dimensions) has the largest area

## Classical isoperimetric problem

- The classical isoperimetric problem seeks to find the shape of a closed curve in the plane that encloses the maximum area for a given perimeter
- The isoperimetric inequality in the plane states that for a region  $\Omega$  with area  $A$  and perimeter  $L$ , the following inequality holds:  $4\pi A \leq L^2$
- Equality holds if and only if the region  $\Omega$  is a disk, confirming that the disk is the optimal shape that maximizes the area for a given perimeter
- The classical isoperimetric problem and inequality have a rich history, with contributions from ancient Greek mathematicians and ongoing research in various fields

## Isoperimetric inequality on manifolds

- The isoperimetric inequality generalizes to Riemannian manifolds,

## 3.5 Conformal metrics

Conformal metrics are a key concept in differential geometry, preserving angles between tangent vectors under transformations. They differ by a positive smooth function called the conformal factor, allowing comparison of geometric structures on manifolds.

Conformal equivalence relates metrics that share angle measurements but may differ in length. This concept is crucial for studying manifold geometry and topology, providing a framework to analyze geometric properties while preserving angular relationships.

### Definition of conformal metrics

- Conformal metrics are a fundamental concept in differential geometry that describe how the notion of angle is preserved under certain transformations
- Two metrics on a manifold are said to be conformal if they differ by a positive smooth function, called the conformal factor
- Conformal metrics play a crucial role in the study of the geometry and topology of manifolds, as they provide a way to compare and relate different geometric structures

## Conformal equivalence of metrics

### Notion of conformal equivalence

- Two Riemannian metrics  $g$  and  $\tilde{g}$  on a manifold  $M$  are conformally equivalent if there exists a positive smooth function  $\lambda : M \rightarrow \mathbb{R}$  such that  $\tilde{g} = \lambda g$
- Conformally equivalent metrics share the same angle measurements between tangent vectors at each point, but may differ in their length measurements
- Conformal equivalence is an equivalence relation on the set of Riemannian metrics on a manifold

### Conformal factor between metrics

- The positive smooth function  $\lambda$  that relates two conformally equivalent metrics is called the conformal factor

- The conformal factor determines how the length scales are related between the two metrics
- Given a metric  $\mathcal{G}$  and a conformal factor  $\lambda$ , the conformally equivalent metric  $\tilde{\mathcal{G}}$  can be obtained by multiplying  $\mathcal{G}$  by  $\lambda$  at each point

## Conformal structure on manifolds

### Definition of conformal structure

- A conformal structure on a manifold  $M$  is an equivalence class of Riemannian metrics under the relation of conformal equivalence
- Conformal structures capture the notion of angle preservation without specifying a particular choice of metric
- A manifold equipped with a conformal structure is called a conformal manifold

### Conformal atlas and transition functions

- A conformal atlas on a manifold  $M$  is a collection of charts  $(U_\alpha, \varphi_\alpha)$  such that the transition functions  $\varphi_\alpha \circ \varphi_\beta^{-1}$  are conformal mappings (angle-preserving)
- The transition functions of a conformal atlas satisfy the cocycle condition, ensuring the consistency of the conformal structure
- Conformal atlases provide a way to describe conformal structures locally and to study their global properties

## Conformal transformations

### Definition and properties

- A conformal transformation between two Riemannian manifolds  $(M, g)$  and  $(N, h)$  is a diffeomorphism  $f: M \rightarrow N$  that preserves angles between tangent vectors
- Conformal transformations map infinitesimal circles to infinitesimal circles, preserving their shape but not necessarily their size
- The composition of two conformal transformations is again a conformal transformation, forming a group structure

### Examples of conformal transformations

- In the complex plane, the Möbius transformations (linear fractional transformations) are conformal transformations that preserve angles and generalize the notion of circle-preserving maps
- Stereographic projection, which maps the sphere minus a point to the plane, is a conformal transformation that preserves angles but not distances
- In higher dimensions, the conformal transformations include similarities (scaling and isometries), inversions with respect to spheres, and compositions thereof

### Conformal group of a manifold

- The conformal group of a Riemannian manifold  $(M, g)$  is the group of all conformal transformations from  $M$  to itself

- The conformal group is a Lie group that acts on the manifold, preserving its conformal structure
- The dimension and structure of the conformal group depend on the dimension and geometry of the manifold (e.g., the conformal group of the sphere  $S^n$  is the Möbius group)

## Conformal invariants

### Notion of conformal invariance

- A quantity or property associated with a Riemannian manifold is said to be conformally invariant if it remains unchanged under conformal transformations of the metric
- Conformal invariants capture the essential features of the conformal structure, independent of the specific choice of metric within the conformal class
- Examples of conformal invariants include the conformal dimension, the conformal Laplacian, and certain curvature quantities

### Examples of conformal invariants

- The conformal dimension of a manifold, which measures the asymptotic growth of the volume of balls, is a conformal invariant that provides information about the large-scale geometry
- The conformal Laplacian, obtained by adding a specific multiple of the scalar curvature to the usual Laplacian, is a conformally invariant differential operator
- The Weyl tensor, which measures the conformal curvature of a manifold, is a conformally invariant tensor that vanishes if and only if the manifold is conformally flat

### Conformal invariants vs metric invariants

- Metric invariants, such as the Ricci curvature and the scalar curvature, depend on the specific choice of metric and can change under conformal transformations
- Conformal invariants, on the other hand, remain unchanged under conformal transformations and capture properties that are intrinsic to the conformal structure
- The study of conformal invariants provides insights into the conformal geometry of manifolds and helps distinguish between different conformal structures

## Conformal curvature tensors

### Weyl conformal curvature tensor

- The Weyl tensor is a conformally invariant tensor that measures the conformal curvature of a Riemannian manifold
- In dimensions greater than 3, the Weyl tensor is the traceless part of the Riemann curvature tensor and has the same symmetries
- The vanishing of the Weyl tensor characterizes conformal flatness, meaning that the manifold is locally conformally equivalent to Euclidean space



## Schouten tensor and its properties

- The Schouten tensor is a conformally covariant tensor that is constructed from the Ricci curvature and the scalar curvature of a Riemannian manifold
- It plays a crucial role in the decomposition of the Riemann curvature tensor into the Weyl tensor and lower-order terms
- The Schouten tensor satisfies certain transformation laws under conformal changes of the metric, making it a useful tool in conformal geometry

## Conformal flatness and vanishing of Weyl tensor

- A Riemannian manifold is said to be conformally flat if it is locally conformally equivalent to Euclidean space
- In dimensions greater than 3, a manifold is conformally flat if and only if its Weyl tensor vanishes identically
- Conformal flatness is a strong condition that implies the existence of a local conformal diffeomorphism to a flat metric, providing a simplification of the local geometry

## Conformal geometry in dimensions 2 and 3

### Uniformization theorem in dimension 2

- The uniformization theorem states that every simply connected Riemann surface is conformally equivalent to one of three model spaces: the Riemann sphere, the complex plane, or the unit disk
- This theorem provides a complete classification of the conformal structures on surfaces and has far-reaching consequences in complex analysis and geometry
- The proof of the uniformization theorem relies on the existence of conformal metrics with constant curvature, which can be obtained by solving certain partial differential equations

### Conformal geometry of surfaces

- In dimension 2, the conformal geometry of surfaces is closely related to complex analysis and the theory of Riemann surfaces
- Conformal structures on surfaces can be described using holomorphic quadratic differentials, which encode the information about the conformal metric and its singularities
- The moduli space of conformal structures on a surface of genus  $g$  is a complex manifold of dimension  $3g - 3$  (for  $g > 1$ ), known as the Teichmüller space

### Yamabe problem and Poincaré conjecture in dimension 3

- The Yamabe problem asks whether every compact Riemannian manifold admits a conformal metric with constant scalar curvature
- In dimension 3, the Yamabe problem is closely related to the Poincaré conjecture, which states that every simply connected closed 3-manifold is homeomorphic to the 3-sphere
- The resolution of the Poincaré conjecture by Perelman in 2003 used techniques from conformal geometry and the Ricci flow, highlighting the deep connections between topology and geometry in

## Applications of conformal geometry

### Conformal geometry in physics

- Conformal geometry plays a significant role in various areas of physics, such as general relativity, string theory, and conformal field theory
- In general relativity, conformal transformations are used to study the causal structure of spacetime and to analyze the behavior of fields under scale transformations
- Conformal field theories, which are quantum field theories with conformal symmetry, have important applications in statistical mechanics and the study of critical phenomena

### Conformal geometry in complex analysis

- Conformal geometry is intimately connected to complex analysis, as holomorphic functions are conformal mappings in the complex plane
- The Riemann mapping theorem, which states that any simply connected proper open subset of the complex plane is conformally equivalent to the unit disk, is a fundamental result in complex analysis
- Conformal mappings and the Schwarz-Christoffel formula are used to solve various boundary value problems in complex analysis and potential theory

### Conformal geometry in geometric topology

- Conformal geometry provides a bridge between the worlds of geometry and topology, allowing the use of analytic techniques to study topological problems
- The uniformization theorem and the Poincaré conjecture are prime examples of how conformal geometry can be used to classify and understand the topology of manifolds
- Conformal structures and their deformation spaces, such as the Teichmüller space of a surface, are important objects of study in geometric topology and low-dimensional topology

## 3.6 Warped product metrics

Warped product metrics are a fascinating generalization of direct product metrics. They allow for the scaling of the fiber manifold's metric by a warping function dependent on the base manifold, leading to rich geometric structures with diverse applications.

These metrics offer unique insights into curvature, geodesics, and isometries. By studying warped products, we gain a deeper understanding of manifold geometry and its connections to physics, particularly in areas like general relativity and cosmology.

### Definition of warped product metrics

- Warped product metrics are a type of metric structure on a product manifold that generalizes the notion of a direct product metric
- They allow for the scaling of the metric on the fiber manifold by a warping function that depends on the point in the base manifold

- Warped product metrics have important applications in various areas of differential geometry and mathematical physics, such as general relativity and the study of Riemannian submersions

## Warping functions

- A warping function is a smooth, positive function  $f: B \rightarrow \mathbb{R}^+$  defined on the base manifold  $B$
- The warping function determines how the metric on the fiber manifold  $F$  is scaled at each point of the base manifold
- The metric on the warped product manifold  $B \times_f F$  is given by  $g = g_B + f^2 g_F$ , where  $g_B$  and  $g_F$  are the metrics on the base and fiber manifolds, respectively
- The choice of the warping function significantly influences the geometric properties of the warped product manifold, such as its curvature and geodesics

## Base manifolds and fiber manifolds

- In a warped product manifold  $B \times_f F$ , the manifold  $B$  is called the base manifold, and the manifold  $F$  is called the fiber manifold
- The base manifold and fiber manifold are Riemannian manifolds with their own metrics, denoted by  $g_B$  and  $g_F$ , respectively
- The dimensions of the base manifold and fiber manifold are denoted by  $\dim B = m$  and  $\dim F = k$ , and the dimension of the warped product manifold is  $m + k$
- The topology of the warped product manifold is the product topology, meaning that open sets in  $B \times_f F$  are products of open sets in  $B$  and  $F$

## Curvature of warped products

- The curvature of a warped product manifold depends on the curvatures of the base manifold and fiber manifold, as well as the warping function
- The Riemann curvature tensor, Ricci tensor, and scalar curvature of a warped product can be expressed in terms of the corresponding quantities on the base and fiber manifolds and the warping function
- The curvature of a warped product manifold can exhibit interesting properties, such as the presence of singularities or the existence of Einstein metrics

## Ricci curvature

- The Ricci curvature of a warped product manifold  $B \times_f F$  can be expressed in terms of the Ricci curvatures of the base manifold  $B$  and the fiber manifold  $F$ , as well as the warping function  $f$  and its derivatives
- For a vector  $X$  tangent to the base manifold, the Ricci curvature is given by  $\text{Ric}(X, X) = \text{Ric}_B(X, X) - k \frac{\nabla^2 f(X, X)}{f}$ , where  $\text{Ric}_B$  is the Ricci curvature of the base manifold and  $k$  is the dimension of the fiber manifold
- For a vector  $Y$  tangent to the fiber manifold, the Ricci curvature is given by  $\text{Ric}(Y, Y) = \text{Ric}_F(Y, Y) - (f\Delta f + (m-1)|\nabla f|^2)g_F(Y, Y)$ , where  $\text{Ric}_F$  is the Ricci curvature of the fiber manifold,  $\Delta f$  is the Laplacian of  $f$ , and  $m$  is the dimension of the base manifold

## Scalar curvature

- The scalar curvature of a warped product manifold  $B \times_f F$  can be expressed in terms of the scalar curvatures of the base manifold  $B$  and the fiber manifold  $F$ , as well as the warping function  $f$  and its derivatives
- The scalar curvature is given by  $S = S_B + \frac{S_F}{f^2} - \frac{2k}{f}\Delta f - k(k-1)\frac{|\nabla f|^2}{f^2}$ , where  $S_B$  and  $S_F$  are the scalar curvatures of the base manifold and fiber manifold, respectively, and  $\Delta f$  is the Laplacian of  $f$
- The scalar curvature of a warped product manifold can be used to study the global geometry of the manifold, such as the existence of Einstein metrics or the behavior of the manifold at infinity

## Sectional curvature

- The sectional curvature of a warped product manifold  $B \times_f F$  depends on the sectional curvatures of the base manifold  $B$  and the fiber manifold  $F$ , as well as the warping function  $f$  and its derivatives
- For a plane spanned by orthonormal vectors  $X$  and  $Y$  tangent to the base manifold, the sectional curvature is given by  $K(X, Y) = K_B(X, Y)$ , where  $K_B$  is the sectional curvature of the base manifold
- For a plane spanned by orthonormal vectors  $V$  and  $W$  tangent to the fiber manifold, the sectional curvature is given by  $K(V, W) = \frac{K_F(V, W) - |\nabla f|^2}{f^2}$ , where  $K_F$  is the sectional curvature of the fiber manifold
- For a plane spanned by a unit vector  $X$  tangent to the base manifold and a unit vector  $V$  tangent to the fiber manifold, the sectional curvature is given by  $K(X, V) = -\frac{\nabla^2 f(X, X)}{f}$

## Geodesics in warped products

- Geodesics in a warped product manifold  $B \times_f F$  are curves that minimize the distance between points in the manifold
- The behavior of geodesics in a warped product manifold is influenced by the warping function and the geodesics of the base manifold and fiber manifold
- Studying geodesics in warped product manifolds is important for understanding the global geometry of the manifold and its implications in various applications, such as general relativity

## Clairaut's theorem

- Clairaut's theorem is a result that describes the behavior of geodesics in warped product manifolds with a warping function that depends only on the distance from a fixed point in the base manifold
- In such manifolds, called rotationally symmetric manifolds, Clairaut's theorem states that the quantity  $r \sin \alpha$  is constant along a geodesic, where  $r$  is the distance from the fixed point and  $\alpha$  is the angle between the geodesic and the radial direction
- Clairaut's theorem provides a useful tool for studying the qualitative behavior of geodesics in rotationally symmetric warped product manifolds, such as surfaces of revolution

## Geodesic equations

- The geodesic equations in a warped product manifold  $B \times_f F$  can be derived using the Christoffel symbols of the warped product metric

- For a curve  $\gamma(t) = (x(t), y(t))$  in  $B \times {}_fF$ , where  $x(t)$  is a curve in the base manifold and  $y(t)$  is a curve in the fiber manifold, the geodesic equations are:
  - $\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k + 2 \frac{f'}{f} \dot{x}^i |\dot{y}|^2 = 0$
  - $\ddot{y}^\alpha + \tilde{\Gamma}_{\beta\gamma}^\alpha \dot{y}^\beta \dot{y}^\gamma - \frac{f'}{f} |\dot{x}|^2 y^\alpha = 0$
- Here,  $\Gamma_{jk}^i$  are the Christoffel symbols of the base manifold,  $\tilde{\Gamma}_{\beta\gamma}^\alpha$  are the Christoffel symbols of the fiber manifold, and  $f'$  denotes the derivative of the warping function along the curve  $x(t)$

## Geodesic completeness

- A Riemannian manifold is said to be geodesically complete if every geodesic can be extended indefinitely in both directions
- The geodesic completeness of a warped product manifold  $B \times {}_fF$  depends on the geodesic completeness of the base manifold  $B$  and the fiber manifold  $F$ , as well as the behavior of the warping function  $f$
- If the base manifold  $B$  and the fiber manifold  $F$  are both geodesically complete and the warping function  $f$  is bounded away from zero, then the warped product manifold  $B \times {}_fF$  is geodesically complete
- However, if either the base manifold or the fiber manifold is not geodesically complete, or if the warping function approaches zero at some point, the warped product manifold may not be geodesically complete

## Isometries of warped products

- An isometry is a distance-preserving map between Riemannian manifolds
- The isometry group of a warped product manifold  $B \times {}_fF$  is related to the isometry groups of the base manifold  $B$  and the fiber manifold  $F$ , as well as the symmetries of the warping function  $f$
- Understanding the isometries of warped product manifolds is important for studying the symmetries and the global geometry of these manifolds

## Vertical isometries

- A vertical isometry of a warped product manifold  $B \times {}_fF$  is an isometry that preserves the fibers  $\{x\} \times F$  for each point  $x$  in the base manifold  $B$
- The vertical isometries of a warped product manifold form a subgroup of the isometry group, called the vertical isometry group
- The vertical isometry group of a warped product manifold is isomorphic to the isometry group of the fiber manifold  $F$
- Vertical isometries play a crucial role in understanding the symmetries of the fiber manifold and their impact on the geometry of the warped product manifold

## Horizontal isometries

- A horizontal isometry of a warped product manifold  $B \times {}_fF$  is an isometry that preserves the leaves  $B \times \{y\}$  for each point  $y$  in the fiber manifold  $F$

- The horizontal isometries of a warped product manifold form a subgroup of the isometry group, called the horizontal isometry group
- The horizontal isometry group of a warped product manifold is related to the isometry group of the base manifold  $B$  and the symmetries of the warping function  $f$
- In some cases, such as when the warping function is constant, the horizontal isometry group of a warped product manifold can be isomorphic to the isometry group of the base manifold

## Warped product isometry groups

- The isometry group of a warped product manifold  $B \times_f F$  is a subgroup of the product of the isometry groups of the base manifold  $B$  and the fiber manifold  $F$
- The structure of the warped product isometry group depends on the compatibility of the isometries of the base manifold and the fiber manifold with the warping function  $f$
- In some cases, the warped product isometry group can be expressed as a semidirect product of the horizontal isometry group and the vertical isometry group
- The warped product isometry group provides valuable insights into the symmetries and the global geometry of the warped product manifold, which are essential for various applications in differential geometry and mathematical physics

## Examples of warped products

- Warped product manifolds arise naturally in various contexts in differential geometry and mathematical physics
- Some notable examples of warped product manifolds include surfaces of revolution, generalized cylinders, and Robertson-Walker spacetimes
- Studying these examples helps to illustrate the geometric properties of warped product manifolds and their applications in different areas of mathematics and physics

## Surfaces of revolution

- A surface of revolution is a warped product manifold obtained by rotating a curve in the  $XY$ -plane around the  $X$ -axis
- The base manifold is an interval  $I \subset \mathbb{R}$ , and the fiber manifold is a circle  $S^1$
- The warping function  $f: I \rightarrow \mathbb{R}^+$  is the distance from the curve to the  $X$ -axis
- The metric on a surface of revolution is given by  $ds^2 = dx^2 + f(x)^2 d\theta^2$ , where  $x$  is the parameter along the base interval and  $\theta$  is the angular coordinate on the circle
- Examples of surfaces of revolution include the sphere, the torus, and the paraboloid

## Generalized cylinders

- A generalized cylinder is a warped product manifold obtained by taking the product of a curve in a Riemannian manifold  $M$  with another Riemannian manifold  $N$
- The base manifold is the curve  $\gamma: I \rightarrow M$ , and the fiber manifold is the Riemannian manifold  $N$
- The warping function  $f: I \rightarrow \mathbb{R}^+$  is a positive function along the curve

- The metric on a generalized cylinder is given by  $ds^2 = dt^2 + f(t)^2 g_N$ , where  $t$  is the parameter along the base curve and  $g_N$  is the metric on the fiber manifold
- Generalized cylinders can be used to model various geometric objects, such as ruled surfaces and warped products of higher-dimensional manifolds

## Robertson-Walker spacetimes

- Robertson-Walker spacetimes are warped product manifolds that model homogeneous and isotropic universes in general relativity
- The base manifold is an interval  $I \subset \mathbb{R}$  representing cosmic time, and the fiber manifold is a three-dimensional space of constant curvature (either spherical, Euclidean, or hyperbolic)
- The warping function  $a : I \rightarrow \mathbb{R}^+$  is called the scale factor and describes the expansion or contraction of the universe over time
- The metric on a Robertson-Walker spacetime is given by  $ds^2 = -dt^2 + a(t)^2 g_k$ , where  $g_k$  is the metric on the space of constant curvature with curvature  $k \in \{1, 0, -1\}$
- Robertson-Walker spacetimes are the foundation of the standard cosmological models, such as the Big Bang model and the inflationary universe model

## Comparison with direct products

- Direct product manifolds are a special case of warped product manifolds where the warping function is constant
- Comparing the properties of warped product manifolds with those of direct product manifolds helps to highlight the effects of the warping function on the geometry of the manifold
- Understanding the similarities and differences between warped products and direct products is important for various applications in differential geometry and mathematical physics

## Metric properties

- In a direct product manifold  $M \times N$ , the metric is given by the sum of the metrics on the factor manifolds:  $g = g_M + g_N$
- In a warped product manifold  $B \times_f F$ , the metric is given by  $g = g_B + f^2 g_F$ , where the metric on the fiber manifold is scaled by the square of the warping function
- The presence of the warping function in the metric of a warped product manifold leads to different geometric properties compared to direct product manifolds, such as the behavior of distances, angles, and curvatures

## Curvature properties

- The curvature of a direct product manifold  $M \times N$  is completely determined by the curvatures of the factor manifolds  $M$  and  $N$
- In a warped product manifold  $B \times_f F$ , the curvature depends on the curvatures of the base manifold  $B$  and the fiber manifold  $F$ , as well as the warping function  $f$  and its derivatives
- The presence of the warping function in the curvature expressions of a warped product manifold can lead to interesting geometric phenomena, such as the existence of singularities or the realization of Einstein metrics, which are not possible in direct product manifolds

## Isometry groups

- The isometry group of a direct product manifold  $M \times N$  is the product of the isometry groups of the factor manifolds:  $\text{Isom}(M \times N) = \text{Isom}(M) \times \text{Isom}(N)$
- In a warped product manifold  $B \times_{\mathcal{F}} F$ , the isometry group is



# Unit 4 – Geodesics and Distance in Metric Geometry

## 4.1 Geodesics and exponential map

Geodesics are curves that locally minimize distance on curved spaces, generalizing straight lines. They're crucial for understanding Riemannian manifolds' geometry, serving as shortest paths between points and satisfying a specific differential equation.

The exponential map connects a manifold's tangent space to the manifold itself. It generates geodesics, preserves vector lengths, and provides a local diffeomorphism. This tool is essential for studying local geometry and geodesic behavior on Riemannian manifolds.

### Definition of geodesics

- Geodesics are fundamental objects in Riemannian geometry that generalize the notion of straight lines to curved spaces
- They play a central role in understanding the intrinsic geometry of Riemannian manifolds and serve as the shortest paths between points on the manifold

### Geodesics as locally length-minimizing curves

- Geodesics are curves that locally minimize the distance between two points on a Riemannian manifold
- For any point on a geodesic, there exists a neighborhood such that the geodesic segment within that neighborhood is the shortest path connecting its endpoints
- This property distinguishes geodesics from other curves on the manifold and highlights their importance in defining the intrinsic geometry

### Geodesic equation

- The geodesic equation is a second-order differential equation that characterizes geodesics on a Riemannian manifold
- In local coordinates, the geodesic equation is given by:

$$\frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

where  $x^i$  are the coordinates of the curve,  $t$  is an affine parameter along the curve, and  $\Gamma_{jk}^i$  are the Christoffel symbols of the Levi-Civita connection - Solving the geodesic equation with appropriate initial conditions yields the geodesics on the manifold

### Uniqueness of geodesics

- Given a point on a Riemannian manifold and a tangent vector at that point, there exists a unique geodesic passing through the point with the given tangent vector
- This uniqueness property ensures that geodesics are well-defined and can be used to study the geometry of the manifold

- The uniqueness of geodesics is a consequence of the existence and uniqueness theorem for ordinary differential equations applied to the geodesic equation

## Geodesics on Riemannian manifolds

- On a Riemannian manifold, geodesics are determined by the metric tensor, which encodes the geometric properties of the manifold
- The metric tensor induces a Levi-Civita connection, which is used to define the geodesic equation
- Examples of geodesics on specific Riemannian manifolds include:
  - Great circles on a sphere
  - Straight lines on Euclidean space
  - Hyperbolic geodesics on hyperbolic space

## Exponential map

- The exponential map is a fundamental tool in Riemannian geometry that relates the tangent space at a point to the manifold itself
- It provides a way to generalize the notion of exponential function to curved spaces and plays a crucial role in studying geodesics and the local geometry of Riemannian manifolds

## Definition of exponential map

- For a point  $P$  on a Riemannian manifold  $M$  and a tangent vector  $V$  in the tangent space  $T_P M$ , the exponential map  $\exp_P : T_P M \rightarrow M$  is defined as follows:
- $\exp_P(V)$  is the point on  $M$  obtained by following the unique geodesic starting at  $P$  with initial velocity  $V$  for a unit time
- The exponential map takes a tangent vector and maps it to a point on the manifold, establishing a local diffeomorphism between the tangent space and the manifold

## Exponential map as geodesic flow

- The exponential map can be interpreted as the geodesic flow on the tangent bundle of the manifold
- For each tangent vector  $V$  at a point  $P$ , the exponential map traces out the geodesic starting at  $P$  with initial velocity  $V$
- This interpretation highlights the role of the exponential map in generating geodesics and studying their properties

## Properties of exponential map

- The exponential map has several important properties:
- It is a local diffeomorphism near the origin of the tangent space
- It preserves the length of tangent vectors:  $d(P, \exp_P(V)) = \|V\|$ , where  $d$  is the Riemannian distance and  $\|V\|$  is the norm of the tangent vector
- It is a radial isometry: the exponential map preserves the length of radial geodesics emanating from the base point

- These properties make the exponential map a powerful tool for studying the local geometry of Riemannian manifolds and understanding the behavior of geodesics

## Injectivity radius of exponential map

- The injectivity radius of the exponential map at a point  $P$  is the largest radius  $r$  such that the exponential map is a diffeomorphism from the open ball of radius  $r$  in the tangent space  $T_P M$  to an open neighborhood of  $P$  on the manifold
- The injectivity radius measures the size of the largest geodesically convex neighborhood around a point
- It provides information about the local injectivity of the exponential map and the existence of conjugate points along geodesics

## Geodesics and topology

- Geodesics play a crucial role in understanding the topological properties of Riemannian manifolds
- The behavior of geodesics, such as their completeness and the existence of conjugate points, is closely related to the global topology of the manifold

## Geodesically convex neighborhoods

- A subset  $U$  of a Riemannian manifold  $M$  is called geodesically convex if for any two points  $p, q \in U$ , there exists a unique geodesic segment connecting  $P$  and  $q$  that is entirely contained in  $U$
- Geodesically convex neighborhoods are important for local analysis on Riemannian manifolds, as they ensure the uniqueness of geodesics and the well-definedness of the exponential map
- The existence of geodesically convex neighborhoods is related to the injectivity radius of the exponential map

## Geodesic completeness

- A Riemannian manifold is said to be geodesically complete if every geodesic can be extended indefinitely in both directions
- Geodesic completeness is a global property that ensures the existence of geodesics between any two points on the manifold
- It has important implications for the topology and geometry of the manifold, such as the Hopf-Rinow theorem

## Hopf-Rinow theorem

- The Hopf-Rinow theorem is a fundamental result in Riemannian geometry that establishes the equivalence between several important properties of Riemannian manifolds:
- Geodesic completeness
- Completeness as a metric space with respect to the Riemannian distance
- The existence of a point such that the exponential map at that point is surjective
- The theorem highlights the deep connection between the geodesic structure, the metric structure, and the topology of Riemannian manifolds

## Hadamard manifolds

- A Hadamard manifold is a simply connected, complete Riemannian manifold with non-positive sectional curvature
- Hadamard manifolds have several important properties related to geodesics:
- They are diffeomorphic to Euclidean space
- The exponential map at any point is a global diffeomorphism
- Any two points can be connected by a unique geodesic
- Examples of Hadamard manifolds include the Euclidean space, the hyperbolic space, and certain symmetric spaces

## Geodesic distance

- The geodesic distance is a metric on a Riemannian manifold that measures the length of the shortest geodesic connecting two points
- It is a fundamental concept in Riemannian geometry and plays a crucial role in studying the metric and topological properties of Riemannian manifolds

## Definition of geodesic distance

- For any two points  $p, q$  on a Riemannian manifold  $M$ , the geodesic distance  $d(p, q)$  is defined as the infimum of the lengths of all piecewise smooth curves connecting  $p$  and  $q$
- If  $\gamma : [a, b] \rightarrow M$  is a geodesic connecting  $p$  and  $q$ , then the geodesic distance is given by:

$$d(p, q) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt$$

where  $g$  is the Riemannian metric tensor and  $\gamma'(t)$  is the tangent vector to the geodesic at  $t$

## Metric space structure induced by geodesic distance

- The geodesic distance induces a metric space structure on the Riemannian manifold
- It satisfies the axioms of a metric:
- Non-negativity:  $d(p, q) \geq 0$  and  $d(p, q) = 0$  if and only if  $p = q$
- Symmetry:  $d(p, q) = d(q, p)$
- Triangle inequality:  $d(p, q) \leq d(p, r) + d(r, q)$  for any points  $p, q, r$  on the manifold
- The metric space structure allows for the study of topological properties of the manifold, such as completeness and compactness, using tools from metric geometry

## Relationship between geodesic and Riemannian distance

- The geodesic distance and the Riemannian distance, defined using the Riemannian metric tensor, are closely related
- In general, the geodesic distance between two points is greater than or equal to the Riemannian distance

- The two distances coincide for nearby points, where the exponential map is a local isometry
- However, for points that are far apart, the geodesic distance may be strictly greater than the Riemannian distance due to the curvature of the manifold

## Cut locus and conjugate points

- The cut locus of a point  $P$  on a Riemannian manifold is the set of points  $Q$  for which the geodesic from  $P$  to  $Q$  is no longer minimizing beyond  $Q$
- Conjugate points are points along a geodesic where the exponential map fails to be a local diffeomorphism
- The presence of cut points and conjugate points affects the behavior of geodesics and the properties of the geodesic distance
- Understanding the cut locus and conjugate points is important for studying the global geometry and topology of Riemannian manifolds

## Applications of geodesics

- Geodesics have numerous applications in various areas of mathematics and physics, showcasing their importance in understanding geometric and physical phenomena

## Shortest paths on surfaces

- Geodesics provide a natural way to define shortest paths on surfaces
- On a smooth surface equipped with a Riemannian metric, geodesics are the curves that minimize the distance between points
- Finding geodesics on surfaces has practical applications in computer graphics, robotics, and navigation systems
- Examples include finding the shortest path between two points on a sphere (great circles) or on a terrain surface

## Geodesics in general relativity

- In general relativity, geodesics play a fundamental role in describing the motion of particles and the propagation of light in curved spacetime
- Freely falling particles and light rays follow geodesics in the spacetime manifold, which is equipped with a pseudo-Riemannian metric
- The geodesic equation in general relativity incorporates the effects of gravity and provides a geometric description of the motion of objects in the presence of gravitational fields
- Studying geodesics in general relativity helps in understanding gravitational lensing, the orbits of planets and satellites, and the behavior of light near massive objects

## Geodesic curvature and Gauss-Bonnet theorem

- Geodesic curvature measures the deviation of a curve from being a geodesic on a surface
- It is an important concept in the study of curves on surfaces and has applications in differential geometry and topology

- The Gauss-Bonnet theorem relates the total geodesic curvature of a closed curve on a surface to the integral of the Gaussian curvature over the enclosed region
- This theorem establishes a deep connection between the intrinsic geometry of a surface and its topological properties, such as the Euler characteristic
- Applications of the Gauss-Bonnet theorem include classifying surfaces, studying the global geometry of surfaces, and understanding the behavior of geodesics on surfaces

## Geodesic flow and dynamical systems

- The geodesic flow is a dynamical system on the tangent bundle of a Riemannian manifold that describes the motion of particles along geodesics
- It is defined using the exponential map and provides a way to study the long-term behavior of geodesics on the manifold
- The properties of the geodesic flow, such as ergodicity and mixing, provide insights into the geometry and dynamics of the manifold
- Studying the geodesic flow has applications in classical mechanics, statistical mechanics, and the theory of dynamical systems
- Techniques from dynamical systems, such as the study of Lyapunov exponents and entropy, can be used to analyze the behavior of geodesics and understand the complexity of the geodesic flow on Riemannian manifolds

## 4.2 Minimizing properties of geodesics

Geodesics are curves that locally minimize distance between points on curved spaces. They generalize straight lines to Riemannian manifolds, playing a crucial role in understanding geometry and topology. Their study reveals fundamental properties of curved spaces.

Minimizing properties of geodesics connect local and global aspects of manifold geometry. While all geodesics minimize locally, global minimization depends on manifold structure. This topic explores existence, uniqueness, and variational approaches to geodesics, providing insights into manifold completeness and curvature.

## Geodesics as shortest paths

- Geodesics are fundamental objects in Metric Differential Geometry that generalize the notion of straight lines in Euclidean space to curved spaces
- The study of geodesics and their properties is essential for understanding the geometry and topology of Riemannian manifolds

## Definition of geodesics

- Geodesics are curves that locally minimize the distance between two points on a Riemannian manifold
- Mathematically, a geodesic is a curve  $\gamma(t)$  that satisfies the geodesic equation:  $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ , where  $\nabla$  is the Levi-Civita connection and  $\dot{\gamma}$  is the tangent vector to the curve
- Intuitively, geodesics are the "straightest" possible curves on a curved space (great circles on a sphere, straight lines on a plane)

## Locally minimizing property

- Geodesics have the property of being locally minimizing, meaning that for any two sufficiently close points on the geodesic, the segment of the geodesic connecting them is the shortest path between those points
- This local minimizing property is a consequence of the geodesic equation and the fact that geodesics have zero acceleration with respect to the Levi-Civita connection
- The local minimizing property is essential for understanding the role of geodesics in determining the geometry of a Riemannian manifold (distance function, curvature)

## Globally minimizing geodesics

- While all geodesics are locally minimizing, not all geodesics are globally minimizing, meaning they may not be the shortest path between any two points on the manifold
- The global minimizing property of a geodesic depends on the global topology and geometry of the manifold (closed geodesics on a sphere, non-minimizing geodesics on a torus)
- The existence and uniqueness of globally minimizing geodesics are important questions in Metric Differential Geometry and are related to the completeness and curvature of the manifold

## Existence of minimizing geodesics

- The existence of minimizing geodesics is a fundamental question in Metric Differential Geometry, as it relates to the completeness and compactness of Riemannian manifolds
- Several important theorems, such as the Hopf-Rinow theorem, provide conditions for the existence of minimizing geodesics on a Riemannian manifold

## Hopf-Rinow theorem

- The Hopf-Rinow theorem is a central result in Metric Differential Geometry that establishes the equivalence between metric completeness and geodesic completeness for connected Riemannian manifolds
- It states that if a connected Riemannian manifold is metrically complete (every Cauchy sequence converges), then it is geodesically complete (every geodesic can be extended indefinitely)
- The theorem also guarantees the existence of minimizing geodesics between any two points on a complete, connected Riemannian manifold

## Completeness and geodesic completeness

- Completeness is a crucial property in Metric Differential Geometry that ensures the well-behaved nature of a Riemannian manifold
- Metric completeness means that every Cauchy sequence of points on the manifold converges to a point on the manifold, while geodesic completeness means that every geodesic can be extended indefinitely
- The Hopf-Rinow theorem establishes the equivalence between these two notions of completeness for connected Riemannian manifolds

## Compact manifolds and minimizing geodesics

- Compact Riemannian manifolds, which are both complete and bounded, always admit minimizing geodesics between any two points
- This is a consequence of the Extreme Value Theorem, which guarantees the existence of a minimum for a continuous function on a compact set
- The existence of minimizing geodesics on compact manifolds has important implications for the study of the geometry and topology of these spaces (closed geodesics, diameter, systole)

## Uniqueness of minimizing geodesics

- The uniqueness of minimizing geodesics is another fundamental question in Metric Differential Geometry, as it relates to the convexity and curvature of Riemannian manifolds
- Several concepts, such as conjugate points and the cut locus, play a crucial role in determining the uniqueness of minimizing geodesics

## Conjugate points and geodesic uniqueness

- Conjugate points are pairs of points along a geodesic where the geodesic fails to be minimizing beyond the first point
- The existence of conjugate points is related to the focusing of geodesics and the presence of positive curvature on the manifold (conjugate points on a sphere, no conjugate points on a plane)
- If a geodesic segment contains no conjugate points, then it is the unique minimizing geodesic between its endpoints

## Cut locus and minimizing properties

- The cut locus of a point on a Riemannian manifold is the set of points where the geodesics emanating from that point cease to be minimizing
- Points on the cut locus are either conjugate points or points where two or more minimizing geodesics intersect (antipodal points on a sphere, cut locus on a torus)
- The cut locus plays a crucial role in understanding the global minimizing properties of geodesics and the geometry of the manifold

## Convexity and uniqueness of geodesics

- Convexity is a geometric property that guarantees the uniqueness of minimizing geodesics between points
- A subset of a Riemannian manifold is said to be convex if every pair of points in the subset is connected by a unique minimizing geodesic that lies entirely within the subset
- The convexity of a manifold or a subset is related to the curvature and the absence of conjugate points (convex neighborhoods in spaces of non-positive curvature)

## Variational approach to geodesics

- The variational approach to geodesics is a powerful tool in Metric Differential Geometry that characterizes geodesics as critical points of an energy functional



- This approach leads to the derivation of the Euler-Lagrange equations for geodesics and the study of Jacobi fields and the second variation formula

## Geodesics as critical points of energy functional

- In the variational approach, geodesics are defined as the critical points of the energy functional  $E[\gamma] = \frac{1}{2} \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt$ , where  $\gamma(t)$  is a curve connecting two points on the manifold
- The energy functional measures the total squared speed of the curve, and its critical points are precisely the geodesics
- This variational characterization of geodesics is analogous to the principle of least action in classical mechanics (geodesics as "paths of least resistance")

## Euler-Lagrange equations for geodesics

- The Euler-Lagrange equations are a system of differential equations that characterize the critical points of a variational problem
- In the context of geodesics, the Euler-Lagrange equations take the form of the geodesic equation:  $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$
- The geodesic equation expresses the fact that the acceleration of a geodesic with respect to the Levi-Civita connection vanishes, which is equivalent to the curve being a critical point of the energy functional

## Jacobi fields and second variation formula

- Jacobi fields are vector fields along a geodesic that describe the infinitesimal variation of the geodesic under small perturbations
- The study of Jacobi fields is crucial for understanding the stability and focusing properties of geodesics, as well as the presence of conjugate points
- The second variation formula is an expression for the second derivative of the energy functional along a geodesic in terms of the Jacobi fields and the curvature of the manifold (positive second variation implies stability, negative second variation implies instability)

## Comparison theorems for geodesics

- Comparison theorems are powerful tools in Metric Differential Geometry that relate the geometry of a Riemannian manifold to the geometry of spaces of constant curvature (spheres, Euclidean spaces, hyperbolic spaces)
- These theorems allow for the comparison of geometric quantities, such as distances, angles, and curvatures, between the manifold and the model spaces

## Rauch comparison theorem

- The Rauch comparison theorem, also known as the Rauch-Berger theorem, compares the length of Jacobi fields along geodesics in a Riemannian manifold with those in a space of constant curvature
- The theorem states that if the sectional curvatures of the manifold are bounded above (resp. below) by a constant  $K$ , then the length of Jacobi fields is bounded above (resp. below) by the length of corresponding Jacobi fields in a space of constant curvature  $K$

- This theorem has important implications for the focusing and divergence of geodesics and the presence of conjugate points

## Toponogov's triangle comparison theorem

- Toponogov's triangle comparison theorem compares the geometry of triangles in a Riemannian manifold with the geometry of triangles in spaces of constant curvature
- The theorem states that if the sectional curvatures of the manifold are bounded above (resp. below) by a constant  $k$ , then the angles and side lengths of geodesic triangles in the manifold are bounded above (resp. below) by the corresponding quantities in a space of constant curvature  $k$
- This theorem has important applications in the study of the global geometry and topology of Riemannian manifolds (diameter, injectivity radius, fundamental group)

## Applications of comparison theorems

- Comparison theorems have numerous applications in Metric Differential Geometry, including:
- Estimating the size and shape of geodesic balls and spheres
- Studying the growth of the volume and area of submanifolds
- Analyzing the behavior of geodesics and the presence of conjugate points
- Proving the existence of closed geodesics and estimating their length
- Investigating the relationship between curvature and topology (Sphere theorem, Preissman's theorem)

## Geodesics in specific geometries

- The study of geodesics in specific geometries is an important aspect of Metric Differential Geometry, as it provides concrete examples and applications of the general theory
- Geodesics in spaces such as spheres, hyperbolic spaces, Lie groups, and symmetric spaces have special properties and are often related to the symmetries and algebraic structures of these spaces

## Geodesics on spheres and hyperbolic spaces

- Spheres and hyperbolic spaces are model spaces of constant positive and negative curvature, respectively, and their geodesics have explicit descriptions
- On the unit sphere  $S^n$ , geodesics are great circles, which are the intersections of the sphere with planes passing through the origin (equator, meridians)
- In the hyperbolic space  $H^n$ , geodesics are the intersections of the space with planes passing through the origin in the Minkowski space  $\mathbb{R}^{1,n}$  (hyperbolic lines, hypercycles)
- The study of geodesics on spheres and hyperbolic spaces is crucial for understanding the geometry of these spaces and their role as comparison spaces in Metric Differential Geometry

## Geodesics in Riemannian submersions

- Riemannian submersions are maps between Riemannian manifolds that preserve the length of horizontal vectors, which are orthogonal to the fibers of the submersion

- Geodesics in the base space of a Riemannian submersion can be lifted to horizontal geodesics in the total space, providing a way to study the geometry of the total space in terms of the geometry of the base and the fibers (Hopf fibration, O'Neill's formula)
- The study of geodesics in Riemannian submersions has applications in the geometry of fiber bundles, homogeneous spaces, and gauge theories

## Geodesics in Lie groups and symmetric spaces

- Lie groups are smooth manifolds equipped with a group structure, and their geometry is closely related to the properties of the associated Lie algebra
- Geodesics in Lie groups are often related to one-parameter subgroups and the exponential map, which connects the Lie algebra to the Lie group (geodesics on the rotation group  $SO(3)$ , the Heisenberg group)
- Symmetric spaces are Riemannian manifolds with a special symmetry property, namely, the existence of an isometry that reverses geodesics (Euclidean spaces, spheres, hyperbolic spaces, Grassmannians)
- The study of geodesics in Lie groups and symmetric spaces is important for understanding the interplay between geometry, algebra, and topology in these spaces

## 4.3 Cut locus and conjugate points

Cut locus and conjugate points are key concepts in Riemannian geometry. They help us understand how geodesics behave on a manifold and provide insights into its global structure and curvature.

Cut locus deals with points where geodesics stop being minimizing paths. Conjugate points, on the other hand, relate to the local behavior of geodesics and are linked to Jacobi fields. Both concepts are crucial for studying manifold geometry and topology.

### Definition of cut locus

- The cut locus is a fundamental concept in Riemannian geometry that plays a crucial role in understanding the global structure of a manifold
- It is closely related to the notion of geodesics, which are the shortest paths between points on a manifold
- The cut locus provides information about the points where geodesics cease to be minimizing, and it helps to characterize the topology and geometry of the manifold

### Cut points

- A cut point of a point  $p$  on a Riemannian manifold  $M$  is a point  $q$  such that there exists a minimizing geodesic from  $p$  to  $q$ , but no geodesic from  $p$  to  $q$  is minimizing beyond  $q$
- Intuitively, a cut point is the first point along a geodesic where it stops being the shortest path between its endpoints
- The distance from  $p$  to its cut point  $q$  is called the cut distance, denoted by  $d(p, q)$

### Relation to geodesics

- The cut locus is intimately connected to the behavior of geodesics on a manifold

- Geodesics are locally minimizing, meaning they are the shortest paths between nearby points
- However, a geodesic may fail to be globally minimizing due to the presence of cut points
- The cut locus determines the maximal domain on which a geodesic remains minimizing

## Cut locus as set of cut points

- The cut locus of a point  $P$  on a manifold  $M$ , denoted by  $Cut(p)$ , is defined as the set of all cut points of  $P$
- It can be seen as the collection of points where geodesics emanating from  $P$  cease to be minimizing
- The cut locus forms a subset of the manifold  $M$  and provides valuable information about its global structure

## Properties of cut locus

- The cut locus exhibits several important properties that shed light on the geometry and topology of the underlying manifold
- Understanding these properties is crucial for gaining insights into the behavior of geodesics and the structure of the manifold
- The properties of the cut locus have implications for various applications, such as optimal transport, computational geometry, and computer graphics

## Closedness

- The cut locus  $Cut(p)$  of a point  $P$  on a complete Riemannian manifold is always a closed subset of the manifold
- This means that the limit of any sequence of points in the cut locus also belongs to the cut locus
- The closedness property ensures that the cut locus is well-behaved and can be studied using topological techniques

## Triangulability

- In many cases, the cut locus of a point on a Riemannian manifold is triangulable
- Triangulability means that the cut locus can be decomposed into a union of simplices (triangles in dimension 2, tetrahedra in dimension 3, and their higher-dimensional analogues)
- The triangulability property allows for the application of combinatorial and computational methods to study the cut locus

## Relation to injectivity radius

- The cut locus is closely related to the concept of injectivity radius in Riemannian geometry
- The injectivity radius at a point  $P$ , denoted by  $inj(p)$ , is the largest radius for which the exponential map at  $P$  is a diffeomorphism (a smooth and invertible map)
- The distance from  $P$  to its cut locus  $Cut(p)$  is equal to the injectivity radius at  $P$
- The injectivity radius provides a measure of how far one can go from a point before encountering cut points or self-intersections of geodesics

## Computation of cut locus

- Computing the cut locus is a challenging problem in Riemannian geometry, as it requires a deep understanding of the global structure of the manifold
- There are various techniques and algorithms developed for computing the cut locus, depending on the specific properties of the manifold and the available computational resources
- The computation of the cut locus has applications in areas such as robotics, computer vision, and medical imaging, where understanding the shortest paths and the global structure of the data is important

## Techniques for specific manifolds

- For certain classes of manifolds with special properties, there exist specific techniques for computing the cut locus
- For example, on Riemannian surfaces (2-dimensional manifolds), the cut locus can be determined using methods from complex analysis and Riemann surface theory
- In the case of Lie groups and symmetric spaces, the cut locus can be described using algebraic and combinatorial techniques based on the underlying symmetries

## Algorithms for general case

- For general Riemannian manifolds, computing the cut locus often requires numerical algorithms and approximation techniques
- One approach is to discretize the manifold using a mesh or a graph and then apply shortest path algorithms (Dijkstra's algorithm) to compute the cut locus
- Another approach is to use the fast marching method, which propagates a wavefront from a source point and tracks the arrival times of the wavefront to determine the cut locus
- More advanced techniques, such as the geodesic shooting algorithm and the cut pursuit algorithm, have been developed to handle more complex manifolds and higher dimensions

## Definition of conjugate points

- Conjugate points are another important concept in Riemannian geometry that provide information about the behavior of geodesics and the curvature of the manifold
- Unlike cut points, which are related to the global minimizing properties of geodesics, conjugate points are connected to the local infinitesimal behavior of geodesics
- The notion of conjugate points is based on the study of Jacobi fields, which describe the variation of geodesics with respect to their initial conditions

## Relation to Jacobi fields

- Jacobi fields are vector fields along a geodesic that satisfy the Jacobi equation, a second-order linear differential equation
- They measure the infinitesimal deviation of nearby geodesics from a given geodesic
- Conjugate points along a geodesic are the points where the Jacobi fields vanish (become zero)

- The presence of conjugate points indicates the existence of infinitesimally close geodesics that intersect the given geodesic

## Multiplicity of conjugate points

- Conjugate points along a geodesic can have different multiplicities, depending on the number of linearly independent Jacobi fields that vanish at that point
- The multiplicity of a conjugate point is a measure of the degeneracy of the geodesic at that point
- Higher multiplicity conjugate points indicate a more severe degeneracy and have implications for the stability and bifurcation of geodesics

## Properties of conjugate points

- Conjugate points exhibit several important properties that provide insights into the geometry and topology of the manifold
- These properties are closely related to the curvature of the manifold and have implications for the global behavior of geodesics
- Understanding the properties of conjugate points is crucial for studying the stability and optimality of geodesics, as well as for understanding the structure of the manifold

## Relation to geodesic completeness

- The presence or absence of conjugate points is related to the notion of geodesic completeness of a Riemannian manifold
- A Riemannian manifold is said to be geodesically complete if every geodesic can be extended indefinitely in both directions
- The absence of conjugate points along all geodesics is a sufficient condition for a manifold to be geodesically complete
- Conversely, the existence of conjugate points may indicate the incompleteness of the manifold or the presence of singularities

## Relation to exponential map

- Conjugate points are closely related to the behavior of the exponential map on a Riemannian manifold
- The exponential map at a point  $P$ , denoted by  $\exp_P$ , maps tangent vectors at  $P$  to points on the manifold by following geodesics starting at  $P$
- A conjugate point  $Q$  of  $P$  along a geodesic corresponds to a critical point of the exponential map  $\exp_P$
- At a conjugate point, the differential of the exponential map becomes singular, indicating a loss of injectivity

## Absence in nonpositive curvature

- In Riemannian manifolds with nonpositive sectional curvature (also known as Hadamard manifolds), conjugate points do not exist
- Nonpositive curvature implies that geodesics diverge from each other, preventing the occurrence of conjugate points

- The absence of conjugate points in nonpositively curved manifolds has important consequences for the global geometry and topology of the manifold
- Many results in geometry, such as the Cartan-Hadamard theorem and the visibility boundary, rely on the absence of conjugate points in nonpositively curved spaces

## Comparison of cut locus vs conjugate points

- Cut locus and conjugate points are two distinct but related concepts in Riemannian geometry that provide different insights into the behavior of geodesics and the structure of the manifold
- While both concepts are concerned with the properties of geodesics, they differ in their definitions, properties, and implications for the geometry of the manifold
- Understanding the similarities and differences between cut locus and conjugate points is important for a comprehensive understanding of Riemannian geometry

## Differences in definitions

- The cut locus is defined in terms of the global minimizing properties of geodesics, while conjugate points are defined in terms of the local infinitesimal behavior of geodesics
- A cut point is a point where a geodesic ceases to be minimizing, while a conjugate point is a point where Jacobi fields along a geodesic vanish
- The cut locus is a subset of the manifold, while conjugate points are specific points along a geodesic

## Similarities in properties

- Both cut locus and conjugate points provide information about the behavior of geodesics on a Riemannian manifold
- The presence of cut points or conjugate points indicates the existence of non-minimizing or degenerate geodesics
- Both concepts are related to the exponential map and its properties, such as injectivity and critical points
- The absence of cut points or conjugate points has implications for the global geometry and topology of the manifold

## Implications for geometry of manifold

- The cut locus and conjugate points offer complementary perspectives on the geometry of a Riemannian manifold
- The cut locus provides information about the global structure of the manifold, such as its diameter, topology, and shortest paths
- Conjugate points, on the other hand, are related to the local curvature of the manifold and the stability of geodesics
- The interplay between cut locus and conjugate points can be used to study the optimality and bifurcation of geodesics, as well as the singularities and incompleteness of the manifold
- Understanding both concepts is essential for a comprehensive analysis of the geometric properties of Riemannian manifolds and their applications in various fields

## 4.4 Riemannian distance function

The Riemannian distance function measures the shortest path between points on a Riemannian manifold. It's defined using the metric tensor and curve lengths, providing a way to quantify the manifold's intrinsic geometry.

This function has key properties like non-negativity, symmetry, and the triangle inequality. It's computed using geodesics and the exponential map, playing a crucial role in understanding manifold geometry and topology.

### Definition of Riemannian distance function

- The Riemannian distance function is a fundamental concept in Riemannian geometry that measures the shortest path between two points on a Riemannian manifold
- It is defined using the metric tensor and the length of curves on the manifold
- The Riemannian distance function provides a way to quantify the intrinsic geometry of the manifold and plays a crucial role in various geometric and analytical aspects of Riemannian geometry

### Riemannian manifolds

- A Riemannian manifold is a smooth manifold equipped with a Riemannian metric, which is a positive definite symmetric (0,2)-tensor field on the manifold
- The Riemannian metric allows one to measure angles, lengths, and volumes on the manifold in a consistent manner
- Examples of Riemannian manifolds include Euclidean spaces, spheres, hyperbolic spaces, and Lie groups with bi-invariant metrics

### Metric tensor

- The metric tensor, denoted by  $g$ , is a smooth (0,2)-tensor field on a Riemannian manifold that assigns an inner product to each tangent space
- In local coordinates  $(x^1, \dots, x^n)$ , the metric tensor is represented by a symmetric positive definite matrix  $(g_{ij})$
- The metric tensor allows one to compute the length of tangent vectors and the angle between them, which is essential for defining the Riemannian distance function

### Length of curves

- Given a smooth curve  $\gamma : [a, b] \rightarrow M$  on a Riemannian manifold  $(M, g)$ , its length is defined as:  
$$L(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} dt$$
- The length of a curve is independent of the parametrization and is invariant under isometries of the Riemannian manifold
- The Riemannian distance between two points is defined as the infimum of the lengths of all curves connecting them



## Properties of Riemannian distance function

- The Riemannian distance function, denoted by  $d : M \times M \rightarrow \mathbb{R}$ , satisfies several important properties that characterize it as a metric on the Riemannian manifold
- These properties reflect the geometric nature of the Riemannian distance and allow for the study of various geometric and topological aspects of the manifold
- The properties of the Riemannian distance function are crucial in understanding the behavior of geodesics, the topology of the manifold, and the interplay between geometry and analysis on Riemannian manifolds

## Non-negativity

- For any two points  $p, q \in M$ , the Riemannian distance  $d(p, q) \geq 0$
- The distance is zero if and only if  $p = q$ , i.e.,  $d(p, q) = 0 \Leftrightarrow p = q$
- Non-negativity ensures that the Riemannian distance function is a meaningful measure of the separation between points on the manifold

## Symmetry

- The Riemannian distance is symmetric, i.e., for any  $p, q \in M$ ,  $d(p, q) = d(q, p)$
- Symmetry reflects the fact that the shortest path from  $p$  to  $q$  is the same as the shortest path from  $q$  to  $p$  (up to parametrization)
- This property is a consequence of the symmetry of the metric tensor  $g$

## Triangle inequality

- For any three points  $p, q, r \in M$ , the Riemannian distance satisfies the triangle inequality:  $d(p, r) \leq d(p, q) + d(q, r)$
- The triangle inequality states that the shortest path between two points is always less than or equal to the sum of the lengths of any other paths connecting them through an intermediate point
- This property is crucial in proving the existence and uniqueness of geodesics and in studying the topology of the manifold

## Positive definiteness

- The Riemannian distance function is positive definite, i.e., for any  $p, q \in M$ ,  $d(p, q) > 0$  if  $p \neq q$
- Positive definiteness ensures that distinct points on the manifold have a non-zero distance between them
- This property is a consequence of the positive definiteness of the metric tensor  $g$  and is essential for the Riemannian distance to be a metric on the manifold

## Computation of Riemannian distance

- Computing the Riemannian distance between two points on a manifold involves finding the shortest path (geodesic) connecting them

- Several techniques and tools are used to compute Riemannian distances, including the study of geodesics, the exponential map, and variational principles
- The computation of Riemannian distances is a central problem in Riemannian geometry and has applications in various fields such as computer vision, robotics, and machine learning

## Geodesics

- Geodesics are locally length-minimizing curves on a Riemannian manifold that generalize the concept of straight lines in Euclidean space
- In local coordinates, geodesics are characterized by the geodesic equation:  $\frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$  where  $\Gamma_{jk}^i$  are the Christoffel symbols of the Levi-Civita connection
- Geodesics play a fundamental role in computing Riemannian distances, as the shortest path between two points is always a geodesic (or a concatenation of geodesic segments)

## Exponential map

- The exponential map  $\exp_p : T_p M \rightarrow M$  at a point  $p \in M$  is a local diffeomorphism that maps tangent vectors to geodesics starting at  $p$
- For a tangent vector  $v \in T_p M$ ,  $\exp_p(v)$  is the point on the manifold obtained by following the geodesic starting at  $p$  with initial velocity  $v$  for a unit time
- The exponential map is a powerful tool for computing Riemannian distances, as it allows one to map the problem of finding geodesics on the manifold to the tangent space

## Minimizing length of curves

- Computing the Riemannian distance between two points  $p, q \in M$  involves finding the curve of minimal length connecting them
- This can be formulated as a variational problem, where one seeks to minimize the energy functional:  $E(\gamma) = \frac{1}{2} \int_a^b g(\gamma'(t), \gamma'(t)) dt$  among all curves  $\gamma : [a, b] \rightarrow M$  with  $\gamma(a) = p$  and  $\gamma(b) = q$
- The minimizing curve is a geodesic, and its length is the Riemannian distance between  $p$  and  $q$
- Numerical methods, such as the shooting method or the fast marching method, can be used to solve this variational problem and compute Riemannian distances

## Applications of Riemannian distance

- The Riemannian distance function has numerous applications in various areas of mathematics and beyond
- It plays a crucial role in understanding the geometry and topology of Riemannian manifolds, as well as in the study of geometric flows, harmonic maps, and eigenvalue problems
- Riemannian distances also find applications in fields such as computer vision, robotics, and machine learning, where they are used for shape analysis, path planning, and data classification

## Geometry of surfaces

- For two-dimensional Riemannian manifolds (surfaces), the Riemannian distance function encodes important geometric properties

- The Gaussian curvature of a surface can be expressed in terms of the Riemannian distance function using the Gauss-Bonnet theorem
- The Riemannian distance also determines the geodesic triangles on the surface, which are essential for understanding the intrinsic geometry of the surface (e.g., the sum of angles in a geodesic triangle is related to the total curvature enclosed by the triangle)

## Shortest paths on manifolds

- The Riemannian distance function provides a natural way to define shortest paths (geodesics) between points on a manifold
- Shortest paths have applications in various fields, such as robotics (path planning), computer graphics (curve and surface parameterization), and medical imaging (tractography in diffusion tensor imaging)
- The computation of shortest paths on manifolds involves solving the geodesic equation or minimizing the length functional, which can be done using numerical methods like the fast marching method or the shooting method

## Isometry groups

- The Riemannian distance function is invariant under isometries of the Riemannian manifold, i.e., diffeomorphisms that preserve the Riemannian metric
- The group of isometries of a Riemannian manifold, denoted by  $\text{Isom}(M, g)$ , is a Lie group that acts on the manifold by preserving distances
- The study of isometry groups is central to understanding the symmetries of Riemannian manifolds and has applications in various areas, such as the classification of homogeneous spaces and the study of Einstein manifolds

## Curvature via distance function

- The Riemannian distance function encodes information about the curvature of the manifold
- The sectional curvature of a Riemannian manifold can be expressed in terms of the second variation of the Riemannian distance function along geodesics
- The Ricci curvature, which measures the average sectional curvature, can be characterized using the Laplacian of the Riemannian distance function (the Bakry-Émery theorem)
- These connections between curvature and the Riemannian distance function are fundamental in the study of comparison geometry and the analysis of geometric flows (e.g., the Ricci flow)

## Relationship to other distance functions

- The Riemannian distance function is a specific instance of a more general class of distance functions on manifolds and metric spaces
- Understanding the relationship between the Riemannian distance and other distance functions helps to contextualize its properties and applications
- Comparing the Riemannian distance to other distances also sheds light on the specific features of Riemannian geometry and its connection to other areas of mathematics

## Euclidean distance vs Riemannian distance

- In Euclidean space  $\mathbb{R}^n$ , the Riemannian distance function coincides with the usual Euclidean distance, which is defined using the dot product
- The Euclidean distance is a special case of the Riemannian distance, corresponding to the flat metric on  $\mathbb{R}^n$
- On general Riemannian manifolds, the Riemannian distance differs from the Euclidean distance due to the curvature of the manifold, which affects the lengths of curves and the behavior of geodesics

## Comparison with Finsler distance

- Finsler geometry is a generalization of Riemannian geometry, where the metric tensor is replaced by a more general object called the Finsler metric, which may depend on both the position and the direction
- The Finsler distance function is defined similarly to the Riemannian distance, but using the Finsler metric instead of the Riemannian metric
- Finsler distances share some properties with Riemannian distances (e.g., non-negativity, symmetry, triangle inequality) but may lack others (e.g., the existence and uniqueness of geodesics)
- Studying the relationship between Riemannian and Finsler distances helps to understand the role of the quadratic nature of the Riemannian metric in the properties of the Riemannian distance function

## Induced distance functions on submanifolds

- Given a Riemannian manifold  $(M, g)$  and a submanifold  $N \subset M$ , the Riemannian metric  $g$  induces a Riemannian metric  $g_N$  on  $N$ , called the induced metric
- The induced metric  $g_N$  gives rise to an induced Riemannian distance function  $d_N$  on the submanifold  $N$
- The induced distance  $d_N(p, q)$  between two points  $p, q \in N$  is always greater than or equal to the ambient distance  $d(p, q)$  in  $M$ , with equality holding if and only if the shortest path between  $p$  and  $q$  in  $M$  lies entirely within  $N$
- Understanding the relationship between the induced distance and the ambient distance is important in the study of submanifold geometry and the behavior of geodesics on submanifolds

## 4.5 Completeness and Hopf-Rinow theorem

Completeness in metric spaces ensures that Cauchy sequences converge within the space. This property is crucial for understanding the behavior of geodesics and the overall structure of Riemannian manifolds.

The Hopf-Rinow theorem connects completeness, geodesic completeness, and the existence of minimal geodesics. It's a cornerstone result in Riemannian geometry, providing insights into the relationship between metric and geometric properties of manifolds.

### Definition of completeness

- Completeness is a fundamental property of metric spaces that ensures the existence of limits for certain sequences
- A metric space is considered complete if every Cauchy sequence in the space converges to a point within the space

- Completeness is a stronger condition than convergence, as it requires all Cauchy sequences to converge, not just some

## Cauchy sequences

- A sequence  $\{x_n\}$  in a metric space  $(X, d)$  is called a Cauchy sequence if for every  $\varepsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that  $d(x_m, x_n) < \varepsilon$  for all  $m, n \geq N$
- Intuitively, a Cauchy sequence is a sequence whose terms become arbitrarily close to each other as the sequence progresses
- Examples of Cauchy sequences include convergent sequences in  $\mathbb{R}$  (real numbers) and  $\mathbb{C}$  (complex numbers)

## Convergent sequences

- A sequence  $\{x_n\}$  in a metric space  $(X, d)$  is said to converge to a point  $x \in X$  if for every  $\varepsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that  $d(x_n, x) < \varepsilon$  for all  $n \geq N$
- Convergent sequences are Cauchy sequences, but the converse is not always true
- In a complete metric space, every Cauchy sequence is convergent

## Completeness in metric spaces

- A metric space  $(X, d)$  is complete if every Cauchy sequence in  $X$  converges to a point in  $X$
- Examples of complete metric spaces include  $\mathbb{R}$  (real numbers),  $\mathbb{C}$  (complex numbers), and  $\mathbb{R}^n$  (Euclidean space)
- The space of rational numbers  $\mathbb{Q}$  is not complete, as there exist Cauchy sequences in  $\mathbb{Q}$  that do not converge to rational numbers (e.g., the sequence of decimal approximations of  $\sqrt{2}$ )

## Geodesics in complete spaces

- Geodesics are the shortest paths between points in a metric space, generalizing the concept of straight lines in Euclidean space
- In a complete Riemannian manifold, the existence and uniqueness of geodesics can be established using the Hopf-Rinow theorem
- Completeness plays a crucial role in the study of geodesics and their properties

## Existence of geodesics

- In a complete Riemannian manifold, the Hopf-Rinow theorem guarantees the existence of a geodesic connecting any two points
- This means that for any two points  $p, q$  in the manifold, there exists a curve  $\gamma : [a, b] \rightarrow M$  such that  $\gamma(a) = p$ ,  $\gamma(b) = q$ , and  $\gamma$  is a geodesic
- The existence of geodesics is a consequence of the completeness of the manifold and the properties of the Riemannian metric

## Uniqueness of geodesics

- In a complete Riemannian manifold, geodesics are locally unique, meaning that for any point  $p$  and any tangent vector  $v$  at  $p$ , there exists a unique geodesic  $\gamma : [a, b] \rightarrow M$  such that  $\gamma(a) = p$  and  $\gamma'(a) = v$
- However, globally, geodesics may not be unique, as there can be multiple geodesics connecting two points (e.g., great circles on a sphere)
- The uniqueness of geodesics is related to the curvature of the manifold and the existence of conjugate points

## Geodesic completeness

- A Riemannian manifold is said to be geodesically complete if every geodesic can be extended indefinitely in both directions
- Geodesic completeness is a stronger condition than completeness as a metric space
- In a geodesically complete manifold, the exponential map  $\exp_p : T_p M \rightarrow M$  is defined on the entire tangent space  $T_p M$  for every point  $p$

## Hopf-Rinow theorem

- The Hopf-Rinow theorem is a fundamental result in Riemannian geometry that establishes the equivalence of several important properties of Riemannian manifolds
- It connects the concepts of completeness, geodesic completeness, and the existence of minimal geodesics between points
- The theorem has significant implications for the study of the geometry and topology of Riemannian manifolds

## Statement of the theorem

- For a connected Riemannian manifold  $M$ , the following statements are equivalent: 1.  $M$  is complete as a metric space 2.  $M$  is geodesically complete 3. Every closed and bounded subset of  $M$  is compact 4. Every pair of points in  $M$  can be connected by a minimal geodesic

## Proof of the theorem

- The proof of the Hopf-Rinow theorem involves several steps and relies on techniques from analysis and differential geometry
- Key steps include:
  - Showing that completeness implies the existence of a minimal geodesic between any two points
  - Proving that the existence of minimal geodesics implies geodesic completeness
  - Establishing the equivalence between completeness and the compactness of closed and bounded subsets

## Implications of the theorem

- The Hopf-Rinow theorem has several important consequences for the study of Riemannian manifolds:

- It allows for the use of techniques from metric geometry in the study of Riemannian manifolds
- It ensures the existence of minimal geodesics, which are crucial for understanding the geometry and topology of the manifold
- It provides a connection between the completeness of the manifold and the behavior of its geodesics

## Relationship between completeness and compactness

- Completeness and compactness are two important properties of metric spaces that are closely related
- In the context of Riemannian manifolds, the Hopf-Rinow theorem establishes a connection between completeness and the compactness of certain subsets
- Understanding the relationship between these properties is crucial for the study of the geometry and topology of metric spaces and Riemannian manifolds

## Definitions of compactness

- A metric space  $(X, d)$  is said to be compact if every open cover of  $X$  has a finite subcover
- Equivalently, a metric space is compact if every sequence in the space has a subsequence that converges to a point in the space
- Compactness is a topological property that captures the notion of a space being "bounded" and "closed"

## Compact vs complete spaces

- Compactness and completeness are distinct properties of metric spaces
- A compact metric space is always complete, but a complete metric space may not be compact
- For example, the real line  $\mathbb{R}$  is complete but not compact, while the closed unit interval  $[0, 1]$  is both complete and compact

## Compact subsets of complete spaces

- In a complete metric space, closed and bounded subsets are compact (Heine-Borel theorem)
- This property is particularly important in the context of Riemannian manifolds, as the Hopf-Rinow theorem establishes the equivalence between completeness and the compactness of closed and bounded subsets
- The compactness of certain subsets can be used to prove the existence of minimal geodesics and other geometric properties of the manifold

## Applications of completeness and Hopf-Rinow theorem

- The concepts of completeness and the Hopf-Rinow theorem have numerous applications in Riemannian geometry and related fields
- They provide a foundation for the study of the geometry and topology of Riemannian manifolds and are used in the analysis of geodesics, curvature, and other geometric properties
- Some specific applications include:

## Riemannian manifolds

- The Hopf-Rinow theorem is a fundamental tool in the study of Riemannian manifolds
- It allows for the use of techniques from metric geometry and provides a connection between the completeness of the manifold and the behavior of its geodesics
- The theorem is used in the proof of various results, such as the Bonnet-Myers theorem and the Hadamard-Cartan theorem

## Geodesic distance and completeness

- In a complete Riemannian manifold, the geodesic distance function  $d(p, q)$  is well-defined for all points  $p, q$  in the manifold
- The completeness of the manifold ensures that the infimum of the lengths of curves connecting two points is always attained by a minimal geodesic
- This property is crucial for the study of the metric structure of the manifold and the behavior of geodesics

## Geometric consequences of completeness

- Completeness and the Hopf-Rinow theorem have several important geometric consequences:
- In a complete Riemannian manifold, the exponential map  $\exp_p : T_p M \rightarrow M$  is surjective for every point  $p$
- Complete Riemannian manifolds with non-positive sectional curvature are diffeomorphic to Euclidean space (Hadamard-Cartan theorem)
- Complete Riemannian manifolds with positive Ricci curvature have finite diameter and are compact (Bonnet-Myers theorem)

## Counterexamples and pathological cases

- While completeness and the Hopf-Rinow theorem provide a solid foundation for the study of Riemannian manifolds, there exist counterexamples and pathological cases that demonstrate the limitations and subtleties of these concepts
- These examples highlight the importance of the assumptions in the theorem and the need for careful analysis when working with non-complete or non-Riemannian spaces

## Incomplete metric spaces

- Incomplete metric spaces, such as the space of rational numbers  $\mathbb{Q}$ , provide examples where Cauchy sequences may not converge
- In these spaces, the Hopf-Rinow theorem does not apply, and the relationship between completeness, geodesics, and compactness breaks down
- Studying incomplete spaces can lead to a deeper understanding of the role of completeness in metric geometry



## Spaces with no geodesics

- There exist metric spaces and even Riemannian manifolds where geodesics may not exist between certain points
- For example, consider the punctured plane  $\mathbb{R}^2 \setminus \{(0, 0)\}$  with the induced Euclidean metric
- In this space, there is no geodesic connecting points on opposite sides of the removed origin, as any curve connecting them must pass through the origin

## Spaces with non-unique geodesics

- In some spaces, geodesics between two points may not be unique, even locally
- This can occur in spaces with non-smooth or degenerate metrics, such as the Manhattan metric on  $\mathbb{R}^2$
- In such cases, the uniqueness part of the Hopf-Rinow theorem does not hold, and the relationship between geodesics and the geometry of the space becomes more complex

## Generalizations and related concepts

- The concepts of completeness and the Hopf-Rinow theorem can be generalized and extended to other settings beyond Riemannian manifolds
- These generalizations provide a broader perspective on the role of completeness in various mathematical contexts and highlight the connections between different areas of geometry and analysis

## Completeness in normed vector spaces

- The notion of completeness can be defined for normed vector spaces, where the metric is induced by the norm
- A normed vector space is complete if every Cauchy sequence in the space converges to a point in the space
- Complete normed vector spaces, also known as Banach spaces, play a fundamental role in functional analysis and have numerous applications in mathematics and physics

## Completeness in topological spaces

- Completeness can also be defined for topological spaces, using the concept of a uniform structure
- A uniform space is complete if every Cauchy net in the space converges to a point in the space
- Complete uniform spaces, such as complete metric spaces and complete topological groups, have important applications in topology and analysis

## Relationship to other geometric properties

- Completeness is closely related to other geometric properties, such as curvature and convexity
- In Riemannian manifolds, the relationship between completeness, curvature, and topology is explored through various theorems, such as the Hadamard-Cartan theorem and the Bonnet-Myers theorem
- In metric spaces, completeness is connected to properties such as total boundedness, compactness, and local compactness, which play important roles in the study of the geometry and topology of these

## 4.6 Spaces of constant curvature

Spaces of constant curvature are fundamental in differential geometry. They're Riemannian manifolds where the sectional curvature is the same everywhere. These spaces include Euclidean (flat), spherical (positive curvature), and hyperbolic (negative curvature) geometries.

These spaces serve as models for more complex geometries. They have simple curvature tensors and unique geodesics. Understanding their properties helps us grasp the relationship between curvature, topology, and the behavior of geodesics in more general settings.

### Definition of constant curvature spaces

- Constant curvature spaces are Riemannian manifolds where the sectional curvature is the same at every point and in every direction
- The curvature tensor and scalar curvature can be expressed in terms of a single constant, which determines the geometry of the space
- Constant curvature spaces are the simplest and most symmetric Riemannian manifolds, serving as important models in differential geometry and physics

### Riemannian manifolds of constant sectional curvature

- A Riemannian manifold has constant sectional curvature if the sectional curvature  $K(P)$  is the same for all 2-dimensional tangent planes  $P$  at every point
- The value of the constant sectional curvature can be positive, zero, or negative, leading to different types of geometry
- Examples of constant sectional curvature manifolds include Euclidean spaces ( $K = 0$ ), spheres ( $K > 0$ ), and hyperbolic spaces ( $K < 0$ )

### Curvature tensor in constant curvature spaces

- The Riemann curvature tensor  $R_{ijkl}$  in a constant curvature space takes a particularly simple form:  $R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$  where  $K$  is the constant sectional curvature and  $g_{ij}$  is the Riemannian metric tensor
- This expression shows that the curvature tensor is completely determined by the metric and the constant  $K$
- The symmetries of the curvature tensor, such as  $R_{ijkl} = -R_{jikl} = -R_{ijlk}$ , are easily verified using this formula

### Scalar curvature in constant curvature spaces

- The scalar curvature  $S$  in a constant curvature space is proportional to the sectional curvature  $K$ :  $S = n(n - 1)K$  where  $n$  is the dimension of the manifold
- This relation demonstrates that the scalar curvature is also constant throughout the space
- In two dimensions, the scalar curvature is twice the Gaussian curvature, while in higher dimensions, it is a measure of the average sectional curvature

## Classification of constant curvature spaces

- Constant curvature spaces can be classified into three main categories based on the sign of their sectional curvature: Euclidean spaces (zero curvature), spherical spaces (positive curvature), and hyperbolic spaces (negative curvature)
- These spaces have distinct geometric properties and can be characterized by their isometry groups and geodesic structures
- The classification of constant curvature spaces is closely related to the uniformization theorem for Riemann surfaces and the geometrization conjecture for 3-manifolds

## Euclidean spaces

- Euclidean spaces  $\mathbb{E}^n$  are the simplest examples of constant curvature spaces, with sectional curvature  $K = 0$
- They are flat, meaning that parallel lines remain parallel and the angles of a triangle sum to 180 degrees
- Euclidean spaces are isotropic and homogeneous, with translations and rotations as isometries

## Spherical spaces

- Spherical spaces  $\mathbb{S}^n$  are constant curvature spaces with positive sectional curvature  $K > 0$
- They are compact and have a finite volume, with great circles as geodesics
- The  $n$ -sphere can be viewed as the set of points equidistant from a central point in  $(n + 1)$ -dimensional Euclidean space

## Hyperbolic spaces

- Hyperbolic spaces  $\mathbb{H}^n$  are constant curvature spaces with negative sectional curvature  $K < 0$
- They have an infinite volume and exhibit unique geometric properties, such as the existence of infinitely many parallel lines through a point not on a given line
- Hyperbolic spaces can be modeled using various representations, such as the hyperboloid model and the Poincaré disk model

## Uniqueness up to isometry

- A fundamental result in the theory of constant curvature spaces is that they are unique up to isometry for a given dimension and sign of curvature
- This means that any two constant curvature spaces with the same dimension and curvature sign are isometric, i.e., there exists a distance-preserving bijection between them
- This uniqueness property highlights the special role of constant curvature spaces in geometry and allows for the study of their properties using various models and representations

## Models of constant curvature spaces

- Several models are used to represent and study constant curvature spaces, each with its own advantages and applications

- These models provide different ways to visualize and analyze the geometric properties of Euclidean, spherical, and hyperbolic spaces
- The relationships between these models can be established through coordinate transformations and isometries, allowing for a deeper understanding of the underlying geometry

## Euclidean model

- The Euclidean model represents Euclidean space  $\mathbb{E}^n$  using Cartesian coordinates  $(x_1, \dots, x_n)$
- The Euclidean metric is given by  $ds^2 = dx_1^2 + \dots + dx_n^2$ , which measures the distance between points
- Euclidean space is flat, and the shortest paths between points are straight lines

## Spherical model

- The spherical model represents the  $n$ -sphere  $\mathbb{S}^n$  as the set of points equidistant from the origin in  $(n + 1)$ -dimensional Euclidean space
- The spherical metric is induced from the Euclidean metric and can be expressed using angular coordinates
- Geodesics on the sphere are great circles, which are the intersection of the sphere with planes passing through the origin

## Hyperboloid model

- The hyperboloid model represents hyperbolic space  $\mathbb{H}^n$  as the upper sheet of a two-sheeted hyperboloid in  $(n + 1)$ -dimensional Minkowski space
- The metric is induced from the Minkowski metric  $ds^2 = -dx_0^2 + dx_1^2 + \dots + dx_n^2$ , with the constraint  $-x_0^2 + x_1^2 + \dots + x_n^2 = -1$
- Geodesics in this model are the intersection of the hyperboloid with planes passing through the origin

## Poincaré disk model

- The Poincaré disk model represents hyperbolic space  $\mathbb{H}^n$  as the interior of a unit disk in Euclidean space
- The hyperbolic metric is given by  $ds^2 = 4(dx_1^2 + \dots + dx_n^2)/(1 - r^2)^2$ , where  $r^2 = x_1^2 + \dots + x_n^2$
- Geodesics in this model are circular arcs perpendicular to the boundary of the disk

## Poincaré half-plane model

- The Poincaré half-plane model represents hyperbolic space  $\mathbb{H}^n$  as the upper half-plane  $\{(x_1, \dots, x_n) : x_n > 0\}$  in Euclidean space
- The hyperbolic metric is given by  $ds^2 = (dx_1^2 + \dots + dx_n^2)/x_n^2$
- Geodesics in this model are semicircles perpendicular to the boundary (the  $x$ -axis) and vertical lines

## Relationships between models

- The models of constant curvature spaces are related by isometries and coordinate transformations

- For example, stereographic projection establishes an isometry between the sphere (minus a point) and Euclidean space
- The Poincaré disk and half-plane models of hyperbolic space are related by a Möbius transformation, which is an isometry of hyperbolic space
- Understanding the relationships between models allows for the transfer of geometric insights and the application of different tools and techniques in the study of constant curvature spaces

## Geodesics in constant curvature spaces

- Geodesics are the shortest paths between points on a Riemannian manifold and play a crucial role in the study of constant curvature spaces
- The properties of geodesics in Euclidean, spherical, and hyperbolic spaces reflect the distinct geometric characteristics of these spaces
- Geodesic equations can be derived using variational principles or the Christoffel symbols of the Levi-Civita connection

## Geodesic equations

- The geodesic equations are a system of second-order differential equations that describe the paths of geodesics on a Riemannian manifold
- In local coordinates  $(x^1, \dots, x^n)$ , the geodesic equations are given by:  $\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0$  where  $\Gamma_{jk}^i$  are the Christoffel symbols of the Levi-Civita connection
- Solutions to the geodesic equations are the parametrized curves that minimize the distance between points on the manifold

## Geodesics in Euclidean spaces

- In Euclidean space  $E^n$ , geodesics are straight lines, as expected from the flat geometry
- The geodesic equations reduce to  $\ddot{x}^i = 0$ , implying that the coordinates of a geodesic are linear functions of the parameter
- The Euclidean distance between points is given by the Pythagorean theorem, and the shortest path is a straight line segment

## Geodesics in spherical spaces

- On the  $n$ -sphere  $S^n$ , geodesics are great circles, which are the intersection of the sphere with planes passing through the origin
- The geodesic equations on the sphere can be derived using the induced metric and the constraint  $x_1^2 + \dots + x_{n+1}^2 = r^2$ , where  $r$  is the radius of the sphere
- The length of a geodesic arc on the sphere is proportional to the angle it subtends at the center, and the shortest path between points is along a great circle

## Geodesics in hyperbolic spaces

- In hyperbolic space  $H^n$ , geodesics have unique properties due to the negative curvature

- In the hyperboloid model, geodesics are the intersection of the hyperboloid with planes passing through the origin
- In the Poincaré disk and half-plane models, geodesics are circular arcs or lines perpendicular to the boundary
- The length of a geodesic segment in hyperbolic space grows exponentially with respect to the Euclidean distance between its endpoints

## Isometry groups of constant curvature spaces

- The isometry group of a Riemannian manifold is the group of all distance-preserving transformations (isometries) of the manifold
- Constant curvature spaces have large and highly symmetric isometry groups, which reflect their homogeneity and isotropy
- The structure and action of isometry groups provide valuable insights into the geometry and symmetries of constant curvature spaces

## Isometries of Euclidean spaces

- The isometry group of Euclidean space  $\mathbb{E}^n$  is the Euclidean group  $E(n)$ , which consists of translations and rotations
- Translations are represented by vectors in  $\mathbb{R}^n$  and form a normal subgroup of  $E(n)$ , while rotations are represented by orthogonal matrices and form the orthogonal group  $O(n)$
- The Euclidean group is a semidirect product of the translation group and the orthogonal group,  $E(n) = \mathbb{R}^n \rtimes O(n)$

## Isometries of spherical spaces

- The isometry group of the  $n$ -sphere  $\mathbb{S}^n$  is the orthogonal group  $O(n+1)$ , consisting of all  $(n+1) \times (n+1)$  orthogonal matrices
- Isometries of the sphere can be represented as rotations in the ambient Euclidean space  $\mathbb{R}^{n+1}$
- The group  $O(n+1)$  has two connected components: the special orthogonal group  $SO(n+1)$  (rotations) and the set of orthogonal matrices with determinant -1 (rotations composed with a reflection)

## Isometries of hyperbolic spaces

- The isometry group of hyperbolic space  $\mathbb{H}^n$  depends on the model being used
- In the hyperboloid model, the isometry group is the Lorentz group  $O(1, n)$ , consisting of linear transformations that preserve the Minkowski metric
- In the Poincaré disk and half-plane models, the isometry group is the group of Möbius transformations that preserve the unit disk or upper half-plane, respectively
- The isometry group of hyperbolic space has a rich structure and includes both continuous isometries (e.g., rotations and translations) and discrete subgroups (e.g., Fuchsian and Kleinian groups)

## Transitive action of isometry groups

- The isometry groups of constant curvature spaces act transitively on the manifold, meaning that for any two points, there exists an isometry that maps one point to the other
- This property reflects the homogeneity of constant curvature spaces and implies that the geometry is the same at every point
- The transitive action of isometry groups also allows for the study of constant curvature spaces using group-theoretic methods and the construction of quotient spaces with interesting geometric and topological properties

## Curvature and topology

- The curvature of a Riemannian manifold is closely related to its topological properties, as demonstrated by the Gauss-Bonnet theorem and its generalizations
- The relationship between curvature and topology plays a crucial role in the classification and understanding of surfaces and higher-dimensional manifolds
- Constant curvature spaces provide important examples and models for studying the interplay between geometry and topology

## Gauss-Bonnet theorem for surfaces

- The Gauss-Bonnet theorem is a fundamental result in differential geometry that relates the Gaussian curvature of a compact, oriented surface to its Euler characteristic
- For a compact, oriented surface  $M$  with boundary  $\partial M$ , the theorem states:  
$$\int_M K dA + \int_{\partial M} k_g ds = 2\pi\chi(M)$$
 where  $K$  is the Gaussian curvature,  $k_g$  is the geodesic curvature of the boundary, and  $\chi(M)$  is the Euler characteristic of the surface
- The theorem has important implications for the topology of surfaces, such as the classification of closed surfaces by their Euler characteristic and the existence of elliptic, parabolic, and hyperbolic geometries on surfaces

## Generalized Gauss-Bonnet theorem

- The Gauss-Bonnet theorem can be generalized to higher-dimensional Riemannian manifolds using the concept of the Pfaffian of the curvature form
- For a compact, oriented, even-dimensional Riemannian manifold  $M$ , the generalized Gauss-Bonnet theorem states:  $\int_M \text{Pf}(\Omega) = \chi(M)$  where  $\text{Pf}(\Omega)$  is the Pfaffian of the curvature form and  $\chi(M)$  is the Euler characteristic of the manifold
- The generalized theorem relates the integral of a curvature invariant to a topological invariant and has applications in the study of characteristic classes and index theorems

## Euler characteristic and curvature

- The Euler characteristic is a topological invariant that measures the "holes" or "connectivity" of a space
- For a compact, oriented surface, the Euler characteristic is related to the genus  $g$  (number of handles) by  $\chi = 2 - 2g$

- The Gauss-Bonnet theorem implies that the integral of the Gaussian curvature over a closed surface is determined by its Euler characteristic, establishing a deep connection between curvature and topology
- In higher dimensions, the Euler characteristic appears in the generalized Gauss-Bonnet theorem and is related to the integral of curvature invariants, such as the Pfaffian of the curvature form

## Compact spaces of constant curvature

- Compact spaces of constant curvature are particularly interesting from both geometric and topological perspectives
- The sphere  $\mathbb{S}^n$  is the only compact, simply connected space of positive constant curvature (up to scaling)
- Compact flat spaces (zero curvature) are quotients of the torus  $\mathbb{T}^n$  by discrete isometry groups and are classified by their fundamental group (which is a Bieberbach group)
- Compact hyperbolic spaces (negative curvature) are more diverse and can be obtained as quotients of hyperbolic space  $\mathbb{H}^n$



# Unit 5 – Riemann Curvature Tensor and Its Applications

## 5.1 Covariant derivative

The covariant derivative extends directional derivatives to curved spaces, allowing differentiation of vector and tensor fields on manifolds. It accounts for curvature and connection, enabling comparison of vectors at different points while preserving geometric properties.

This concept is crucial for understanding how fields change along curves and surfaces on manifolds. The covariant derivative incorporates the connection, describing vector transport along curves, and satisfies key properties like linearity and the Leibniz rule.

### Definition of covariant derivative

- The covariant derivative is a fundamental concept in Metric Differential Geometry that extends the notion of directional derivatives to curved spaces
- It provides a way to differentiate vector fields and tensor fields on a manifold while preserving the geometric properties of the manifold
- The covariant derivative takes into account the curvature and connection of the manifold, allowing for the comparison of vectors and tensors at different points

### Motivation for covariant derivative

- In Euclidean space, the usual directional derivative is sufficient for differentiating vector fields, but on curved manifolds, it fails to capture the intrinsic geometry
- The covariant derivative is introduced to address this issue by incorporating the connection, which describes how vectors are transported along curves on the manifold
- It enables the study of how vector fields and tensor fields change along curves and surfaces, which is crucial for understanding the geometry of the manifold

### Formal definition

- Let  $M$  be a smooth manifold with a connection  $\nabla$ . The covariant derivative of a vector field  $X$  with respect to another vector field  $Y$  is denoted by  $\nabla_Y X$
- The covariant derivative satisfies the following properties: 1. Linearity:  $\nabla_{aY_1 + bY_2} X = a\nabla_{Y_1} X + b\nabla_{Y_2} X$   
2. Leibniz rule:  $\nabla_Y(fX) = Y(f)X + f\nabla_Y X$ , where  $f$  is a smooth function on  $M$   
3. Compatibility with the tangent bundle:  $\nabla_Y X$  is a vector field on  $M$
- The connection  $\nabla$  determines how the covariant derivative behaves and is chosen based on the geometric properties of the manifold

## Properties of covariant derivative

- The covariant derivative satisfies several important properties that make it a useful tool in Metric Differential Geometry: 1. Linearity in both arguments:

$$\nabla_{aY_1 + bY_2}(cX_1 + dX_2) = ac\nabla_{Y_1}X_1 + ad\nabla_{Y_1}X_2 + bc\nabla_{Y_2}X_1 + bd\nabla_{Y_2}X_2$$

- 2. Product rule:  
 $\nabla_Y(X_1 \otimes X_2) = (\nabla_Y X_1) \otimes X_2 + X_1 \otimes (\nabla_Y X_2)$  3. Compatibility with the metric (for Riemannian manifolds):  $Y\langle X_1, X_2 \rangle = \langle \nabla_Y X_1, X_2 \rangle + \langle X_1, \nabla_Y X_2 \rangle$

- These properties ensure that the covariant derivative behaves consistently with the geometric structure of the manifold and allows for the generalization of various calculus concepts to curved spaces

## Covariant derivative of vector fields

- The covariant derivative of vector fields is a fundamental operation in Metric Differential Geometry that describes how vector fields change along curves on a manifold
- It takes into account the curvature and connection of the manifold, allowing for the comparison of vectors at different points
- The covariant derivative of a vector field  $X$  with respect to another vector field  $Y$  is denoted by  $\nabla_Y X$

## Parallel transport

- Parallel transport is a concept closely related to the covariant derivative that describes how vectors are transported along curves on a manifold while preserving their geometric properties
- A vector field  $X$  is said to be parallel along a curve  $Y$  if its covariant derivative along the tangent vector field of  $Y$  is zero, i.e.,  $\nabla_{\dot{Y}} X = 0$
- Parallel transport allows for the comparison of vectors at different points on the manifold by transporting them along curves in a way that preserves their angles and lengths

## Covariant derivative along a curve

- The covariant derivative of a vector field  $X$  along a curve  $Y$  is defined as  $\frac{DX}{dt} = \nabla_{\dot{Y}} X$ , where  $\dot{Y}$  is the tangent vector field of  $Y$
- It measures how the vector field  $X$  changes as it is transported along the curve  $Y$
- The covariant derivative along a curve is a generalization of the usual derivative of a vector-valued function in Euclidean space to curved manifolds

## Covariant derivative in local coordinates

- In local coordinates, the covariant derivative of a vector field  $X = X^i \partial_i$  with respect to another vector field  $Y = Y^j \partial_j$  is given by  $\nabla_Y X = (Y^j \partial_j X^i + Y^j X^k \Gamma_{jk}^i) \partial_i$
- Here,  $\Gamma_{jk}^i$  are the Christoffel symbols, which are the components of the connection in the chosen coordinate system
- The Christoffel symbols encode information about the curvature of the manifold and how vectors are transported along curves

## Covariant derivative of tensor fields

- The covariant derivative can be extended to tensor fields on a manifold, allowing for the differentiation of more general geometric objects
- The covariant derivative of a tensor field  $T$  with respect to a vector field  $Y$  is denoted by  $\nabla_Y T$  and satisfies certain properties that ensure compatibility with the tensor product and contraction operations
- The covariant derivative of tensor fields is crucial for studying the intrinsic geometry of the manifold and formulating physical theories in curved spaces

## Covariant derivative of functions

- Functions on a manifold can be considered as tensor fields of rank 0
- The covariant derivative of a function  $f$  with respect to a vector field  $Y$  is simply the directional derivative of  $f$  along  $Y$ , i.e.,  $\nabla_Y f = Yf = Y^i \partial_i f$
- This definition is consistent with the usual notion of directional derivatives in Euclidean space

## Covariant derivative of 1-forms

- A 1-form  $\omega$  is a tensor field of rank (0, 1) that assigns a real number to each vector at each point on the manifold
- The covariant derivative of a 1-form  $\omega$  with respect to a vector field  $Y$  is defined by the relation  $(\nabla_Y \omega)(X) = Y(\omega(X)) - \omega(\nabla_Y X)$  for any vector field  $X$
- In local coordinates, the covariant derivative of a 1-form  $\omega = \omega_i dx^i$  is given by  $\nabla_Y \omega = (Y^j \partial_j \omega_i - Y^j \omega_k \Gamma_{ji}^k) dx^i$

## Covariant derivative of general tensor fields

- The covariant derivative can be extended to tensor fields of arbitrary rank  $(k, l)$  using the Leibniz rule and the covariant derivatives of vector fields and 1-forms
- For a tensor field  $T = T_{j_1 \dots j_l}^{i_1 \dots i_k} \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l}$ , the covariant derivative with respect to a vector field  $Y$  is given by:

$$\nabla_Y T = (Y^m \partial_m T_{j_1 \dots j_l}^{i_1 \dots i_k} + \sum_{r=1}^k Y^m T_{j_1 \dots j_l}^{i_1 \dots i_k} \Gamma_{mp}^{i_r} - \sum_{s=1}^l Y^m T_{j_1 \dots j_l}^{i_1 \dots i_k} \Gamma_{mj_s}^q) \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l}$$

- The covariant derivative of tensor fields satisfies the Leibniz rule and is compatible with the tensor product and contraction operations, making it a powerful tool for studying the geometry of the manifold

## Torsion and curvature

- Torsion and curvature are two fundamental concepts in Metric Differential Geometry that describe the intrinsic properties of a manifold and its connection
- They are closely related to the covariant derivative and provide important information about the geometric structure of the manifold
- Understanding torsion and curvature is crucial for studying the geometry of curved spaces and formulating physical theories in such spaces

## Definition of torsion

- The torsion of a connection  $\nabla$  on a manifold  $M$  is a tensor field  $T$  of rank (1, 2) that measures the failure of the connection to be symmetric
- It is defined by  $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$  for any vector fields  $X$  and  $Y$  on  $M$
- In local coordinates, the components of the torsion tensor are given by  $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$
- A connection is said to be torsion-free or symmetric if its torsion tensor vanishes identically, i.e.,  $T \equiv 0$

## Torsion-free connections

- A connection  $\nabla$  is called torsion-free or symmetric if its torsion tensor vanishes identically, i.e.,  $T(X, Y) = 0$  for all vector fields  $X$  and  $Y$  on the manifold
- Torsion-free connections are particularly important in Metric Differential Geometry because they are compatible with the notion of parallel transport and geodesics
- The Levi-Civita connection, which is the unique torsion-free connection compatible with the metric on a Riemannian manifold, plays a central role in the study of Riemannian geometry

## Riemann curvature tensor

- The Riemann curvature tensor is a tensor field  $R$  of rank (1, 3) that measures the curvature of a manifold with a connection
- It is defined by  $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$  for any vector fields  $X, Y$ , and  $Z$  on the manifold
- In local coordinates, the components of the Riemann curvature tensor are given by  $R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk}$
- The Riemann curvature tensor provides information about the intrinsic curvature of the manifold and how it affects the parallel transport of vectors and the behavior of geodesics

## Symmetries of Riemann curvature tensor

- The Riemann curvature tensor satisfies several symmetry properties that reflect the geometric properties of the manifold: 1. Antisymmetry in the first two indices:  $R_{ijkl} = -R_{jikl}$  2. Antisymmetry in the last two indices:  $R_{ijkl} = -R_{ijlk}$  3. Symmetry under the exchange of pairs of indices:  $R_{ijkl} = R_{klij}$  4. First Bianchi identity:  $R_{ijkl} + R_{jkil} + R_{kijl} = 0$
- These symmetries reduce the number of independent components of the Riemann curvature tensor and provide constraints on the geometry of the manifold

## Levi-Civita connection

- The Levi-Civita connection is a unique torsion-free connection on a Riemannian manifold that is compatible with the metric
- It is named after the Italian mathematicians Tullio Levi-Civita and Gregorio Ricci-Curbastro, who introduced the concept in the early 20th century
- The Levi-Civita connection plays a central role in Riemannian geometry and is widely used in various applications, including general relativity and other geometric theories of physics

## Metric compatibility

- A connection  $\nabla$  on a Riemannian manifold  $(M, g)$  is said to be compatible with the metric  $g$  if it satisfies  $\nabla_X g = 0$  for all vector fields  $X$  on  $M$
- In local coordinates, this condition is equivalent to  $\partial_k g_{ij} = \Gamma_{ki}^m g_{mj} + \Gamma_{kj}^m g_{im}$
- Metric compatibility ensures that the covariant derivative of the metric tensor vanishes, which means that the metric is preserved under parallel transport

## Christoffel symbols

- The Christoffel symbols are the components of the Levi-Civita connection in a given coordinate system
- They are denoted by  $\Gamma_{jk}^i$  and are defined in terms of the metric tensor  $g_{ij}$  and its derivatives:  
$$\Gamma_{jk}^i = \frac{1}{2} g^{im} (\partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk})$$
- The Christoffel symbols are symmetric in the lower indices, i.e.,  $\Gamma_{jk}^i = \Gamma_{kj}^i$ , due to the torsion-free property of the Levi-Civita connection

## Uniqueness of Levi-Civita connection

- The Levi-Civita connection is the unique connection on a Riemannian manifold that is both torsion-free and compatible with the metric
- This uniqueness property is known as the fundamental theorem of Riemannian geometry
- The existence and uniqueness of the Levi-Civita connection allow for the consistent definition of geometric quantities, such as curvature and geodesics, on a Riemannian manifold

## Covariant derivative in terms of Christoffel symbols

- The covariant derivative of a vector field  $X = X^i \partial_i$  with respect to another vector field  $Y = Y^j \partial_j$  using the Levi-Civita connection is given by:  $\nabla_Y X = (Y^j \partial_j X^i + Y^j X^k \Gamma_{jk}^i) \partial_i$
- Similarly, the covariant derivative of a 1-form  $\omega = \omega_i dx^i$  is given by:  $\nabla_Y \omega = (Y^j \partial_j \omega_i - Y^j \omega_k \Gamma_{ji}^k) dx^i$
- The Christoffel symbols completely determine the covariant derivative and, consequently, the geometric properties of the Riemannian manifold

## Geodesics

- Geodesics are the generalization of straight lines to curved spaces and play a fundamental role in Metric Differential Geometry
- They are defined as curves that parallel transport their own tangent vector, which means that the covariant derivative of the tangent vector along the curve vanishes
- Geodesics are the shortest paths between points on a manifold and are used to study the intrinsic geometry of the space

## Definition of geodesics

- A curve  $\gamma: I \rightarrow M$  on a manifold  $M$  with a connection  $\nabla$  is called a geodesic if its tangent vector  $\dot{\gamma}$  is parallel transported along the curve, i.e.,  $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$

- In local coordinates, this condition leads to the geodesic equation:  $\ddot{\gamma}^j + \Gamma_{jk}^i \dot{\gamma}^j \dot{\gamma}^k = 0$
- Geodesics are the curves that minimize the distance between two points on the manifold, provided they are sufficiently close

## Geodesic equation

- The geodesic equation is a system of second-order ordinary differential equations that describes the motion of a particle along a geodesic on a manifold with a given connection
- In local coordinates, the geodesic equation is given by:  $\ddot{\gamma}^i + \Gamma_{jk}^i \dot{\gamma}^j \dot{\gamma}^k = 0$

## 5.2 Riemann curvature tensor

The Riemann curvature tensor is a key concept in Riemannian geometry, measuring how a manifold's geometry differs from flat space. It encodes crucial information about curvature, affecting geodesics, parallel transport, and the manifold's overall shape.

Understanding the Riemann tensor is vital for grasping the intricate relationship between geometry and physics. It's central to Einstein's theory of general relativity, where spacetime curvature represents gravity, and has applications in string theory and cosmology.

### Definition of Riemann curvature tensor

- The Riemann curvature tensor is a fundamental object in Riemannian geometry that measures the curvature of a Riemannian manifold
- It provides a way to quantify the extent to which the geometry of a manifold deviates from that of Euclidean space
- The study of the Riemann curvature tensor is central to the field of Metric Differential Geometry, as it encodes important geometric and topological properties of manifolds

### Motivation for studying curvature

- Curvature plays a crucial role in understanding the geometry of surfaces and higher-dimensional manifolds
- The notion of curvature allows us to distinguish between different types of geometries, such as flat, positively curved, and negatively curved spaces
- Studying curvature is essential for various applications in physics, such as general relativity, where the curvature of spacetime is related to the presence of matter and energy

### Components and notation

- The Riemann curvature tensor is a (1,3)-tensor denoted by  $R(X, Y)Z$  or  $R_{ijk}^l$  in component notation
- It takes three vector fields  $X$ ,  $Y$ , and  $Z$  as input and produces a vector field as output
- The components of the Riemann curvature tensor can be expressed in terms of the Christoffel symbols  $\Gamma_{ij}^k$  and their partial derivatives

### Relationship to metric tensor

- The Riemann curvature tensor is defined in terms of the metric tensor  $g_{ij}$  and its derivatives

- The metric tensor determines the inner product structure on the tangent spaces of a Riemannian manifold
- The Christoffel symbols, which appear in the expression for the Riemann curvature tensor, are derived from the metric tensor and its first derivatives

## Geometric interpretation

- The Riemann curvature tensor has several geometric interpretations that provide insight into the curvature properties of a Riemannian manifold
- These interpretations help us understand how curvature affects the behavior of geodesics and the geometry of submanifolds

## Deviation of geodesics

- Geodesics are the "straightest" possible curves on a manifold, generalizing the notion of straight lines in Euclidean space
- The Riemann curvature tensor measures the deviation of geodesics from being parallel
- If two initially parallel geodesics on a curved manifold are extended, they will generally not remain parallel, and the Riemann curvature tensor quantifies this deviation

## Sectional curvature

- Sectional curvature is a scalar measure of curvature that depends on a plane (a two-dimensional subspace of the tangent space) at each point of the manifold
- It is obtained by evaluating the Riemann curvature tensor on a pair of orthonormal vectors spanning the plane
- Sectional curvature provides information about the curvature of two-dimensional submanifolds (geodesic surfaces) of the manifold

## Ricci curvature and scalar curvature

- Ricci curvature is a contraction of the Riemann curvature tensor, obtained by summing over two of its indices
- It is a symmetric  $(0,2)$ -tensor that measures the average curvature in different directions at each point
- Scalar curvature is a further contraction of the Ricci curvature, resulting in a scalar function on the manifold
- Scalar curvature represents an overall measure of the curvature at each point, taking into account all directions

## Symmetries and identities

- The Riemann curvature tensor satisfies several important symmetries and identities that reflect its geometric properties and constrain its components
- These symmetries and identities are essential for simplifying calculations and deriving further results in Riemannian geometry

## Skew-symmetry and interchange symmetry

- The Riemann curvature tensor is skew-symmetric in its last two indices:  $R_{ijkl} = -R_{ijlk}$
- It also satisfies the interchange symmetry:  $R_{ijkl} = R_{klij}$
- These symmetries reduce the number of independent components of the Riemann curvature tensor, making it more tractable to work with

## First and second Bianchi identities

- The first Bianchi identity states that the cyclic sum of the Riemann curvature tensor over three indices vanishes:  $R_{ijkl} + R_{iklj} + R_{iljk} = 0$
- The second Bianchi identity involves the covariant derivatives of the Riemann curvature tensor and relates them to the covariant derivatives of the Ricci tensor
- These identities provide important constraints on the Riemann curvature tensor and are used in various geometric and physical applications

## Ricci identity for covariant derivatives

- The Ricci identity relates the commutator of covariant derivatives acting on a tensor to the Riemann curvature tensor
- It states that the difference between taking covariant derivatives in two different orders is given by the action of the Riemann curvature tensor on the tensor
- The Ricci identity is a crucial tool for manipulating covariant derivatives and deriving further identities involving the Riemann curvature tensor

## Curvature and parallel transport

- Parallel transport is a way of moving vectors along curves on a manifold while preserving their direction
- The Riemann curvature tensor is closely related to the concept of parallel transport and its properties

## Parallel transport around loops

- If a vector is parallel transported along a closed loop on a curved manifold, it may not return to its initial value
- The change in the vector after parallel transport around a loop is determined by the Riemann curvature tensor
- The non-vanishing of the Riemann curvature tensor implies that parallel transport is path-dependent

## Holonomy groups and holonomy algebras

- The holonomy group at a point on a manifold is the group of linear transformations obtained by parallel transporting vectors along all possible closed loops starting and ending at that point
- The holonomy algebra is the corresponding Lie algebra of the holonomy group
- The Riemann curvature tensor provides information about the holonomy group and its properties, such as its dimension and structure



## Flat manifolds and vanishing curvature

- A manifold is said to be flat if its Riemann curvature tensor vanishes identically
- In a flat manifold, parallel transport is path-independent, and the holonomy group consists only of the identity transformation
- Examples of flat manifolds include Euclidean spaces and tori, which have zero curvature everywhere

## Curvature and geodesic deviation

- The Riemann curvature tensor plays a fundamental role in describing the behavior of geodesics and their deviation from one another
- Geodesic deviation is a measure of how nearby geodesics spread apart or converge as they are extended on a curved manifold

## Jacobi fields and Jacobi equation

- Jacobi fields are vector fields along a geodesic that describe the infinitesimal variation of the geodesic under small perturbations
- The Jacobi equation is a second-order differential equation that governs the evolution of Jacobi fields along a geodesic
- The Riemann curvature tensor appears in the Jacobi equation, indicating its influence on the behavior of geodesic deviation

## Conjugate points and focal points

- Conjugate points are pairs of points along a geodesic where Jacobi fields vanish, indicating the existence of nearby geodesics that intersect the original geodesic at both points
- Focal points are points where a family of geodesics emanating from a single point (or a submanifold) converge
- The occurrence of conjugate points and focal points is related to the curvature of the manifold, as described by the Riemann curvature tensor

## Rauch comparison theorem

- The Rauch comparison theorem relates the behavior of Jacobi fields on a manifold to those on a reference manifold with known curvature properties
- It states that if the sectional curvature of a manifold is bounded above (or below) by a constant, then the norm of Jacobi fields is bounded by the corresponding Jacobi fields on a space of constant curvature
- The Rauch comparison theorem allows us to estimate the size of Jacobi fields and the distance between geodesics based on curvature bounds

## Curvature and topology

- The Riemann curvature tensor is intimately connected to the topology of a manifold

- Curvature invariants derived from the Riemann curvature tensor, such as the Gauss-Bonnet integrand and characteristic classes, provide information about the topological properties of a manifold

## Gauss-Bonnet theorem for surfaces

- The Gauss-Bonnet theorem is a fundamental result that relates the total Gaussian curvature (a scalar curvature) of a closed surface to its Euler characteristic (a topological invariant)
- It states that the integral of the Gaussian curvature over the surface, plus the integral of the geodesic curvature along the boundary (if present), equals  $2\pi$  times the Euler characteristic
- The Gauss-Bonnet theorem demonstrates a deep connection between curvature and topology in the case of surfaces

## Generalized Gauss-Bonnet theorem

- The generalized Gauss-Bonnet theorem extends the classical Gauss-Bonnet theorem to higher-dimensional manifolds
- It involves the Pfaffian of the Riemann curvature tensor, which is a higher-dimensional analog of the Gaussian curvature
- The generalized Gauss-Bonnet theorem relates the integral of the Pfaffian over the manifold to its Euler characteristic, providing a link between curvature and topology in higher dimensions

## Chern-Weil theory and characteristic classes

- Chern-Weil theory is a powerful framework that constructs topological invariants (characteristic classes) of vector bundles using curvature forms
- The Riemann curvature tensor can be used to define connection forms and curvature forms on the tangent bundle of a manifold
- Characteristic classes, such as the Chern classes and Pontryagin classes, are obtained by taking certain combinations of the curvature forms and integrating them over the manifold
- These characteristic classes provide information about the topological structure of the manifold and its vector bundles

## Curvature in special geometries

- There are several special classes of Riemannian manifolds characterized by specific curvature properties
- These special geometries have important applications in mathematics and physics and exhibit interesting geometric and topological features

## Constant curvature spaces

- A constant curvature space is a Riemannian manifold whose sectional curvature is constant at every point and in every direction
- Examples of constant curvature spaces include Euclidean spaces (zero curvature), spheres (positive curvature), and hyperbolic spaces (negative curvature)
- Constant curvature spaces have a high degree of symmetry and are used as model spaces in geometry and cosmology

## Einstein manifolds and Ricci-flat manifolds

- An Einstein manifold is a Riemannian manifold whose Ricci curvature tensor is proportional to the metric tensor
- In other words, the Ricci curvature is constant in all directions at each point
- Ricci-flat manifolds are a special case of Einstein manifolds where the Ricci curvature vanishes identically
- Einstein manifolds and Ricci-flat manifolds play a significant role in general relativity and the study of gravitational physics

## Kähler manifolds and Ricci curvature

- Kähler manifolds are complex manifolds equipped with a compatible Riemannian metric and a closed, non-degenerate 2-form (the Kähler form)
- The Ricci curvature of a Kähler manifold satisfies special properties, such as being a closed  $(1,1)$ -form
- The study of Ricci curvature in Kähler geometry has led to important results, such as the Calabi conjecture and the Yau-Tian-Donaldson conjecture, which relate the existence of Kähler-Einstein metrics to the stability of the manifold

## Curvature and physics

- The Riemann curvature tensor and its related concepts have profound implications in various areas of physics
- Curvature plays a central role in the description of gravity, spacetime, and the dynamics of the universe

## Einstein field equations in general relativity

- In general relativity, spacetime is modeled as a four-dimensional pseudo-Riemannian manifold, and gravity is described by the curvature of this manifold
- The Einstein field equations relate the Ricci curvature tensor and the scalar curvature of spacetime to the energy-momentum tensor of matter and radiation
- The Riemann curvature tensor appears in the Einstein field equations through its contractions, the Ricci tensor, and the scalar curvature

## Energy conditions and singularity theorems

- Energy conditions are assumptions about the properties of matter and energy in general relativity, expressed in terms of the energy-momentum tensor
- The strong energy condition, which states that the sum of energy density and pressure is non-negative, can be formulated using the Ricci curvature tensor
- Singularity theorems, such as the Penrose singularity theorem and the Hawking-Penrose singularity theorem, use the Riemann curvature tensor and energy conditions to prove the existence of singularities (such as black holes) in spacetime under certain conditions

## Curvature in string theory and M-theory

- String theory and M-theory are theoretical frameworks that attempt to unify gravity with other fundamental forces of nature
- In these theories, spacetime is modeled as a higher-dimensional manifold, and the dynamics of strings and membranes are influenced by the curvature of this manifold
- The Riemann curvature tensor and its generalizations (such as the Riemann-Christoffel tensor and the Riemann-Cartan tensor) play a crucial role in describing the geometry of the extra dimensions and the coupling of strings and membranes to the background spacetime

### 5.3 Sectional, Ricci, and scalar curvatures

Curvatures in Riemannian geometry measure how a manifold deviates from being flat. Sectional curvature measures the Gaussian curvature of geodesic planes, Ricci curvature averages sectional curvatures, and scalar curvature provides a single value for total curvature at each point.

These curvatures are interconnected and offer complementary information about a manifold's geometry. They have intuitive interpretations: sectional curvature controls geodesic spreading, Ricci curvature affects volume growth, and scalar curvature relates to large-scale volume behavior and general relativity.

#### Definitions of curvatures

- Curvatures are fundamental invariants in Riemannian geometry that measure the deviation of a manifold from being flat
- They provide important information about the intrinsic geometry and shape of a Riemannian manifold

#### Sectional curvature

- Measures the Gaussian curvature of geodesic planes in a Riemannian manifold
- Determined by the Riemannian curvature tensor  $R(X, Y, Z, W)$  evaluated on pairs of orthonormal vectors  $X, Y$
- Formula:  $K(X, Y) = \frac{R(X, Y, Y, X)}{|X \wedge Y|^2}$
- Provides the most complete description of curvature at a point

#### Ricci curvature

- Obtained by taking the trace of the Riemannian curvature tensor
- Measures the average sectional curvature in each tangent direction
- Formula:

$$Ric(X, Y) = \sum_{i=1}^n R(X, e_i, e_i, Y)$$

for an orthonormal basis  $\{e_i\}$

- Captures the amount of volume distortion in the manifold

## Scalar curvature

- Further contraction of the Ricci curvature
- Defined as the trace of the Ricci tensor
- Formula:

$$S = \sum_{i=1}^n Ric(e_i, e_i)$$

for an orthonormal basis  $\{e_i\}$

- Provides a single real number that represents the total curvature at each point

## Relationships between curvatures

- The different notions of curvature are interconnected and provide complementary information about the geometry of a Riemannian manifold

## Ricci curvature vs sectional curvature

- Ricci curvature is the average of sectional curvatures over all planes containing a given direction
- If the sectional curvature is constant (space forms), then the Ricci curvature is proportional to the metric tensor
- Positive (negative) Ricci curvature implies the existence of some positive (negative) sectional curvatures

## Scalar curvature vs Ricci curvature

- Scalar curvature is the trace of the Ricci tensor
- Positive (negative) scalar curvature implies the existence of some positive (negative) Ricci curvatures
- In dimension 2, the scalar curvature is twice the Gaussian curvature
- The scalar curvature appears in the Hilbert-Einstein functional in general relativity

## Geometric interpretations

- The different curvatures have intuitive geometric interpretations that help understand their meaning

## Sectional curvature interpretation

- Measures the Gaussian curvature of geodesic planes
- Controls the spreading or focusing of geodesics and the growth of Jacobi fields
- Positive (negative) sectional curvature implies that geodesics tend to converge (diverge)

## Ricci curvature interpretation

- Measures the average convergence or divergence of geodesics in each direction
- Controls the volume growth of small balls in the manifold

- Positive (negative) Ricci curvature implies that the volume of balls grows slower (faster) than in Euclidean space

## Scalar curvature interpretation

- Provides a rough measure of the total curvature at each point
- Related to the asymptotic behavior of the volume of large balls
- Appears in the Hilbert-Einstein functional, which is the action functional for general relativity

## Computations and formulas

- Computing the different curvatures involves the Riemannian curvature tensor and its contractions

## Sectional curvature formula

- $K(X, Y) = \frac{R(X, Y, Y, X)}{|X \wedge Y|^2}$  for orthonormal vectors  $X, Y$
- Can be computed using the Christoffel symbols and their derivatives
- In local coordinates:  $K(\partial_i, \partial_j) = R_{ijij}$

## Ricci curvature formula

•

$$Ric(X, Y) = \sum_{i=1}^n R(X, e_i, e_i, Y)$$

for an orthonormal basis  $\{e_i\}$

- In local coordinates:

$$Ric_{ij} = \sum_{k=1}^n R_{ikjk}$$

- Can be expressed in terms of the Christoffel symbols and the metric tensor

## Scalar curvature formula

•

$$S = \sum_{i=1}^n Ric(e_i, e_i)$$

for an orthonormal basis  $\{e_i\}$

- In local coordinates:  $S = g^{ij} Ric_{ij}$
- Can be computed using the metric tensor and its derivatives

## Curvature in special cases

- Certain classes of Riemannian manifolds have special curvature properties that simplify their geometry

## Curvature of surfaces

- For 2-dimensional Riemannian manifolds (surfaces), the sectional curvature and Gaussian curvature coincide
- The scalar curvature is twice the Gaussian curvature
- Gauss's Theorema Egregium states that the Gaussian curvature is an intrinsic invariant

## Curvature of space forms

- Space forms are Riemannian manifolds with constant sectional curvature (spheres, Euclidean spaces, hyperbolic spaces)
- The Ricci curvature is proportional to the metric tensor:  $Ric = (n - 1)Kg$
- The scalar curvature is a constant multiple of the sectional curvature:  $S = n(n - 1)K$

## Curvature of product manifolds

- The sectional curvature of a product manifold  $(M_1 \times M_2, g_1 \oplus g_2)$  is determined by the curvatures of the factors
- The Ricci curvature of a product manifold is the direct sum of the Ricci curvatures of the factors
- The scalar curvature of a product manifold is the sum of the scalar curvatures of the factors

## Curvature and topology

- The curvature of a Riemannian manifold is closely related to its topology and can provide obstructions to the existence of certain geometric structures

## Influence on Betti numbers

- The Betti numbers  $b_k(M)$  measure the rank of the homology groups  $H_k(M; \mathbb{R})$
- The Bochner technique relates the Ricci curvature to the vanishing of Betti numbers
- Positive Ricci curvature implies the vanishing of Betti numbers in certain degrees

## Influence on fundamental group

- The fundamental group  $\pi_1(M)$  captures the non-trivial loops in a manifold
- The Cartan-Hadamard theorem states that a complete, simply connected manifold with non-positive sectional curvature is diffeomorphic to  $\mathbb{R}^n$
- Negative Ricci curvature implies the fundamental group is infinite

## Gauss-Bonnet theorem

- Relates the total integral of the Gaussian curvature to the Euler characteristic  $\chi(M)$  for compact surfaces
- Formula:  $\int_M K dA = 2\pi\chi(M)$
- Generalizes to higher dimensions using the Pfaffian of the curvature form

## Variational properties

- Curvatures arise naturally in variational problems on the space of Riemannian metrics

## Hilbert-Einstein functional

- The Hilbert-Einstein functional is the action functional for general relativity
- Defined as  $S(g) = \int_M (S_g + L_m) dV_g$ , where  $L_m$  is the Lagrangian density of matter
- Critical points of the functional are Einstein metrics, which satisfy  $Ric_g = \lambda g$  for some constant  $\lambda$

## Yamabe problem

- Aims to find a metric of constant scalar curvature in a given conformal class
- Equivalent to solving the nonlinear PDE  $-\Delta_g u + \frac{n-2}{4(n-1)} S_g u = \lambda u^{\frac{n+2}{n-2}}$
- Solved affirmatively by Yamabe, Trudinger, Aubin, and Schoen

## Ricci flow

- A geometric evolution equation that deforms a Riemannian metric in the direction of its Ricci curvature
- Defined as  $\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)}$
- Used by Perelman to prove the Poincaré conjecture and Thurston's geometrization conjecture

## Applications and examples

- Curvature plays a central role in various areas of mathematics and physics

## General relativity

- In general relativity, spacetime is modeled as a 4-dimensional Lorentzian manifold
- The Einstein field equations relate the Ricci curvature to the energy-momentum tensor of matter
- Solutions to the Einstein equations describe the geometry of spacetime in the presence of matter and energy

## Comparison geometry

- Comparison theorems relate the curvature of a manifold to the geometry of model spaces (space forms)
- The Rauch comparison theorem compares Jacobi fields on a manifold to those on a space form
- The Bishop-Gromov volume comparison theorem compares the volume growth of balls to that in a space form

## Sphere theorems

- Sphere theorems provide sufficient conditions for a manifold to be homeomorphic or diffeomorphic to a sphere



- The 1/4-pinched sphere theorem states that a complete, simply connected manifold with  $\frac{1}{4} < K \leq 1$  is homeomorphic to a sphere
- The positive mass theorem in general relativity is a type of sphere theorem in Lorentzian geometry

## 5.4 Symmetries of the curvature tensor

The Riemann curvature tensor is a key concept in Riemannian geometry, encoding the curvature of a manifold. It has several symmetry properties that constrain its components and provide insights into the geometry. These symmetries reduce the tensor's independent components and relate it to other curvature measures.

Understanding these symmetries is crucial for studying geodesics, parallel transport, and manifold structure. The Riemann tensor's skew symmetry, interchange symmetry, and Bianchi identities all play important roles in simplifying calculations and revealing geometric relationships. These properties have significant implications for manifold geometry and topology.

### Symmetries of Riemann curvature tensor

- The Riemann curvature tensor is a fundamental object in Riemannian geometry that encodes the curvature properties of a manifold
- It satisfies several symmetry properties that constrain its components and provide insights into the geometry of the manifold
- Understanding these symmetries is crucial for studying the behavior of geodesics, parallel transport, and the overall structure of the manifold

### Skew symmetry in first two arguments

- The Riemann curvature tensor is skew-symmetric in its first two arguments:  

$$R(X, Y, Z, W) = -R(Y, X, Z, W)$$
- This property arises from the fact that the Lie bracket of vector fields is skew-symmetric:  

$$[X, Y] = -[Y, X]$$
- Skew symmetry reduces the number of independent components of the Riemann tensor
- In an  $n$ -dimensional manifold, the number of independent components is reduced by a factor of  $\frac{1}{2}n(n-1)$

### Skew symmetry in last two arguments

- The Riemann curvature tensor is also skew-symmetric in its last two arguments:  

$$R(X, Y, Z, W) = -R(X, Y, W, Z)$$
- This property follows from the definition of the Riemann tensor in terms of the Levi-Civita connection and its compatibility with the metric
- Skew symmetry in the last two arguments further reduces the number of independent components of the Riemann tensor
- In an  $n$ -dimensional manifold, the total reduction in independent components due to both skew symmetries is  $\frac{1}{4}n^2(n^2 - 1)$

## Interchange symmetry of pairs of arguments

- The Riemann curvature tensor satisfies an interchange symmetry between pairs of arguments:  
 $R(X, Y, Z, W) = R(Z, W, X, Y)$
- This symmetry relates the curvature experienced by different pairs of vectors and reflects the symmetry of the manifold's geometry
- The interchange symmetry is a consequence of the first Bianchi identity and the skew symmetries of the Riemann tensor
- It plays a crucial role in deriving other symmetries and identities involving the Riemann tensor

## First Bianchi identity

- The first Bianchi identity is a fundamental relation satisfied by the Riemann curvature tensor:  
 $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$
- It relates the curvature tensor evaluated on different permutations of vector fields
- The first Bianchi identity is a consequence of the Jacobi identity for the Lie bracket of vector fields and the properties of the Levi-Civita connection
- It provides a constraint on the components of the Riemann tensor and is used in deriving other identities and symmetries

## Cyclic permutation of first Bianchi identity

- The cyclic permutation of the first Bianchi identity states that:  
 $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$
- This identity is obtained by cyclically permuting the vector fields in the first Bianchi identity
- It provides an alternative representation of the constraint imposed by the first Bianchi identity on the Riemann tensor
- The cyclic permutation is useful in deriving further symmetries and identities involving the Riemann tensor and its contractions

## Symmetries of Ricci curvature tensor

- The Ricci curvature tensor is a contraction of the Riemann curvature tensor, obtained by summing over two of its indices
- It captures information about the average curvature in different directions at a point on the manifold
- The Ricci tensor inherits symmetries from the Riemann tensor, which simplify its form and provide insights into the geometry of the manifold

## Symmetry of Ricci tensor

- The Ricci tensor is symmetric, meaning that  $Ric(X, Y) = Ric(Y, X)$  for any vector fields  $X$  and  $Y$
- This symmetry follows from the interchange symmetry and the skew symmetries of the Riemann tensor
- The symmetric nature of the Ricci tensor reduces the number of its independent components

- In an  $n$ -dimensional manifold, the Ricci tensor has  $\frac{1}{2}n(n+1)$  independent components

## Relationship to Riemann curvature tensor

- The Ricci tensor is defined as the contraction of the Riemann tensor:  $Ric(X, Y) = \text{tr}(Z \mapsto R(X, Z)Y)$
- It can be expressed in terms of the components of the Riemann tensor as  $R_{ij} = R^k_{ikj}$ , where  $R^k_{ikj}$  are the components of the Riemann tensor in a local coordinate system
- The Ricci tensor provides a way to extract scalar curvature information from the Riemann tensor
- The scalar curvature is obtained by further contracting the Ricci tensor with the metric:  $R = g^{ij}R_{ij}$

## Contracted second Bianchi identity

- The contracted second Bianchi identity is a differential identity satisfied by the Riemann curvature tensor and its contractions
- It relates the divergence of the Riemann tensor to the covariant derivatives of the Ricci tensor and the metric
- The contracted Bianchi identity has important implications for the geometry of the manifold and the behavior of curvature

## Divergence of Riemann curvature tensor

- The divergence of the Riemann curvature tensor is given by  $\nabla^m R_{ijkm} = \nabla_i R_{jk} - \nabla_j R_{ik}$
- This identity relates the covariant derivatives of the Riemann tensor to the covariant derivatives of the Ricci tensor
- It is obtained by contracting the second Bianchi identity, which involves the covariant derivatives of the Riemann tensor
- The divergence identity is crucial in deriving the contracted Bianchi identity for the Ricci tensor

## Contracted Bianchi identity for Ricci tensor

- The contracted Bianchi identity for the Ricci tensor states that  $\nabla^i R_{ij} = \frac{1}{2} \nabla_j R$ , where  $R$  is the scalar curvature
- It relates the divergence of the Ricci tensor to the gradient of the scalar curvature
- The contracted Bianchi identity is obtained by contracting the divergence identity for the Riemann tensor with the metric
- It has important consequences for the geometry of the manifold, such as the conservation of energy-momentum in general relativity

## Implications of symmetries

- The symmetries of the Riemann and Ricci curvature tensors have significant implications for the geometry and topology of the manifold
- They provide constraints on the components of the curvature tensors, reduce the number of independent components, and relate curvature quantities to each other

- Understanding these implications is essential for studying the properties of Riemannian and pseudo-Riemannian manifolds

## Reduction of independent components

- The skew symmetries and interchange symmetry of the Riemann tensor greatly reduce the number of its independent components
- In an  $n$ -dimensional manifold, the number of independent components is reduced from  $n^4$  to  $\frac{1}{12}n^2(n^2 - 1)$
- The symmetry of the Ricci tensor further reduces the number of its independent components to  $\frac{1}{2}n(n + 1)$
- These reductions simplify computations and provide constraints on the possible values of curvature components

## Relationship to sectional curvature

- The sectional curvature is a scalar quantity that measures the Gaussian curvature of two-dimensional submanifolds (sections) of the manifold
- It can be expressed in terms of the Riemann curvature tensor as  $K(X, Y) = \frac{R(X, Y, X, Y)}{|X|^2|Y|^2 - \langle X, Y \rangle^2}$
- The symmetries of the Riemann tensor imply that the sectional curvature is independent of the choice of basis for the section
- This property allows the sectional curvature to capture intrinsic geometric information about the manifold

## Relationship to scalar curvature

- The scalar curvature is the simplest curvature invariant of a Riemannian manifold, obtained by contracting the Ricci tensor with the metric
- It provides an average measure of the curvature at each point of the manifold
- The symmetries of the Ricci tensor and the contracted Bianchi identity relate the scalar curvature to other curvature quantities
- For example, in Einstein manifolds, the Ricci tensor is proportional to the metric, implying a constant scalar curvature

## Symmetries in special cases

- The symmetries of the Riemann and Ricci curvature tensors can take on special forms in certain types of manifolds or under specific conditions
- These special cases often provide additional insights into the geometry of the manifold and simplify the study of its properties
- Examples of special cases include Riemannian vs Lorentzian metrics, constant curvature spaces, and Einstein manifolds

## Symmetries in Riemannian vs Lorentzian metrics

- Riemannian metrics have a positive-definite metric tensor, while Lorentzian metrics have an indefinite signature  $(-, +, +, +)$
- The symmetries of the Riemann and Ricci tensors hold in both cases, but their implications may differ
- In Riemannian manifolds, the symmetries ensure that sectional curvatures are always real-valued
- In Lorentzian manifolds, the indefinite metric allows for the existence of null vectors and can lead to different curvature properties

## Symmetries in constant curvature spaces

- Constant curvature spaces are manifolds where the sectional curvature is the same for all two-dimensional subspaces at each point
- In these spaces, the Riemann tensor takes on a particularly simple form:  $R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$ , where  $K$  is the constant sectional curvature
- The symmetries of the Riemann tensor are automatically satisfied in constant curvature spaces due to the highly symmetric nature of the curvature
- Examples of constant curvature spaces include spheres (positive curvature), Euclidean spaces (zero curvature), and hyperbolic spaces (negative curvature)

## Symmetries in Einstein manifolds

- Einstein manifolds are characterized by the condition that the Ricci tensor is proportional to the metric tensor:  $R_{ij} = \lambda g_{ij}$ , where  $\lambda$  is a constant
- In Einstein manifolds, the contracted Bianchi identity implies that the scalar curvature is constant:  $\nabla_i R = 0$
- The symmetries of the Riemann and Ricci tensors in Einstein manifolds lead to simplified expressions for curvature quantities and have implications for the topology of the manifold
- For example, compact Einstein manifolds with positive scalar curvature are necessarily of constant positive sectional curvature

## 5.5 Bianchi identities

Bianchi identities are crucial mathematical relations that constrain the Riemann curvature tensor in differential geometry. They play a key role in understanding geometric properties of manifolds and the behavior of physical theories on curved spacetimes.

These identities have far-reaching applications in physics and mathematics, particularly in general relativity and gauge theories. They're fundamental in deriving Einstein's field equations and ensuring the conservation of energy-momentum, which is essential for the physical validity of these theories.

### Definition of Bianchi identities

- Bianchi identities are a set of mathematical relations that constrain the Riemann curvature tensor and its covariant derivatives in differential geometry
- They play a crucial role in understanding the geometric properties of manifolds and the behavior of physical theories formulated on curved spacetimes

- Bianchi identities are named after the Italian mathematician Luigi Bianchi who discovered them in the late 19th century

## Bianchi identities for Riemann curvature tensor

- The Bianchi identities for the Riemann curvature tensor  $R_{abcd}$  can be expressed as  $\nabla_{[a}R_{bc]de} = 0$
- $\nabla_a$  denotes the covariant derivative with respect to the coordinate  $x^a$
- The square brackets  $[\dots]$  indicate antisymmetrization over the enclosed indices
- This identity relates the covariant derivatives of the Riemann tensor and imposes a constraint on its components

## First vs second Bianchi identity

- The Bianchi identities can be classified into two types: the first Bianchi identity and the second Bianchi identity
- The first Bianchi identity is an algebraic identity that follows from the symmetries of the Riemann tensor:  $R_{abcd} + R_{acdb} + R_{adbc} = 0$
- The second Bianchi identity,  $\nabla_{[a}R_{bc]de} = 0$ , involves the covariant derivatives of the Riemann tensor and is a differential identity

## Geometric interpretation

- The Bianchi identities have important geometric implications for the curvature and parallel transport on manifolds
- They provide a deeper understanding of the intrinsic properties of curved spaces and the behavior of geodesics

## Bianchi identities and parallel transport

- The Bianchi identities are closely related to the concept of parallel transport on a manifold
- Parallel transport refers to the process of moving a vector along a curve while preserving its direction relative to the manifold's curvature
- The Bianchi identities ensure the consistency and integrability of parallel transport, guaranteeing that the result is independent of the path taken

## Relationship to curvature and torsion

- The Bianchi identities establish a connection between the Riemann curvature tensor and the torsion tensor of a manifold
- In the presence of torsion, the Bianchi identities take a modified form that includes additional terms involving the torsion tensor
- The vanishing of torsion simplifies the Bianchi identities and leads to the standard form  $\nabla_{[a}R_{bc]de} = 0$

## Derivation of Bianchi identities

- The Bianchi identities can be derived using various mathematical techniques, depending on the formalism and the specific context in which they are studied

- Two common approaches are the derivation using Ricci calculus and alternative derivations based on differential forms or tensor analysis

## Derivation using Ricci calculus

- Ricci calculus is a powerful tool for manipulating tensors and deriving identities in differential geometry
- By applying the Ricci identity to the Riemann tensor and using its symmetries, one can derive the second Bianchi identity
- The derivation involves careful bookkeeping of indices and exploiting the properties of the covariant derivative and the Riemann tensor

## Alternative derivations

- Alternative derivations of the Bianchi identities exist, such as those based on differential forms and exterior calculus
- These approaches provide a more abstract and coordinate-independent formulation of the identities
- Derivations using tensor analysis and the properties of the Levi-Civita connection can also lead to the Bianchi identities

## Applications of Bianchi identities

- The Bianchi identities have far-reaching applications in various areas of physics and mathematics, particularly in the context of general relativity and gauge theories
- They play a crucial role in understanding the dynamics of spacetime and the behavior of fundamental forces

## Bianchi identities in general relativity

- In general relativity, the Bianchi identities are fundamental in deriving the Einstein field equations, which describe the relationship between spacetime curvature and matter-energy content
- The contracted Bianchi identities, obtained by contracting indices of the second Bianchi identity, lead to the conservation of energy-momentum tensor
- The Bianchi identities ensure the consistency and compatibility of the Einstein field equations

## Bianchi identities and Einstein field equations

- The Einstein field equations,  $G_{ab} = 8\pi T_{ab}$ , relate the Einstein tensor  $G_{ab}$  to the energy-momentum tensor  $T_{ab}$
- The Bianchi identities imply that the divergence of the Einstein tensor vanishes,  $\nabla^a G_{ab} = 0$
- Consequently, the Bianchi identities guarantee the conservation of energy-momentum,  $\nabla^a T_{ab} = 0$ , which is a necessary condition for the physical validity of the theory

## Bianchi identities in gauge theories

- Bianchi identities also play a significant role in gauge theories, such as Yang-Mills theory and electromagnetism

- In these theories, the Bianchi identities impose constraints on the field strength tensors and ensure the consistency of the gauge fields
- The Bianchi identities in gauge theories are analogous to the Bianchi identities for the Riemann tensor in differential geometry

## Generalizations of Bianchi identities

- The concept of Bianchi identities can be generalized to various mathematical structures beyond the Riemann curvature tensor
- These generalizations extend the scope and applicability of Bianchi identities to different geometric settings and physical theories

## Bianchi identities for other curvature tensors

- Bianchi identities can be formulated for other curvature tensors, such as the Weyl tensor, the Ricci tensor, and the scalar curvature
- These generalized Bianchi identities impose constraints on the respective curvature quantities and their covariant derivatives
- The specific form of the Bianchi identities depends on the symmetries and properties of the curvature tensor under consideration

## Bianchi identities in higher dimensions

- The Bianchi identities can be extended to manifolds of higher dimensions beyond the usual four-dimensional spacetime
- In higher dimensions, the Bianchi identities take a similar form but involve a larger number of indices and more complex symmetry properties
- The study of Bianchi identities in higher dimensions is relevant for theories of extra dimensions, such as string theory and Kaluza-Klein theory

## Bianchi identities and conservation laws

- The Bianchi identities have important implications for conservation laws in physics, particularly in the context of energy-momentum conservation and charge conservation
- These conservation laws are crucial for the consistency and physical validity of theories describing fundamental interactions and the behavior of matter

## Bianchi identities and energy-momentum conservation

- In general relativity, the contracted Bianchi identities lead to the conservation of the energy-momentum tensor,  $\nabla^a T_{ab} = 0$
- This conservation law expresses the fact that energy and momentum cannot be created or destroyed, but only transferred between different forms
- The Bianchi identities ensure that the energy-momentum tensor is divergence-free, which is a necessary condition for the conservation of energy and momentum



## Bianchi identities and charge conservation

- In gauge theories, such as electromagnetism and Yang-Mills theory, the Bianchi identities imply the conservation of charge
- The Bianchi identities for the field strength tensor,  $\nabla_{[a}F_{bc]} = 0$ , lead to the continuity equation for the electric current density,  $\nabla_a j^a = 0$
- This conservation law states that the total electric charge in a closed system remains constant over time, reflecting the fundamental principle of charge conservation

## Computational aspects of Bianchi identities

- The calculation and manipulation of Bianchi identities often involve complex tensor algebra and symbolic computations
- Various computational tools and software packages have been developed to assist in the symbolic manipulation and simplification of Bianchi identities

## Calculating Bianchi identities in specific metrics

- In practice, the Bianchi identities are often evaluated in the context of specific metric tensors that describe the geometry of a particular spacetime or manifold
- The calculation of Bianchi identities in specific metrics involves substituting the components of the metric tensor into the general expressions for the Riemann tensor and its covariant derivatives
- This process can be computationally intensive, especially for metrics with complicated functional forms or high symmetry

## Software for symbolic computation of Bianchi identities

- Several software packages and computer algebra systems have been developed to automate the symbolic computation of Bianchi identities
- These tools can handle the complex tensor algebra involved in the calculations and provide efficient methods for simplifying and manipulating the resulting expressions
- Examples of software used for symbolic computation of Bianchi identities include Mathematica, Maple, and specialized tensor algebra packages like xAct for Mathematica
- These computational tools greatly facilitate the study and application of Bianchi identities in various contexts, allowing researchers to focus on the physical and geometric insights rather than the technical details of the calculations

## 5.6 Constant curvature and Einstein manifolds

Constant curvature and Einstein manifolds are key concepts in Riemannian geometry. These special types of manifolds have uniform curvature properties, making them ideal for studying the relationship between geometry and topology.

Constant curvature manifolds include spheres, hyperbolic spaces, and flat spaces. Einstein manifolds, with Ricci tensors proportional to their metrics, play a crucial role in general relativity and cosmology. Understanding these manifolds provides insights into spacetime structure and universe evolution.

## Constant curvature manifolds

- Constant curvature manifolds are Riemannian manifolds whose sectional curvatures are constant at every point
- The curvature of these manifolds is determined by a single real number, which can be positive, negative, or zero
- Studying constant curvature manifolds provides insights into the relationship between curvature and topology in Riemannian geometry

## Sectional curvature

- Sectional curvature measures the curvature of a manifold along 2-dimensional planes (sections) tangent to the manifold at a point
- Defined as  $K(X, Y) = \frac{\langle R(X, Y)Y, X \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2}$ , where  $R$  is the Riemann curvature tensor and  $X, Y$  are linearly independent tangent vectors
- Constant sectional curvature implies that the curvature is the same for all 2-dimensional sections at every point (e.g., spheres, hyperbolic spaces)

## Ricci curvature

- Ricci curvature is the trace of the Riemann curvature tensor, obtained by contracting the tensor along two indices
- Measures the average sectional curvature in all directions at a point
- For constant curvature manifolds, the Ricci tensor is proportional to the metric tensor:  
 $Ric = (n - 1)Kg$ , where  $K$  is the constant sectional curvature and  $n$  is the dimension of the manifold

## Scalar curvature

- Scalar curvature is the trace of the Ricci tensor, obtained by contracting the Ricci tensor with the metric tensor
- Represents the average of the Ricci curvature at a point
- For constant curvature manifolds, the scalar curvature is constant and equal to  $R = n(n - 1)K$ , where  $K$  is the constant sectional curvature and  $n$  is the dimension of the manifold

## Positive vs negative curvature

- Positive curvature manifolds have a constant sectional curvature  $K > 0$  (e.g., spheres)
- Geodesics converge, and the sum of angles in a triangle is greater than  $\pi$
- Compact without boundary
- Negative curvature manifolds have a constant sectional curvature  $K < 0$  (e.g., hyperbolic spaces)
- Geodesics diverge, and the sum of angles in a triangle is less than  $\pi$
- Non-compact or have a boundary

## Flat manifolds

- Flat manifolds have a constant sectional curvature  $K = 0$  (e.g., Euclidean spaces)
- The Riemann curvature tensor vanishes identically, and the manifold is locally isometric to Euclidean space
- Geodesics are straight lines, and the sum of angles in a triangle is equal to  $\pi$

## Hyperbolic vs elliptic geometry

- Hyperbolic geometry describes the geometry of manifolds with constant negative curvature
- Parallel postulate is replaced by the existence of multiple parallel lines through a point not on a given line
- Poincaré disk and upper half-plane models
- Elliptic geometry describes the geometry of manifolds with constant positive curvature
- Parallel postulate is replaced by the non-existence of parallel lines
- Spherical geometry is an example of elliptic geometry

## Models of constant curvature

- Models of constant curvature provide a way to visualize and study the properties of manifolds with constant sectional curvature
- These models are often constructed as subsets of Euclidean space or projective space with a specific metric

## Spherical geometry

- Spherical geometry is the geometry of the 2-dimensional sphere  $S^2$  with constant positive curvature
- Lines are great circles, and the sum of angles in a triangle is greater than  $\pi$
- Can be generalized to higher dimensions as  $S^n$ , the  $n$ -dimensional sphere embedded in  $(n + 1)$ -dimensional Euclidean space

## Euclidean space

- Euclidean space  $\mathbb{R}^n$  is the model for flat manifolds with constant zero curvature
- Geodesics are straight lines, and the metric is the standard Euclidean metric
- The Euclidean group  $E(n) = O(n) \ltimes \mathbb{R}^n$  acts transitively on  $\mathbb{R}^n$

## Hyperbolic space

- Hyperbolic space  $\mathbb{H}^n$  is the model for manifolds with constant negative curvature
- Can be constructed using the hyperboloid model, Poincaré disk model, or upper half-space model
- Geodesics are hyperbolic lines, and the sum of angles in a triangle is less than  $\pi$

## Isometry groups

- Isometry groups are the groups of transformations that preserve the metric of a constant curvature manifold
- For spherical geometry, the isometry group is the orthogonal group  $O(n+1)$
- For Euclidean space, the isometry group is the Euclidean group  $E(n) = O(n) \ltimes \mathbb{R}^n$
- For hyperbolic space, the isometry group is the orthogonal group  $O(n, 1)$  or its connected component  $SO^+(n, 1)$

## Characterizations of constant curvature

- Several mathematical properties and theorems characterize manifolds with constant curvature
- These characterizations provide insights into the geometric and topological structure of constant curvature manifolds

## Schur's lemma

- Schur's lemma states that if a connected Riemannian manifold of dimension  $n \geq 3$  has a constant sectional curvature at each point, then the sectional curvature is constant on the entire manifold
- Implies that for manifolds of dimension  $n \geq 3$ , pointwise constant sectional curvature is equivalent to global constant sectional curvature

## Killing vector fields

- Killing vector fields are vector fields that generate isometries of a Riemannian manifold
- For constant curvature manifolds, the dimension of the space of Killing vector fields is  $\frac{n(n+1)}{2}$ , where  $n$  is the dimension of the manifold
- The existence of a maximal number of Killing vector fields characterizes constant curvature manifolds

## Jacobi fields

- Jacobi fields describe the behavior of geodesics on a Riemannian manifold
- For constant curvature manifolds, Jacobi fields along geodesics have a specific form depending on the sign of the curvature
- For positive curvature:  $J(t) = a \cos(\sqrt{K}t) + b \sin(\sqrt{K}t)$
- For negative curvature:  $J(t) = a \cosh(\sqrt{-K}t) + b \sinh(\sqrt{-K}t)$
- For zero curvature:  $J(t) = at + b$

## Topological restrictions

- The curvature of a manifold imposes restrictions on its topology
- Compact manifolds with constant positive curvature are diffeomorphic to spherical space forms  $S^n/\Gamma$ , where  $\Gamma$  is a finite subgroup of  $O(n+1)$
- Complete, simply connected manifolds with constant zero curvature are isometric to Euclidean space  $\mathbb{R}^n$

- Complete, simply connected manifolds with constant negative curvature are isometric to hyperbolic space  $\mathbb{H}^n$

## Einstein manifolds

- Einstein manifolds are Riemannian or pseudo-Riemannian manifolds whose Ricci tensor is proportional to the metric tensor
- They are named after Albert Einstein due to their importance in general relativity

## Definition and properties

- A Riemannian or pseudo-Riemannian manifold  $(M, g)$  is an Einstein manifold if its Ricci tensor satisfies  $Ric = \lambda g$  for some constant  $\lambda$
- The constant  $\lambda$  is related to the scalar curvature by  $R = n\lambda$ , where  $n$  is the dimension of the manifold
- Einstein manifolds have constant scalar curvature

## Ricci tensor

- The Ricci tensor is a symmetric (0,2) tensor obtained by contracting the Riemann curvature tensor
- For an Einstein manifold, the Ricci tensor determines the full Riemann curvature tensor
- The Ricci tensor satisfies the contracted Bianchi identities, which imply that  $\lambda$  is constant for Einstein manifolds

## Vacuum Einstein equations

- In general relativity, the vacuum Einstein equations describe the geometry of spacetime in the absence of matter and energy
- The vacuum Einstein equations are given by  $Ric = 0$ , which implies that the manifold is Ricci-flat
- Solutions to the vacuum Einstein equations are important in the study of black holes and gravitational waves

## Examples of Einstein manifolds

- Constant curvature manifolds (spheres, Euclidean spaces, hyperbolic spaces) are Einstein manifolds
- Calabi-Yau manifolds, which are complex Kähler manifolds with vanishing first Chern class, are Ricci-flat and thus Einstein
- Certain homogeneous spaces, such as irreducible symmetric spaces and isotropy irreducible spaces, are Einstein manifolds

## Connections between curvature and topology

- The curvature of a Riemannian manifold has significant implications for its topology
- Several important theorems establish connections between the curvature and topological properties of manifolds

## Myers' theorem

- Myers' theorem states that if a complete Riemannian manifold has Ricci curvature bounded below by  $(n - 1)K > 0$ , then it is compact and has diameter at most  $\frac{\pi}{\sqrt{K}}$
- Implies that complete manifolds with positive Ricci curvature are compact
- Provides a link between positive curvature and finite diameter

## Cheeger-Gromoll splitting theorem

- The Cheeger-Gromoll splitting theorem states that if a complete Riemannian manifold with non-negative Ricci curvature contains a line (a complete geodesic that minimizes distance between any two of its points), then it splits isometrically as a product  $\mathbb{R} \times N$
- Implies that complete, non-compact manifolds with non-negative Ricci curvature have a specific topological structure

## Sphere theorems

- Sphere theorems provide conditions under which a Riemannian manifold is homeomorphic to a sphere
- The 1/4-pinched sphere theorem states that if a complete, simply connected Riemannian manifold has sectional curvature in the range  $[1, 4]$ , then it is homeomorphic to a sphere
- The Diameter sphere theorem states that if a complete Riemannian manifold has sectional curvature in the range  $[1, 4]$  and diameter greater than  $\pi/2$ , then it is homeomorphic to a sphere

## Gauss-Bonnet theorem

- The Gauss-Bonnet theorem relates the curvature of a compact, oriented, even-dimensional Riemannian manifold to its Euler characteristic
- For a compact, oriented surface  $M$ , the theorem states that  $\int_M K dA = 2\pi\chi(M)$ , where  $K$  is the Gaussian curvature,  $dA$  is the area element, and  $\chi(M)$  is the Euler characteristic
- Provides a link between the total curvature and the topology of even-dimensional manifolds

## Applications and related topics

- The study of constant curvature and Einstein manifolds has significant applications in various areas of mathematics and physics
- These manifolds play a crucial role in understanding the geometry of spacetime and the evolution of the universe

## General relativity

- Einstein manifolds, particularly Ricci-flat manifolds, are fundamental in the formulation of general relativity
- The Einstein field equations describe the relationship between the curvature of spacetime and the distribution of matter and energy

- Solutions to the Einstein equations, such as black holes and cosmological models, provide insights into the nature of gravity and the structure of the universe

## Cosmological models

- Constant curvature manifolds are used to model the large-scale geometry of the universe
- The Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which describes homogeneous and isotropic spacetimes, has constant spatial curvature (positive, negative, or zero)
- The curvature of the universe determines its ultimate fate (e.g., closed, open, or flat universe models)

## Yamabe problem

- The Yamabe problem asks whether every compact Riemannian manifold admits a metric with constant scalar curvature
- The problem was solved affirmatively by Richard Schoen and Shing-Tung Yau in the positive case and by Thierry Aubin and Neil Trudinger in the remaining cases
- The Yamabe problem is related to the study of Einstein manifolds and the conformal geometry of Riemannian manifolds

## Ricci flow

- Ricci flow is a geometric evolution equation that deforms the metric of a Riemannian manifold in the direction of its Ricci curvature
- Introduced by Richard Hamilton, Ricci flow has been used to prove significant results in geometric topology, such as the Poincaré conjecture and the geometrization conjecture for 3-manifolds
- Ricci flow can be used to study the long-time behavior of Einstein manifolds and the existence of Einstein metrics on compact manifolds

# Unit 6 – Isometries and Killing Vector Fields

## 6.1 Isometries and Riemannian isometry groups

Isometries are transformations that preserve distances and angles in geometric spaces. They're crucial for understanding symmetries in Riemannian manifolds, including rotations, translations, and reflections in various geometries like Euclidean, hyperbolic, and spherical.

The Riemannian isometry group consists of all isometries of a manifold, forming a Lie group with specific properties. This group's structure reveals important information about the manifold's geometry, including its curvature, symmetries, and topological features.

### Definition of isometries

- Isometries are transformations that preserve the metric structure of a space, maintaining distances and angles between points
- In the context of Metric Differential Geometry, isometries play a crucial role in understanding the symmetries and properties of Riemannian manifolds

### Preserving metric properties

- Isometries preserve the length of curves and the angle between tangent vectors at each point
- For any two points  $p$  and  $q$  in a metric space  $(M, d)$ , an isometry  $f: M \rightarrow M$  satisfies  $d(f(p), f(q)) = d(p, q)$
- Isometries also preserve other metric properties such as curvature and volume

### Isometries as distance-preserving maps

- Isometries can be characterized as distance-preserving maps between metric spaces
- Given two metric spaces  $(M_1, d_1)$  and  $(M_2, d_2)$ , a map  $f: M_1 \rightarrow M_2$  is an isometry if  $d_2(f(p), f(q)) = d_1(p, q)$  for all  $p, q \in M_1$
- Isometries are always injective (one-to-one) and continuous

### Local vs global isometries

- Local isometries preserve the metric structure in a neighborhood of each point, but may not be globally distance-preserving
- Global isometries, also known as metric isometries, preserve distances between all pairs of points in the entire space
- Every global isometry is a local isometry, but the converse is not always true (consider the map  $f(x) = x$  from  $\mathbb{R}$  to  $\mathbb{R}_{\geq 0}$ )



## Examples of isometries

### Euclidean isometries

- Isometries of Euclidean space  $\mathbb{R}^n$  include translations, rotations, and reflections
- Translations shift all points by a fixed vector, preserving distances and angles (parallel transport)
- Rotations are linear transformations that preserve the origin and the length of vectors (orthogonal matrices)
- Reflections flip points across a hyperplane, preserving distances but reversing orientation

### Hyperbolic isometries

- Isometries of the hyperbolic plane  $\mathbb{H}^2$  and hyperbolic space  $\mathbb{H}^n$  include translations along geodesics, rotations around points, and reflections across geodesics
- Hyperbolic translations shift points along a geodesic by a fixed hyperbolic distance, preserving the distance between any two points on the geodesic
- Hyperbolic rotations rotate points around a fixed point by a given angle, preserving distances and angles
- Hyperbolic reflections flip points across a geodesic, preserving distances but reversing orientation

### Isometries on surfaces

- Isometries of surfaces, such as the sphere  $\mathbb{S}^2$  or the torus  $\mathbb{T}^2$ , can be classified based on their fixed points and the topology of the surface
- On the sphere, isometries include rotations around any axis passing through the center and reflections across any great circle
- On the torus, isometries include translations along the two fundamental loops and rotations by multiples of  $\pi/2$  around certain points
- Isometries on surfaces can be studied using the concept of covering spaces and the action of the fundamental group on the universal cover

## Properties of isometries

### Isometries as diffeomorphisms

- Isometries between smooth Riemannian manifolds are always smooth diffeomorphisms
- The differential of an isometry at each point is an orthogonal linear transformation, preserving the inner product on tangent spaces
- The inverse of an isometry is also an isometry, making the set of isometries a group under composition

### Isometries and geodesics

- Isometries map geodesics to geodesics, preserving the length and minimizing properties of geodesics

- If  $\gamma : [a, b] \rightarrow M$  is a geodesic in a Riemannian manifold  $M$  and  $f : M \rightarrow M$  is an isometry, then  $f \circ \gamma$  is also a geodesic
- Isometries preserve the Levi-Civita connection and the curvature tensor of a Riemannian manifold

## Fixed points and invariant sets

- A point  $p \in M$  is a fixed point of an isometry  $f : M \rightarrow M$  if  $f(p) = p$
- The set of fixed points of an isometry can be empty, a single point, or a submanifold of  $M$
- An invariant set under an isometry  $f$  is a subset  $A \subseteq M$  such that  $f(A) = A$
- The fixed point set and invariant sets of an isometry provide information about its geometric and topological properties

## Riemannian isometry groups

### Definition and structure

- The set of all isometries of a Riemannian manifold  $(M, g)$  forms a group under composition, called the isometry group and denoted by  $\text{Isom}(M, g)$
- The isometry group is a subgroup of the diffeomorphism group  $\text{Diff}(M)$  and inherits a natural topology from it
- The isometry group can be finite, discrete, or continuous, depending on the geometry and topology of the manifold

### Lie group properties

- For connected Riemannian manifolds, the isometry group is a Lie group, with a smooth group operation and a compatible smooth structure
- The Lie algebra of the isometry group consists of Killing vector fields, which are vector fields that generate one-parameter subgroups of isometries
- The dimension of the isometry group is bounded by  $\frac{n(n+1)}{2}$ , where  $n$  is the dimension of the manifold, with equality achieved only for spaces of constant curvature

### Orbits and isotropy subgroups

- The orbit of a point  $p \in M$  under the action of the isometry group is the set  $\text{Isom}(M, g) \cdot p = \{f(p) \mid f \in \text{Isom}(M, g)\}$
- Orbits partition the manifold into equivalence classes of points related by isometries
- The isotropy subgroup (or stabilizer) of a point  $p$  is the subgroup of isometries that fix  $p$ :  $\text{Isom}_p(M, g) = \{f \in \text{Isom}(M, g) \mid f(p) = p\}$
- The orbit of  $p$  is diffeomorphic to the quotient space  $\text{Isom}(M, g)/\text{Isom}_p(M, g)$

## Classification of isometries

### Rotations and translations

- Rotations are isometries that fix a point (in Euclidean and hyperbolic spaces) or a geodesic (in spherical geometry) and rotate the space around it
- Translations are isometries that move all points by a fixed distance in a given direction, without any fixed points
- In Euclidean space, rotations and translations generate the entire isometry group, while in hyperbolic and spherical geometries, they form proper subgroups

### Reflections and glide reflections

- Reflections are isometries that flip the space across a hyperplane (in Euclidean space) or a geodesic (in hyperbolic and spherical geometries)
- Glide reflections are compositions of a reflection and a translation in the direction parallel to the reflecting hyperplane or geodesic
- Reflections and glide reflections are orientation-reversing isometries, while rotations and translations are orientation-preserving

### Discrete vs continuous isometry groups

- Discrete isometry groups have a discrete topology and are typically generated by a finite set of isometries, such as reflections or rotations of a specific angle
- Continuous isometry groups have a continuous topology and often arise from the action of a Lie group on the manifold
- Examples of discrete isometry groups include the symmetry groups of regular polyhedra and the fundamental groups of compact hyperbolic surfaces
- Examples of continuous isometry groups include the orthogonal group  $O(n)$  acting on Euclidean space and the Lorentz group  $O(1, n - 1)$  acting on Minkowski spacetime

## Isometry groups of model spaces

### Isometry group of Euclidean space

- The isometry group of Euclidean space  $\mathbb{R}^n$  is the Euclidean group  $E(n)$ , which is the semi-direct product of the orthogonal group  $O(n)$  and the translation group  $\mathbb{R}^n$
- The orthogonal group  $O(n)$  consists of all linear isometries (rotations and reflections) of  $\mathbb{R}^n$ , while the translation group  $\mathbb{R}^n$  acts by adding a constant vector to each point
- The dimension of the Euclidean group is  $\frac{n(n+1)}{2}$ , which is the maximum possible dimension for the isometry group of an  $n$ -dimensional Riemannian manifold

### Isometry groups of spheres

- The isometry group of the  $n$ -dimensional sphere  $\mathbb{S}^n$  is the orthogonal group  $O(n + 1)$ , acting on the sphere as a submanifold of  $\mathbb{R}^{n+1}$

- The orientation-preserving isometries of  $\mathbb{S}^n$  form the special orthogonal group  $SO(n + 1)$ , which is the identity component of  $O(n + 1)$
- The isometry group of  $\mathbb{S}^n$  acts transitively on the sphere, meaning that any two points can be mapped to each other by an isometry

## Isometry groups of hyperbolic spaces

- The isometry group of the  $n$ -dimensional hyperbolic space  $\mathbb{H}^n$  is the orthogonal group  $O(1, n)$  of the Minkowski space  $\mathbb{R}^{1, n}$ , acting on  $\mathbb{H}^n$  as a submanifold
- The orientation-preserving isometries of  $\mathbb{H}^n$  form the special orthogonal group  $SO(1, n)$ , which is the identity component of  $O(1, n)$
- The isometry group of  $\mathbb{H}^n$  acts transitively on the space and preserves the ideal boundary  $\partial\mathbb{H}^n$ , which is a sphere of dimension  $n - 1$

## Applications of isometries

### Symmetries in geometry and physics

- Isometries are used to study symmetries in geometry, such as the symmetries of regular polyhedra, lattices, and tilings
- In physics, isometries of spacetime (Poincaré group) and internal spaces (gauge groups) play a fundamental role in the formulation of theories such as special and general relativity, quantum mechanics, and quantum field theory
- Noether's theorem relates continuous symmetries (isometries) to conservation laws, such as the conservation of energy, momentum, and charge

### Isometric embeddings and rigidity

- An isometric embedding is a map between Riemannian manifolds that preserves the metric tensor
- The Nash embedding theorem states that every compact Riemannian manifold can be isometrically embedded into a Euclidean space of sufficiently high dimension
- Rigidity theorems, such as the Mostow rigidity theorem and the Margulis superrigidity theorem, study the existence and uniqueness of isometric embeddings or maps between certain classes of manifolds

### Isospectral manifolds and inverse problems

- Two Riemannian manifolds are called isospectral if they have the same spectrum of the Laplace-Beltrami operator
- Isospectral manifolds share many geometric and topological properties, but may not be isometric
- The study of isospectral manifolds is related to inverse spectral problems, which aim to determine the geometry of a manifold from its spectrum
- Isometries play a role in the construction and classification of isospectral manifolds, as they preserve the spectrum of the Laplace-Beltrami operator

## 6.2 Killing vector fields

Killing vector fields are essential in understanding symmetries in Riemannian geometry. These special vector fields preserve the metric tensor, generating isometries that leave the manifold's structure unchanged. They provide insights into the fundamental properties of geometric spaces.

Killing fields are characterized by Killing's equation, which relates to the Lie derivative of the metric tensor. By solving this equation, we can identify the symmetries of a manifold and understand its isometries, leading to important applications in physics and differential geometry.

### Definition of Killing vector fields

- Killing vector fields are special vector fields on a Riemannian or pseudo-Riemannian manifold that preserve the metric tensor
- Intuitively, Killing fields generate isometries, which are transformations that leave the metric unchanged
- Formally, a vector field  $X$  is a Killing field if the Lie derivative of the metric tensor  $g$  with respect to  $X$  vanishes:  $\mathcal{L}_X g = 0$

### Lie derivative of metric tensor

- The Lie derivative  $\mathcal{L}_X g$  measures the change in the metric tensor  $g$  along the flow generated by the vector field  $X$
- It captures the infinitesimal change in the metric when the manifold is slightly deformed in the direction of  $X$
- The Lie derivative can be expressed in terms of the covariant derivative:  $(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$

### Killing's equation

- Killing's equation is the fundamental equation that characterizes Killing vector fields
- In local coordinates, Killing's equation takes the form:  $\nabla_i X_j + \nabla_j X_i = 0$
- This equation expresses the condition that the metric tensor remains unchanged along the flow of the Killing field

### Coordinate expression of Killing's equation

- In a coordinate basis, Killing's equation can be written using Christoffel symbols:  
$$\partial_i X_j + \partial_j X_i - 2\Gamma_{ij}^k X_k = 0$$
- This form of Killing's equation involves partial derivatives of the components of the Killing field and the Christoffel symbols of the metric
- Solving this system of partial differential equations allows us to find the Killing fields on a given manifold

### Isometries and Killing fields

- Isometries are transformations that preserve the metric structure of a manifold

- Every Killing field generates a one-parameter group of isometries, and conversely, every one-parameter group of isometries has an associated Killing field
- The flow of a Killing field consists of isometries, which means that the metric remains unchanged along the integral curves of the Killing field

## One-parameter groups of isometries

- A one-parameter group of isometries is a family of isometries  $\phi_t$  parameterized by a real number  $t$ , satisfying certain properties:
- $\phi_0$  is the identity transformation
- $\phi_t \circ \phi_s = \phi_{t+s}$  for all  $t, s \in \mathbb{R}$
- The map  $(t, p) \mapsto \phi_t(p)$  is smooth for all  $p$  in the manifold
- The Killing field associated with a one-parameter group of isometries is obtained by differentiating the isometries with respect to the parameter  $t$  at  $t = 0$

## Orbits of Killing fields

- The orbit of a point  $p$  under a Killing field  $X$  is the set of all points that can be reached by following the integral curves of  $X$  starting from  $p$
- Orbits of Killing fields provide a way to understand the symmetries and structure of the manifold
- The dimension of the orbit (orbit type) can vary from point to point, depending on the nature of the Killing field

## Fixed points of isometries

- A fixed point of an isometry  $\phi$  is a point  $p$  such that  $\phi(p) = p$
- Fixed points of isometries correspond to zeros of the associated Killing field
- The behavior of the Killing field near a fixed point can provide information about the local geometry of the manifold

## Algebra of Killing fields

- The set of all Killing fields on a manifold forms a Lie algebra under the Lie bracket operation
- The Lie bracket of two Killing fields is another Killing field, which means that Killing fields are closed under the Lie bracket
- The dimension of the Lie algebra of Killing fields is related to the symmetries and isometries of the manifold

## Lie bracket of Killing fields

- The Lie bracket of two vector fields  $X$  and  $Y$  is defined as  $[X, Y] = XY - YX$ , where  $XY$  denotes the directional derivative of  $Y$  along  $X$
- For Killing fields  $X$  and  $Y$ , the Lie bracket  $[X, Y]$  is also a Killing field
- The Lie bracket captures the non-commutativity of the flows generated by the Killing fields

## Constant length of Killing fields

- Killing fields have constant length along their integral curves
- The inner product  $\langle X, X \rangle$  of a Killing field  $X$  with itself is constant along the flow of  $X$
- This property follows from the fact that Killing fields preserve the metric tensor

## Killing fields vs conformal Killing fields

- Conformal Killing fields are a generalization of Killing fields that preserve the metric tensor up to a scalar factor
- A vector field  $X$  is a conformal Killing field if  $\mathcal{L}_X g = \lambda g$  for some scalar function  $\lambda$
- Killing fields are a special case of conformal Killing fields, where  $\lambda = 0$

## Examples of Killing fields

- Killing fields arise naturally in many important geometries and provide insights into their symmetries and properties
- Studying examples of Killing fields helps develop intuition and understanding of their behavior

## Killing fields in Euclidean space

- In Euclidean space  $\mathbb{R}^n$ , the Killing fields correspond to translations and rotations
- Translation Killing fields are of the form  $X = (a_1, \dots, a_n)$ , where  $a_i$  are constants
- Rotation Killing fields are of the form  $X = (-y, x, 0, \dots, 0)$ ,  $(0, -z, y, 0, \dots, 0)$ , etc., representing rotations in different planes

## Killing fields on spheres

- On the unit sphere  $S^n$ , the Killing fields generate rotations around various axes
- The dimension of the space of Killing fields on  $S^n$  is  $\frac{1}{2}n(n+1)$
- For example, on the 2-sphere  $S^2$ , there are three linearly independent Killing fields corresponding to rotations around the  $X$ ,  $Y$ , and  $Z$  axes

## Killing fields in hyperbolic space

- In hyperbolic space  $\mathbb{H}^n$ , the Killing fields correspond to hyperbolic translations and rotations
- The dimension of the space of Killing fields in  $\mathbb{H}^n$  is  $\frac{1}{2}n(n+1)$
- Hyperbolic translations move points along geodesics, while hyperbolic rotations preserve certain hypersurfaces

## Physical significance of Killing fields

- Killing fields have important applications in physics, particularly in the context of general relativity and gauge theories
- They provide a mathematical framework for understanding symmetries and conservation laws in physical systems

## Conservation laws from Killing fields

- In general relativity, Killing fields are closely related to conservation laws and conserved quantities
- If a spacetime admits a Killing field, then there exists a corresponding conserved quantity along the geodesics of the spacetime
- For example, time translation symmetry leads to conservation of energy, while rotational symmetry leads to conservation of angular momentum

## Computing Killing fields

- Finding Killing fields on a given manifold involves solving Killing's equation, which is a system of partial differential equations
- The process of computing Killing fields depends on the specific geometry and coordinates used

## Solving Killing's equation

- To find Killing fields, one needs to solve Killing's equation  $\nabla_i X_j + \nabla_j X_i = 0$  for the components of the vector field  $X$
- In practice, this often involves using the coordinate expression of Killing's equation and solving the resulting system of PDEs
- Symmetry considerations and the properties of the manifold can simplify the process of solving Killing's equation

## Dimension of space of Killing fields

- The maximum number of linearly independent Killing fields on a manifold is determined by its dimension and geometric properties
- For an  $n$ -dimensional Riemannian manifold, the maximum dimension of the space of Killing fields is  $\frac{1}{2}n(n+1)$
- However, not all manifolds admit the maximum number of Killing fields, and the actual dimension depends on the specific geometry

## 6.3 Lie derivative and Lie brackets

The Lie derivative and Lie bracket are key concepts in differential geometry, describing how tensor fields change along vector fields. These tools provide insights into the infinitesimal behavior of geometric objects and are essential for studying symmetries and conservation laws.

Understanding Lie derivatives and brackets is crucial for grasping the structure of manifolds and their symmetries. These concepts connect to broader themes in differential geometry, such as Lie groups and algebras, and have applications in mathematical physics and other fields.

### Definition of Lie derivative

- The Lie derivative is a fundamental concept in differential geometry that describes how a tensor field changes along the flow of a vector field
- It generalizes the concept of directional derivative to tensor fields and provides a way to study the infinitesimal behavior of geometric objects under the action of a vector field



- The Lie derivative is denoted by  $\mathcal{L}_X$ , where  $X$  is the vector field along which the derivative is taken

## Lie derivative of functions

- For a smooth function  $f$  on a manifold  $M$ , the Lie derivative of  $f$  along a vector field  $X$  is simply the directional derivative of  $f$  along  $X$
- It is defined as  $\mathcal{L}_X f = X(f) = df(X)$ , where  $df$  is the differential of  $f$
- The Lie derivative of a function measures the rate of change of the function along the flow of the vector field  $X$

## Lie derivative of vector fields

- The Lie derivative of a vector field  $Y$  along another vector field  $X$  is a new vector field that describes how  $Y$  changes along the flow of  $X$
- It is defined as  $\mathcal{L}_X Y = [X, Y]$ , where  $[X, Y]$  is the Lie bracket of  $X$  and  $Y$
- The Lie derivative of a vector field measures the infinitesimal change in  $Y$  as it is transported along the flow of  $X$

## Lie derivative of differential forms

- The Lie derivative can also be extended to differential forms, which are antisymmetric tensor fields
- For a differential form  $\omega$ , the Lie derivative along a vector field  $X$  is defined as  $\mathcal{L}_X \omega = i_X d\omega + d(i_X \omega)$ , where  $i_X$  is the interior product with  $X$  and  $d$  is the exterior derivative
- The Lie derivative of a differential form describes how the form changes along the flow of the vector field  $X$

## Properties of Lie derivative

- The Lie derivative satisfies several important properties that make it a useful tool in the study of differential geometry and mathematical physics
- These properties include linearity, product rule, and commutation with the exterior derivative
- Understanding these properties is crucial for manipulating and simplifying expressions involving Lie derivatives

## Linearity of Lie derivative

- The Lie derivative is a linear operator, which means it satisfies the following properties for tensor fields  $S$  and  $T$  and scalar  $a$ :
- $\mathcal{L}_X(S + T) = \mathcal{L}_X S + \mathcal{L}_X T$
- $\mathcal{L}_X(aS) = a\mathcal{L}_X S$
- Linearity allows us to compute Lie derivatives of linear combinations of tensor fields by taking the linear combination of their Lie derivatives

## Product rule for Lie derivative

- The Lie derivative satisfies a product rule similar to the product rule for ordinary derivatives

- For tensor fields  $S$  and  $T$ , the Lie derivative of their tensor product is given by  $\mathcal{L}_X(S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes (\mathcal{L}_X T)$
- The product rule simplifies the computation of Lie derivatives of tensor products and is useful in many applications

## Commutation with exterior derivative

- The Lie derivative commutes with the exterior derivative  $d$ , which means that  $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X\omega)$  for any differential form  $\omega$
- This property is important in the study of differential forms and their behavior under the action of vector fields
- Commutation with the exterior derivative allows us to interchange the order of taking the Lie derivative and the exterior derivative, simplifying many calculations

## Geometric interpretation of Lie derivative

- The Lie derivative has a natural geometric interpretation in terms of the flow of a vector field and the infinitesimal transport of geometric objects along this flow
- This interpretation provides a more intuitive understanding of the Lie derivative and its role in describing the behavior of tensor fields under the action of vector fields
- The geometric interpretation also connects the Lie derivative to important concepts in differential geometry, such as Lie groups and Lie algebras

## Lie derivative as infinitesimal flow

- The Lie derivative of a tensor field along a vector field  $X$  can be interpreted as the infinitesimal change in the tensor field as it is transported along the flow of  $X$
- More precisely, if  $\phi_t$  is the flow generated by  $X$ , then  $\mathcal{L}_X T = \left. \frac{d}{dt} \right|_{t=0} (\phi_t^* T)$ , where  $\phi_t^*$  is the pullback of  $\phi_t$
- This interpretation allows us to understand the Lie derivative as a measure of how a tensor field changes under the infinitesimal action of a vector field

## Relation to Lie groups and Lie algebras

- The Lie derivative is closely related to the concept of Lie groups and Lie algebras in differential geometry
- A Lie group is a smooth manifold that is also a group, with the group operations being smooth maps
- The tangent space at the identity of a Lie group has the structure of a Lie algebra, which is a vector space equipped with a Lie bracket operation
- Vector fields on a Lie group that are left-invariant (or right-invariant) form a Lie algebra under the Lie bracket operation, and the Lie derivative along these vector fields can be used to study the structure and properties of the Lie group

## Lie brackets

- The Lie bracket is a binary operation on vector fields that plays a fundamental role in the study of Lie derivatives and the geometry of manifolds
- It measures the failure of two vector fields to commute and provides a way to generate new vector fields from existing ones
- The Lie bracket is closely related to the concept of Lie algebras and is an essential tool in the study of symmetries and conservation laws in mathematical physics

## Definition of Lie bracket

- The Lie bracket of two vector fields  $X$  and  $Y$  on a manifold  $M$  is a new vector field  $[X, Y]$  defined by  $[X, Y](f) = X(Y(f)) - Y(X(f))$  for any smooth function  $f$  on  $M$
- Alternatively, the Lie bracket can be defined in terms of the Lie derivative as  $[X, Y] = \mathcal{L}_X Y$
- The Lie bracket measures the extent to which the vector fields  $X$  and  $Y$  fail to commute, i.e., the difference between applying  $X$  then  $Y$  and applying  $Y$  then  $X$  to a function

## Lie bracket of vector fields

- The Lie bracket of vector fields satisfies several important properties, such as skew-symmetry and bilinearity
- In local coordinates, the Lie bracket of vector fields  $X = X^i \frac{\partial}{\partial x^i}$  and  $Y = Y^j \frac{\partial}{\partial x^j}$  is given by  $[X, Y] = (X(Y^i) - Y(X^i)) \frac{\partial}{\partial x^i}$
- The Lie bracket of vector fields can be used to study the integrability of distributions and the existence of symmetries on a manifold

## Jacobi identity for Lie brackets

- The Lie bracket satisfies the Jacobi identity, which states that  $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$  for any vector fields  $X$ ,  $Y$ , and  $Z$
- The Jacobi identity is a fundamental property of Lie brackets and Lie algebras, and it plays a crucial role in many applications
- The Jacobi identity ensures that the Lie bracket operation is compatible with the vector space structure of the space of vector fields on a manifold

## Properties of Lie brackets

- The Lie bracket of vector fields satisfies several important properties that make it a powerful tool in the study of differential geometry and mathematical physics
- These properties include skew-symmetry, bilinearity, and the relation to the commutator of derivations
- Understanding these properties is essential for manipulating expressions involving Lie brackets and for studying the structure of Lie algebras

## Skew-symmetry of Lie bracket

- The Lie bracket is skew-symmetric, which means that  $[X, Y] = -[Y, X]$  for any vector fields  $X$  and  $Y$

- Skew-symmetry implies that  $[X, X] = 0$  for any vector field  $X$ , which is known as the alternating property
- Skew-symmetry is a fundamental property of Lie brackets and is closely related to the antisymmetry of the exterior product of differential forms

## Bilinearity of Lie bracket

- The Lie bracket is bilinear, which means it is linear in each argument separately
- For vector fields  $X, Y, Z$ , and scalars  $a$  and  $b$ , we have:
  - $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
  - $[X, aY + bZ] = a[X, Y] + b[X, Z]$
- Bilinearity allows us to compute Lie brackets of linear combinations of vector fields by taking the corresponding linear combinations of their Lie brackets

## Relation to commutator of derivations

- The Lie bracket of vector fields is closely related to the commutator of derivations on the algebra of smooth functions on a manifold
- For vector fields  $X$  and  $Y$ , the Lie bracket  $[X, Y]$  acts on smooth functions as the commutator of the derivations  $X$  and  $Y$ , i.e.,  $[X, Y](f) = X(Y(f)) - Y(X(f))$
- This relation provides an algebraic interpretation of the Lie bracket and connects it to the concept of derivations on algebras

## Relation between Lie derivative and Lie bracket

- The Lie derivative and Lie bracket are closely related concepts in differential geometry, and their relationship is captured by several important formulas and identities
- Understanding the connection between these two concepts is crucial for studying the geometry of manifolds and the behavior of tensor fields under the action of vector fields
- The most important relations between the Lie derivative and Lie bracket are the definition of the Lie derivative in terms of the Lie bracket and Cartan's magic formula

## Lie derivative as Lie bracket with vector field

- The Lie derivative of a vector field  $Y$  along another vector field  $X$  can be defined as the Lie bracket of  $X$  and  $Y$ , i.e.,  $\mathcal{L}_X Y = [X, Y]$
- This definition provides a direct link between the Lie derivative and the Lie bracket, and it allows us to interpret the Lie derivative as a measure of the non-commutativity of the flows generated by the vector fields  $X$  and  $Y$
- The Lie derivative of other tensor fields can also be expressed in terms of Lie brackets and the contraction operator, which generalizes the connection between the Lie derivative and Lie bracket

## Cartan's magic formula

- Cartan's magic formula is an important identity that relates the Lie derivative, exterior derivative, and interior product of differential forms

- For a differential form  $\omega$  and a vector field  $X$ , Cartan's magic formula states that  $\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega)$ , where  $i_X$  is the interior product with  $X$  and  $d$  is the exterior derivative
- This formula provides a powerful tool for computing Lie derivatives of differential forms and understanding their behavior under the action of vector fields
- Cartan's magic formula is a fundamental result in differential geometry and has numerous applications in mathematical physics, such as in the study of symplectic and contact manifolds

## Applications of Lie derivative and Lie bracket

- The Lie derivative and Lie bracket have numerous applications in differential geometry, mathematical physics, and other areas of mathematics
- These concepts play a crucial role in the study of symmetries, conservation laws, and the integrability of distributions on manifolds
- Some of the most important applications of the Lie derivative and Lie bracket include the study of Frobenius theorem, the Lie algebra of vector fields on manifolds, and the geometry of Lie groups

## Symmetries and conservation laws

- The Lie derivative and Lie bracket are essential tools in the study of symmetries and conservation laws in mathematical physics
- A vector field  $X$  on a manifold  $M$  is called a symmetry of a tensor field  $T$  if  $\mathcal{L}_X T = 0$ , which means that  $T$  is invariant under the flow generated by  $X$
- Symmetries of a system of differential equations can be used to find conserved quantities and simplify the analysis of the system
- The Lie bracket of symmetry vector fields forms a Lie algebra, which captures the structure of the symmetry group of the system

## Frobenius theorem and integrability

- The Frobenius theorem is a fundamental result in differential geometry that characterizes the integrability of distributions on manifolds
- A distribution  $\Delta$  on a manifold  $M$  is a smooth assignment of a subspace of the tangent space at each point of  $M$
- The Frobenius theorem states that a distribution  $\Delta$  is integrable (i.e., it corresponds to a foliation of  $M$ ) if and only if it is involutive, which means that the Lie bracket of any two vector fields in  $\Delta$  also belongs to  $\Delta$
- The Lie bracket plays a crucial role in the formulation and proof of the Frobenius theorem, and it provides a powerful tool for studying the integrability of distributions and the existence of local coordinate systems adapted to a given distribution

## Lie algebra of vector fields on manifolds

- The space of vector fields on a manifold  $M$  has the structure of a Lie algebra under the Lie bracket operation
- This Lie algebra, denoted by  $\mathfrak{X}(M)$ , captures the infinitesimal symmetries of the manifold and plays a fundamental role in the study of the geometry and topology of  $M$

- Many important geometric structures on manifolds, such as Riemannian metrics, symplectic forms, and Poisson brackets, can be studied using the Lie algebra of vector fields and its representations
- The Lie algebra of vector fields is also closely related to the concept of Lie groups and their actions on manifolds, providing a bridge between the infinitesimal and global aspects of symmetry in differential geometry

## 6.4 Homogeneous spaces

Homogeneous spaces are manifolds with high symmetry, where a Lie group acts transitively. They're key in Metric Differential Geometry, connecting algebra and geometry. Examples include spheres, projective spaces, and Grassmannians.

These spaces allow us to study geometric properties through algebraic structures. We'll explore their definition, Riemannian metrics, curvature, and classification. We'll also examine compact and noncompact cases, and their roles in various mathematical and physical applications.

### Definition of homogeneous spaces

- Homogeneous spaces are an important class of manifolds that exhibit a high degree of symmetry and play a central role in the study of Metric Differential Geometry
- A homogeneous space is a manifold  $M$  on which a Lie group  $G$  acts transitively, meaning that for any two points  $p, q \in M$ , there exists an element  $g \in G$  such that  $g \cdot p = q$
- The study of homogeneous spaces provides a rich interplay between the algebraic structure of Lie groups and the geometric properties of manifolds

### Transitive action of Lie groups

- A Lie group  $G$  acts transitively on a manifold  $M$  if for any two points  $p, q \in M$ , there exists an element  $g \in G$  such that  $g \cdot p = q$
- Transitive actions allow us to study the geometry of the manifold  $M$  through the algebraic properties of the Lie group  $G$
- Examples of transitive actions include the action of the rotation group  $SO(3)$  on the 2-sphere  $S^2$  and the action of the Euclidean group  $E(n)$  on  $\mathbb{R}^n$

### Isotropy subgroup and coset space

- For a point  $p \in M$ , the isotropy subgroup (or stabilizer)  $H_p$  is the subgroup of  $G$  that fixes  $p$ , i.e.,  $H_p = \{g \in G : g \cdot p = p\}$
- The coset space  $G/H_p$  is the set of left cosets  $\{gH_p : g \in G\}$ , which can be identified with the orbit of  $p$  under the action of  $G$
- The coset space  $G/H_p$  inherits a unique smooth structure from  $G$  and  $M$ , making it a homogeneous space

### Examples of homogeneous spaces

- The  $n$ -sphere  $S^n$  is a homogeneous space under the action of the orthogonal group  $O(n+1)$
- The projective space  $\mathbb{RP}^n$  is a homogeneous space under the action of the projective linear group  $PGL(n+1, \mathbb{R})$

- Grassmann manifolds  $Gr(k, n)$ , which parametrize  $k$ -dimensional subspaces of  $\mathbb{R}^n$ , are homogeneous spaces under the action of the general linear group  $GL(n, \mathbb{R})$

## Riemannian metrics on homogeneous spaces

- The study of Riemannian metrics on homogeneous spaces is a central theme in Metric Differential Geometry, as it allows us to investigate the interplay between the symmetries of the space and its geometric properties
- A Riemannian metric on a homogeneous space  $M = G/H$  is called  $G$ -invariant if it is preserved under the action of  $G$ , i.e., the group action consists of isometries

## Invariant metrics and reductive homogeneous spaces

- A homogeneous space  $M = G/H$  is called reductive if the Lie algebra  $\mathfrak{g}$  of  $G$  admits a direct sum decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ , where  $\mathfrak{h}$  is the Lie algebra of  $H$  and  $\mathfrak{m}$  is an  $Ad(H)$ -invariant complement
- Reductive homogeneous spaces allow for a natural construction of  $G$ -invariant Riemannian metrics by choosing an  $Ad(H)$ -invariant inner product on  $\mathfrak{m}$
- Examples of reductive homogeneous spaces include symmetric spaces and naturally reductive spaces

## Geodesics in homogeneous spaces

- Geodesics in a homogeneous space  $M = G/H$  with a  $G$ -invariant Riemannian metric can be studied using the algebraic structure of the Lie group  $G$
- Geodesics through the origin  $eH \in M$  are given by the exponential map  $\exp : \mathfrak{m} \rightarrow M$ , where  $\mathfrak{m}$  is the complement of  $\mathfrak{h}$  in the reductive decomposition of  $\mathfrak{g}$
- The homogeneity of the space allows for the extension of geodesics through any point using the group action

## Curvature of homogeneous spaces

- The curvature tensor of a homogeneous space with a  $G$ -invariant Riemannian metric can be computed using the Lie bracket structure of the Lie algebra  $\mathfrak{g}$
- Homogeneous spaces with constant sectional curvature, such as the sphere  $S^n$  and the hyperbolic space  $\mathbb{H}^n$ , play a special role in geometry
- The study of curvature on homogeneous spaces leads to important results in Metric Differential Geometry, such as the classification of homogeneous Einstein manifolds

## Classification of homogeneous spaces

- The classification of homogeneous spaces is a central problem in Metric Differential Geometry, as it provides a systematic way to understand the structure and properties of these spaces
- Several important classes of homogeneous spaces have been studied extensively, each with its own unique geometric and algebraic properties

## Symmetric spaces

- A symmetric space is a connected Riemannian manifold  $M$  such that for each point  $p \in M$ , there exists an isometry  $\sigma_p : M \rightarrow M$  that fixes  $p$  and reverses geodesics through  $p$
- Symmetric spaces are homogeneous spaces with a rich algebraic structure and have been completely classified using the theory of semisimple Lie algebras
- Examples of symmetric spaces include Euclidean spaces, spheres, hyperbolic spaces, and Grassmann manifolds

## Naturally reductive spaces

- A homogeneous space  $M = G/H$  with a  $G$ -invariant Riemannian metric is called naturally reductive if there exists a reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  such that  $\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle = 0$  for all  $X, Y, Z \in \mathfrak{m}$
- Naturally reductive spaces form a larger class than symmetric spaces and have geodesic symmetries at each point
- Examples of naturally reductive spaces include odd-dimensional spheres with the round metric and compact Lie groups with bi-invariant metrics

## Normal homogeneous spaces

- A homogeneous space  $M = G/H$  is called normal if the Lie algebra  $\mathfrak{g}$  of  $G$  admits a reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  such that  $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$
- Normal homogeneous spaces have a canonical connection called the Nomizu connection, which is closely related to the geometry of the space
- Examples of normal homogeneous spaces include isotropy irreducible spaces and generalized flag manifolds

## Compact homogeneous spaces

- Compact homogeneous spaces are an important class of manifolds in Metric Differential Geometry, as they exhibit rich geometric and topological properties
- The study of compact homogeneous spaces often involves the interplay between Lie group theory, representation theory, and differential geometry

## Relationship with Lie groups

- Every compact homogeneous space  $M$  can be realized as a coset space  $G/H$ , where  $G$  is a compact Lie group and  $H$  is a closed subgroup of  $G$
- The structure and properties of compact homogeneous spaces are closely related to the representation theory of compact Lie groups
- Examples of compact homogeneous spaces include spheres, projective spaces, and flag manifolds

## Killing fields and isometries

- A Killing field on a Riemannian manifold  $M$  is a vector field  $X$  that generates a one-parameter family of isometries, i.e., the flow of  $X$  preserves the Riemannian metric



- On a compact homogeneous space  $M = G/H$ , the Killing fields correspond to the elements of the Lie algebra  $\mathfrak{g}$  of  $G$
- The study of Killing fields and isometries on compact homogeneous spaces leads to important results in geometry and topology, such as the Bott periodicity theorem

## Applications in geometry and physics

- Compact homogeneous spaces play a significant role in various areas of geometry and physics
- In Riemannian geometry, compact homogeneous spaces serve as model spaces and provide examples of manifolds with specific curvature properties (constant curvature, Einstein metrics, etc.)
- In mathematical physics, compact homogeneous spaces appear in the study of gauge theories, string theory, and quantum mechanics (e.g., as target spaces for nonlinear sigma models)

## Noncompact homogeneous spaces

- Noncompact homogeneous spaces are another important class of manifolds in Metric Differential Geometry, with applications in various areas of mathematics and physics
- The study of noncompact homogeneous spaces often involves the interplay between Lie group theory, geometric analysis, and differential equations

## Solvmanifolds and nilmanifolds

- A solvmanifold is a homogeneous space  $M = G/H$ , where  $G$  is a solvable Lie group and  $H$  is a closed subgroup
- A nilmanifold is a homogeneous space  $M = G/H$ , where  $G$  is a nilpotent Lie group and  $H$  is a closed subgroup
- Solvmanifolds and nilmanifolds provide important examples of noncompact homogeneous spaces with rich geometric and dynamical properties

## Homogeneous Einstein metrics

- An Einstein metric on a Riemannian manifold  $M$  is a metric whose Ricci curvature is proportional to the metric tensor, i.e.,  $\text{Ric} = \lambda g$  for some constant  $\lambda$
- The study of homogeneous Einstein metrics on noncompact homogeneous spaces is an active area of research in Metric Differential Geometry
- Examples of noncompact homogeneous spaces admitting Einstein metrics include solvmanifolds, nilmanifolds, and symmetric spaces of noncompact type

## Role in geometric analysis and relativity

- Noncompact homogeneous spaces play a significant role in geometric analysis and relativity
- In geometric analysis, noncompact homogeneous spaces appear in the study of harmonic analysis, heat kernels, and geometric flows (e.g., Ricci flow)
- In relativity, noncompact homogeneous spaces serve as models for cosmological spacetimes and provide examples of solutions to the Einstein field equations (e.g., Bianchi spacetimes)

## Connections and curvature on homogeneous spaces

- The study of connections and curvature on homogeneous spaces is a central theme in Metric Differential Geometry, as it provides a way to understand the intrinsic geometry of these spaces
- Homogeneous spaces often admit a natural class of connections that are compatible with the symmetries of the space, leading to simplifications in the computation of curvature

## Invariant connections and Nomizu construction

- An invariant connection on a homogeneous space  $M = G/H$  is a connection that is preserved under the action of the Lie group  $G$
- The Nomizu construction provides a way to obtain invariant connections on reductive homogeneous spaces using the Lie algebra structure of  $\mathfrak{g}$  and the reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$
- The canonical connection on a reductive homogeneous space, obtained via the Nomizu construction, plays a crucial role in the study of the geometry of the space

## Ricci curvature and scalar curvature

- The Ricci curvature and scalar curvature are important geometric invariants that measure the amount of curvature in a Riemannian manifold
- On homogeneous spaces, the Ricci and scalar curvatures can be computed using the structure constants of the Lie algebra  $\mathfrak{g}$  and the invariant metric
- The study of Ricci and scalar curvatures on homogeneous spaces leads to important results in Metric Differential Geometry, such as the classification of homogeneous Einstein manifolds and the Alekseevsky conjecture

## Homogeneous structures and holonomy

- A homogeneous structure on a manifold  $M$  is a reduction of the frame bundle of  $M$  to a subgroup  $G$  of the general linear group  $GL(n, \mathbb{R})$
- The holonomy group of a connection on a manifold  $M$  is the group of parallel transports along loops based at a point
- Homogeneous spaces with invariant connections provide examples of manifolds with reduced holonomy groups, which are of great interest in Metric Differential Geometry and mathematical physics (e.g., Calabi-Yau manifolds,  $G_2$  manifolds)

## Homogeneous spaces in Lie theory

- The study of homogeneous spaces is closely related to Lie theory, as homogeneous spaces are constructed using Lie groups and their subgroups
- Lie theory provides powerful tools for understanding the structure and properties of homogeneous spaces, and conversely, homogeneous spaces serve as important examples and applications of Lie theoretic concepts

## Relationship with representation theory

- The representation theory of Lie groups plays a crucial role in the study of homogeneous spaces

- The isotropy representation of a homogeneous space  $M = G/H$  is the representation of the isotropy subgroup  $H$  on the tangent space  $T_e M$ , which determines the local geometry of the space
- The study of spherical representations and multiplicity-free spaces provides a deep connection between representation theory and the geometry of homogeneous spaces

## Homogeneous spaces as manifolds with symmetry

- Homogeneous spaces are manifolds equipped with a transitive action of a Lie group, which provides a natural notion of symmetry
- The presence of symmetry on homogeneous spaces allows for the use of powerful techniques from Lie theory and representation theory to study their geometry and topology
- Examples of manifolds with symmetry that can be studied using the theory of homogeneous spaces include symmetric spaces, flag manifolds, and isotropy irreducible spaces

## Applications in harmonic analysis and PDEs

- Homogeneous spaces play a significant role in harmonic analysis and the study of partial differential equations (PDEs)
- The symmetries of homogeneous spaces allow for the use of Fourier analysis and representation theory to study function spaces and differential operators on these manifolds
- Important examples of PDEs that can be studied using the theory of homogeneous spaces include the Laplace-Beltrami equation, the heat equation, and the wave equation on symmetric spaces and other homogeneous manifolds

## 6.5 Symmetric spaces

Symmetric spaces are a special class of Riemannian manifolds with high symmetry. They generalize Euclidean spaces and spheres, exhibiting large groups of isometries that preserve their geometry. This makes them important objects of study in differential geometry.

Symmetric spaces are closely related to Lie groups and can be classified into compact, noncompact, and Euclidean types. Their curvature properties play a crucial role in determining their geometry and topology, distinguishing them from general Riemannian manifolds.

### Definition of symmetric spaces

- Symmetric spaces are a special class of Riemannian manifolds that possess a high degree of symmetry, making them important objects of study in differential geometry
- The concept of symmetric spaces generalizes the notion of Euclidean spaces and spheres, which exhibit a large group of isometries that preserve the geometry of the space

### Riemannian manifolds with symmetry

- A Riemannian manifold is a smooth manifold equipped with a Riemannian metric, which is a positive definite inner product on each tangent space
- Symmetric spaces are Riemannian manifolds with an additional symmetry property: for each point  $p$  in the manifold, there exists an isometry (a distance-preserving map) that fixes  $p$  and reverses geodesics through  $p$

- This symmetry condition implies that the geometry of the manifold is the same in all directions around each point

## Geodesic symmetries

- In a symmetric space, for each point  $P$  and each geodesic  $Y$  passing through  $P$ , there exists an isometry  $S_P$  called the geodesic symmetry at  $P$  that fixes  $P$  and reverses the direction of  $Y$
- Geodesic symmetries are a crucial tool in the study of symmetric spaces, as they relate the local geometry around a point to the global geometry of the space
- The existence of geodesic symmetries at every point is a strong condition that distinguishes symmetric spaces from general Riemannian manifolds

## Involutive isometries

- An involutive isometry is an isometry  $\sigma$  such that  $\sigma^2 = \text{id}$ , meaning that applying the isometry twice yields the identity map
- In a symmetric space, the geodesic symmetries at each point are involutive isometries
- The involutive property of geodesic symmetries is essential in the classification and structure theory of symmetric spaces, as it relates to the algebraic properties of the isometry group

## Lie groups and symmetric spaces

- Lie groups are smooth manifolds that are also groups, where the group operations (multiplication and inversion) are smooth maps
- Symmetric spaces are closely related to Lie groups, as the isometry group of a symmetric space is a Lie group, and many symmetric spaces can be realized as homogeneous spaces of Lie groups

## Lie groups as symmetric spaces

- Every Lie group  $G$  can be viewed as a symmetric space by considering the map  $\sigma_g : G \rightarrow G$  given by  $\sigma_g(h) = gh^{-1}g$  for  $g, h \in G$
- The map  $\sigma_g$  is an involutive isometry of  $G$  that fixes the point  $g$ , making  $G$  a symmetric space with respect to any left-invariant Riemannian metric
- This construction provides a rich source of examples of symmetric spaces and highlights the interplay between the group structure and the geometric structure

## Lie algebra of isometries

- The Lie algebra  $\mathfrak{g}$  of a Lie group  $G$  is the tangent space at the identity element, equipped with a Lie bracket operation
- In the context of symmetric spaces, the Lie algebra of the isometry group plays a crucial role in understanding the infinitesimal symmetries of the space
- The Lie algebra of isometries decomposes into a direct sum of the Lie algebra of the isotropy subgroup (the subgroup fixing a point) and its orthogonal complement, which can be identified with the tangent space at the fixed point

## Exponential map and symmetries

- The exponential map  $\exp : \mathfrak{g} \rightarrow G$  is a smooth map that relates the Lie algebra to the Lie group
- In a symmetric space, the exponential map can be used to construct geodesics and to relate the infinitesimal symmetries (Lie algebra elements) to global symmetries (isometries)
- The geodesic symmetries in a symmetric space can be expressed in terms of the exponential map and the Lie algebra of the isometry group, providing a powerful tool for studying the geometry of the space

## Classification of symmetric spaces

- Symmetric spaces can be classified into three main types based on their curvature and completeness properties: compact type, noncompact type, and Euclidean type
- The classification of symmetric spaces is a major result in differential geometry and has important implications for the study of these spaces and their applications

### Compact type

- A symmetric space is of compact type if it is complete and has non-negative sectional curvature
- Examples of compact type symmetric spaces include spheres, projective spaces, and Grassmannians
- Compact type symmetric spaces have a rich geometry and topology, and they are related to important objects in algebraic topology and representation theory, such as homology and cohomology groups and representations of compact Lie groups

### Noncompact type

- A symmetric space is of noncompact type if it is complete, simply connected, and has non-positive sectional curvature
- Examples of noncompact type symmetric spaces include hyperbolic spaces and symmetric spaces associated with semisimple Lie groups
- Noncompact type symmetric spaces have important applications in number theory, representation theory, and the study of discrete subgroups of Lie groups, such as lattices and arithmetic groups

### Euclidean type

- A symmetric space is of Euclidean type if it is complete and has zero sectional curvature
- Euclidean spaces and flat tori are examples of Euclidean type symmetric spaces
- Euclidean type symmetric spaces are the simplest type of symmetric spaces and serve as a building block for the study of more general symmetric spaces through the process of taking products and quotients

## Curvature of symmetric spaces

- The curvature of a symmetric space plays a crucial role in determining its geometry and topology
- Symmetric spaces have special curvature properties that distinguish them from general Riemannian manifolds and make them more tractable objects of study

## Constant curvature spaces

- A space has constant curvature if its sectional curvature is the same at every point and for every plane section
- Examples of constant curvature spaces include Euclidean spaces (zero curvature), spheres (positive curvature), and hyperbolic spaces (negative curvature)
- Symmetric spaces of compact type and noncompact type generalize the notion of constant curvature spaces, as they have non-negative and non-positive sectional curvature, respectively

## Sectional curvature

- Sectional curvature is a measure of the curvature of a Riemannian manifold that assigns a real number to each plane section (a two-dimensional subspace of the tangent space) at each point
- In a symmetric space, the sectional curvature determines the local geometry and the behavior of geodesics
- The sign of the sectional curvature (positive, negative, or zero) is an important invariant of a symmetric space and is used in the classification of these spaces

## Ricci curvature

- Ricci curvature is another measure of the curvature of a Riemannian manifold, obtained by taking the trace of the sectional curvature tensor
- In a symmetric space, the Ricci curvature is closely related to the mean curvature of geodesic spheres and the volume growth of balls
- The Ricci curvature of a symmetric space is used in the study of harmonic functions, heat kernels, and the spectrum of the Laplace-Beltrami operator

## Examples of symmetric spaces

- There are many important examples of symmetric spaces that arise in various branches of mathematics and physics
- These examples illustrate the richness and diversity of symmetric spaces and their applications

## Spheres and hyperbolic spaces

- Spheres  $S^n$  and hyperbolic spaces  $H^n$  are the most basic examples of symmetric spaces of compact and noncompact type, respectively
- These spaces have constant positive (spheres) or negative (hyperbolic) sectional curvature and are the model spaces for the study of geometry in positive and negative curvature
- Spheres and hyperbolic spaces play a central role in topology, geometry, and the study of discrete groups, such as Kleinian groups and Fuchsian groups

## Grassmannians and flag manifolds

- The Grassmannian  $Gr(k, n)$  is the space of  $k$ -dimensional linear subspaces of  $\mathbb{R}^n$  (or  $\mathbb{C}^n$ )

- Flag manifolds are generalizations of Grassmannians, where instead of a single subspace, one considers a sequence of nested subspaces (a flag)
- Grassmannians and flag manifolds are important examples of symmetric spaces of compact type and have applications in algebraic geometry, representation theory, and the study of vector bundles and characteristic classes

## Symmetric spaces of exceptional type

- In addition to the classical examples of symmetric spaces (such as spheres, Grassmannians, and symmetric spaces associated with classical Lie groups), there are also exceptional symmetric spaces that do not fit into these families
- Examples of exceptional symmetric spaces include the Cayley projective plane (associated with the exceptional Lie group  $E_6$ ) and the octonionic projective plane (associated with the exceptional Lie group  $E_8$ )
- These exceptional symmetric spaces have unique geometric properties and are related to the study of exceptional Lie groups, which play a significant role in the classification of simple Lie algebras and the study of symmetries in high dimensions

## Geometry of symmetric spaces

- The geometry of symmetric spaces is a rich and well-developed area of differential geometry, with many deep connections to other areas of mathematics
- The study of geodesics, Jacobi fields, and the cut locus provides insight into the global structure of symmetric spaces and their singularities

## Geodesics and parallel transport

- Geodesics are the generalization of straight lines to curved spaces and are the shortest paths between points in a symmetric space
- In a symmetric space, geodesics are intimately related to the symmetries of the space, as the geodesic symmetries at each point map geodesics to geodesics
- Parallel transport is a way of moving vectors along curves (such as geodesics) while preserving their angle and length, and it plays a crucial role in the study of the curvature and holonomy of symmetric spaces

## Jacobi fields and conjugate points

- Jacobi fields are vector fields along a geodesic that describe the behavior of nearby geodesics
- In a symmetric space, Jacobi fields can be used to study the stability of geodesics and the existence of conjugate points (points where nearby geodesics intersect)
- The study of Jacobi fields and conjugate points is important in understanding the global geometry of symmetric spaces and their singularities, such as the cut locus

## Cut locus and injectivity radius

- The cut locus of a point  $p$  in a symmetric space is the set of points where geodesics starting from  $p$  cease to be minimizing (i.e., they are no longer the shortest paths)

- The injectivity radius of a symmetric space is the largest radius for which the exponential map at each point is a diffeomorphism (a smooth bijection with a smooth inverse)
- The cut locus and injectivity radius provide important information about the global geometry of a symmetric space and are related to the study of geodesic convexity, the behavior of balls and spheres, and the singularities of the space

## Harmonic analysis on symmetric spaces

- Harmonic analysis is the study of functions and their Fourier transforms on symmetric spaces
- The tools of harmonic analysis, such as invariant differential operators and spherical functions, provide a powerful framework for understanding the geometry and representation theory of symmetric spaces

## Invariant differential operators

- Invariant differential operators are differential operators on a symmetric space that commute with the action of the isometry group
- Examples of invariant differential operators include the Laplace-Beltrami operator and the Dirac operator
- The study of invariant differential operators is central to the harmonic analysis on symmetric spaces, as they provide a way to construct and analyze eigenfunctions and to study the spectrum of the space

## Spherical functions and Fourier transform

- Spherical functions are special functions on a symmetric space that are eigenfunctions of invariant differential operators and are invariant under the action of the isotropy subgroup at a point
- The spherical Fourier transform is a generalization of the classical Fourier transform to symmetric spaces, using spherical functions as the basis for the transform
- The spherical Fourier transform provides a way to analyze functions on symmetric spaces in terms of their frequency components and is a powerful tool in harmonic analysis and representation theory

## Representations of isometry groups

- The isometry group of a symmetric space is a Lie group that acts transitively on the space
- Representations of the isometry group (i.e., homomorphisms from the group to the group of linear transformations on a vector space) play a crucial role in the harmonic analysis on symmetric spaces
- The study of representations of isometry groups is related to the construction of spherical functions, the decomposition of  $L^2$  spaces on symmetric spaces, and the study of the spectrum of invariant differential operators

## Applications of symmetric spaces

- Symmetric spaces have numerous applications in various areas of mathematics and physics, showcasing their importance and versatility
- These applications range from the study of automorphic forms and representation theory to integrable systems and mathematical physics



## Representation theory and automorphic forms

- Symmetric spaces are closely related to the representation theory of Lie groups and the study of automorphic forms
- Automorphic forms are functions on symmetric spaces that satisfy certain transformation properties with respect to discrete subgroups of the isometry group (such as arithmetic groups)
- The study of automorphic forms is a central topic in number theory and has important applications in the theory of  $L$ -functions, the Langlands program, and the study of modular forms and Shimura varieties

## Integrable systems and soliton equations

- Symmetric spaces have important applications in the theory of integrable systems and soliton equations
- Many integrable systems, such as the Korteweg-de Vries (KdV) equation and the nonlinear Schrödinger equation, can be viewed as infinite-dimensional symmetric spaces
- The geometry of symmetric spaces provides a framework for understanding the integrability and soliton solutions of these equations, as well as their symmetries and conservation laws

## Symmetric spaces in physics

- Symmetric spaces arise naturally in various areas of mathematical physics, such as general relativity, string theory, and quantum field theory
- In general relativity, symmetric spaces are used to model homogeneous and isotropic spacetimes, such as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which describes the large-scale structure of the universe
- In string theory and supergravity, symmetric spaces appear as the target spaces for nonlinear sigma models and as the moduli spaces of supersymmetric solutions
- The geometry and topology of symmetric spaces play a crucial role in understanding the symmetries and dualities of these physical theories, as well as their quantum corrections and nonperturbative effects

## 6.6 Isometric group actions

Isometric group actions are powerful tools for understanding symmetries in metric spaces. They preserve distances between points, allowing us to analyze geometric structures based on their symmetry properties. This concept is crucial for classifying spaces and studying their transformations.

By exploring orbits, isotropy subgroups, and quotient spaces, we gain insights into the behavior of these actions. This knowledge has wide-ranging applications in geometry, topology, and physics, helping us uncover fundamental properties of space and symmetry.

### Isometric group actions

- Isometric group actions play a crucial role in understanding the symmetries and transformations of metric spaces while preserving distances between points
- The study of isometric group actions allows for the classification and analysis of geometric structures based on their symmetry properties

## Definition of isometric group actions

- An isometric group action is a map  $\phi : G \times X \rightarrow X$  where  $G$  is a group and  $X$  is a metric space
- The map satisfies the following conditions:
  1.  $\phi(e, x) = x$  for all  $x \in X$ , where  $e$  is the identity element of  $G$
  2.  $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$  for all  $g_1, g_2 \in G$  and  $x \in X$
- For each  $g \in G$ , the map  $\phi_g : X \rightarrow X$  defined by  $\phi_g(x) = \phi(g, x)$  is an isometry of  $X$
- This means that  $d(\phi_g(x), \phi_g(y)) = d(x, y)$  for all  $x, y \in X$ , where  $d$  is the metric on  $X$

## Examples of isometric group actions

- The group of rotations  $SO(n)$  acts isometrically on the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  by matrix multiplication
- Each rotation preserves distances between points in  $\mathbb{R}^n$
- The group of translations acts isometrically on Euclidean space by shifting points by a fixed vector
- Translations preserve distances between points and maintain the geometric structure of the space

## Orbits of isometric group actions

- The orbit of a point  $x \in X$  under an isometric group action  $\phi$  is the set  $\{gx : g \in G\}$ , where  $gx = \phi(g, x)$
- Orbits partition the space  $X$  into disjoint subsets, each consisting of points that can be mapped to one another by elements of  $G$
- The orbit space  $X/G$  is the set of all orbits of the action
- The orbit space provides a way to study the quotient space of  $X$  under the equivalence relation induced by the group action

## Isotropy subgroups of isometric group actions

- The isotropy subgroup (or stabilizer) of a point  $x \in X$  under an isometric group action  $\phi$  is the subgroup  $G_x = \{g \in G : gx = x\}$
- The isotropy subgroup consists of all elements of  $G$  that fix the point  $x$
- The conjugacy class of an isotropy subgroup is an invariant of the group action
- Conjugate isotropy subgroups correspond to points in the same orbit

## Properties of isometric group actions

- The properties of isometric group actions provide insight into the behavior and structure of the action and its effect on the underlying metric space

## Proper vs improper actions

- An isometric group action is proper if for every compact subset  $K \subset X$ , the set  $\{g \in G : gK \cap K \neq \emptyset\}$  is compact in  $G$

- Proper actions have well-behaved quotient spaces and orbits
- Improper actions may have orbits that are not closed or have non-Hausdorff quotient spaces

## Discrete vs continuous actions

- A discrete isometric group action is one in which the group  $G$  is a discrete group (e.g., the integers or a finite group)
- Discrete actions often lead to orbits that are discrete subsets of the space
- A continuous isometric group action is one in which the group  $G$  is a continuous group (e.g., Lie groups like  $SO(n)$  or  $\mathbb{R}$ )
- Continuous actions can lead to orbits that are continuous submanifolds of the space

## Free vs non-free actions

- An isometric group action is free if for every  $x \in X$ , the isotropy subgroup  $G_x$  is trivial (consists only of the identity element)
- In a free action, no non-identity element of  $G$  fixes any point of  $X$
- A non-free action has non-trivial isotropy subgroups for some points in  $X$
- Non-free actions may have fixed points or points with non-trivial stabilizers

## Quotient spaces of isometric group actions

- Quotient spaces arise naturally when studying isometric group actions, as they allow for the identification of points that are equivalent under the action

## Definition of quotient space

- The quotient space  $X/G$  of a metric space  $X$  under an isometric group action  $\phi$  is the set of all orbits of the action
- Points in the same orbit are considered equivalent in the quotient space
- The quotient map  $\pi : X \rightarrow X/G$  sends each point  $x \in X$  to its orbit  $Gx$

## Metric on quotient space

- A metric  $d_{X/G}$  can be defined on the quotient space  $X/G$  by setting  $d_{X/G}(Gx, Gy) = \inf \{d(x', y') : x' \in Gx, y' \in Gy\}$
- This metric measures the distance between orbits as the infimum of distances between points in the orbits
- The quotient map  $\pi : X \rightarrow X/G$  is a metric submersion, meaning that it preserves distances between points in  $X$  that are mapped to different orbits in  $X/G$

## Properties of quotient spaces

- The topology of the quotient space  $X/G$  is determined by the quotient topology, where a subset  $U \subset X/G$  is open if and only if its preimage  $\pi^{-1}(U)$  is open in  $X$
- If the isometric group action is proper and free, the quotient space  $X/G$  is a Hausdorff metric space

- In this case, the quotient map  $\pi : X \rightarrow X/G$  is a covering map

## Applications of isometric group actions

- Isometric group actions have numerous applications across various branches of mathematics and physics, providing a framework for understanding symmetries and invariance properties

## Isometric group actions in geometry

- Isometric group actions are used to classify and study geometric structures based on their symmetries
- For example, the classification of Euclidean space forms relies on the study of isometric group actions on Euclidean space
- The study of isometric group actions on Riemannian manifolds leads to the development of important concepts such as Killing vector fields and homogeneous spaces

## Isometric group actions in topology

- Isometric group actions provide a way to construct new topological spaces from existing ones through the process of taking quotients
- Quotient spaces under isometric group actions often have interesting topological properties and can be used to construct examples of manifolds and orbifolds
- The study of free and proper isometric group actions is closely related to the theory of principal bundles and fiber bundles in topology

## Isometric group actions in physics

- Isometric group actions play a fundamental role in the study of symmetries in physical systems
- The invariance of physical laws under certain groups of transformations (e.g., the Poincaré group in special relativity) is described using isometric group actions
- The study of gauge theories in physics heavily relies on the concept of principal bundles, which are constructed using free and proper isometric group actions on the total space

# Unit 7 – Connections and Parallel Transport

## 7.1 Affine connections

Affine connections are essential tools in Riemannian geometry and general relativity. They enable us to differentiate vector fields, define parallel transport, and construct geodesics on manifolds. These concepts form the foundation for understanding intrinsic geometry and its relation to curvature and torsion.

By providing a coordinate-independent way to compare vectors at different points, affine connections allow us to explore the shape and structure of manifolds. They play a crucial role in various areas of mathematics and physics, from differential geometry to general relativity and beyond.

### Affine connections

- Play a crucial role in the study of Riemannian geometry and general relativity by providing a way to differentiate vector fields and define parallel transport
- Allow for the comparison of vectors at different points on a manifold and the construction of geodesics, which are the "straightest possible" curves on a manifold
- Form the foundation for understanding the intrinsic geometry of a manifold and how it relates to the curvature and torsion of the space

### Motivation for affine connections

- Need for a notion of differentiation on manifolds that is independent of the choice of coordinates
- Desire to compare vectors at different points on a manifold and define parallel transport
- Importance in understanding the intrinsic geometry of a manifold and its relation to curvature and torsion

### Definition of affine connection

- An affine connection on a manifold  $M$  is a map  $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$  satisfying certain properties
- $\nabla_{fX}Y = f\nabla_XY$  for all smooth functions  $f$  and vector fields  $X, Y$
- $\nabla_X(fY) = f\nabla_XY + X(f)Y$  for all smooth functions  $f$  and vector fields  $X, Y$
- $\nabla_X(Y + Z) = \nabla_XY + \nabla_XZ$  for all vector fields  $X, Y, Z$
- Provides a way to differentiate vector fields on a manifold

### Covariant derivative

- The covariant derivative of a vector field  $Y$  with respect to another vector field  $X$  is denoted by  $\nabla_XY$
- Measures the rate of change of  $Y$  along the direction of  $X$
- Satisfies the properties of an affine connection
- Allows for the comparison of vectors at different points on a manifold

## Parallel transport

- Parallel transport is the process of moving a vector along a curve while preserving its "direction" with respect to the affine connection
- A vector field  $Y$  is parallel along a curve  $\gamma$  if  $\nabla_{\dot{\gamma}} Y = 0$ , where  $\dot{\gamma}$  is the tangent vector to the curve
- Parallel transport allows for the comparison of vectors at different points on a manifold
- Depends on the choice of affine connection

## Geodesics

- Geodesics are curves that parallel transport their own tangent vector
- Mathematically, a curve  $\gamma$  is a geodesic if  $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$
- Geodesics are the "straightest possible" curves on a manifold with respect to the affine connection
- Play a crucial role in understanding the geometry of a manifold (shortest paths between points)

## Torsion tensor

- The torsion tensor  $T$  of an affine connection  $\nabla$  is defined as  $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$
- Measures the failure of the connection to be symmetric
- An affine connection is called torsion-free if  $T = 0$
- Torsion plays a role in understanding the geometry of a manifold and its relation to the affine connection

## Curvature tensor

- The curvature tensor  $R$  of an affine connection  $\nabla$  is defined as  $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$
- Measures the failure of parallel transport to be path-independent
- Provides information about the intrinsic curvature of the manifold
- Plays a crucial role in understanding the geometry of a manifold and its relation to the affine connection (Riemann tensor in general relativity)

## Levi-Civita connection

- The Levi-Civita connection is a unique torsion-free affine connection that is compatible with a given Riemannian metric
- Defined by the Koszul formula:  
$$2\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle Z, X \rangle - Z\langle X, Y \rangle - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle$$
- Provides a natural way to differentiate vector fields on a Riemannian manifold
- Used to define geodesics and parallel transport in Riemannian geometry

## Affine connections vs metric connections

- An affine connection is a more general concept than a metric connection
- A metric connection is an affine connection that is compatible with a given metric (Levi-Civita connection)
- Not all affine connections are metric connections, but all metric connections are affine connections
- The choice of affine connection can have significant implications for the geometry of a manifold

## Affine connections in local coordinates

- In local coordinates, an affine connection is characterized by its Christoffel symbols
- The Christoffel symbols  $\Gamma_{ij}^k$  are defined by  $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k\partial_k$
- The Christoffel symbols determine how the basis vectors change as one moves along a curve
- The Christoffel symbols can be used to compute the covariant derivative, parallel transport, and geodesics in local coordinates

## Christoffel symbols

- The Christoffel symbols  $\Gamma_{ij}^k$  are the components of an affine connection in local coordinates
- Symmetric in the lower indices if and only if the connection is torsion-free
- For the Levi-Civita connection, the Christoffel symbols can be expressed in terms of the metric tensor:  
$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$
- Play a crucial role in computing the covariant derivative, parallel transport, and geodesics in local coordinates

## Affine connections on vector bundles

- The concept of an affine connection can be generalized to vector bundles over a manifold
- A connection on a vector bundle  $E$  is a map  $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$  satisfying certain properties
- $\nabla_{fX}S = f\nabla_X S$  for all smooth functions  $f$ , vector fields  $X$ , and sections  $S$  of  $E$
- $\nabla_X(fS) = f\nabla_X S + X(f)S$  for all smooth functions  $f$ , vector fields  $X$ , and sections  $S$  of  $E$
- $\nabla_X(s + t) = \nabla_X s + \nabla_X t$  for all vector fields  $X$  and sections  $s, t$  of  $E$
- Allows for the differentiation of sections of a vector bundle along vector fields on the base manifold

## Affine connections and Riemannian metrics

- On a Riemannian manifold  $(M, g)$ , there is a unique torsion-free affine connection compatible with the metric, called the Levi-Civita connection
- The Levi-Civita connection provides a natural way to differentiate vector fields and define parallel transport on a Riemannian manifold
- The curvature tensor of the Levi-Civita connection is closely related to the Riemann curvature tensor, which encodes information about the intrinsic geometry of the manifold

- The study of affine connections and Riemannian metrics is central to understanding the geometry of Riemannian manifolds

## Affine connections and Lie groups

- On a Lie group  $G$ , there is a natural affine connection called the Maurer-Cartan connection
- The Maurer-Cartan connection is defined using the left-invariant vector fields on  $G$
- The Maurer-Cartan connection is flat (has zero curvature) and torsion-free
- The study of affine connections on Lie groups is important in understanding the geometry of homogeneous spaces and symmetric spaces

## Affine connections in general relativity

- In general relativity, spacetime is modeled as a 4-dimensional Lorentzian manifold  $(M, g)$
- The Levi-Civita connection of the spacetime metric plays a crucial role in the formulation of the theory
- The geodesics of the Levi-Civita connection represent the paths of freely falling particles in the absence of external forces
- The curvature tensor of the Levi-Civita connection, known as the Riemann tensor, encodes information about the gravitational field and the geometry of spacetime
- The study of affine connections in general relativity is essential for understanding the nature of gravity and the structure of spacetime

## 7.2 Levi-Civita connection

The Levi-Civita connection is a key concept in Riemannian geometry, providing a way to define parallel transport and covariant derivatives on curved spaces. It's the unique connection that's both metric-compatible and torsion-free, making it essential for studying a manifold's intrinsic geometry.

This connection allows us to generalize calculus concepts to curved spaces and is crucial for understanding geodesics, curvature, and parallel transport. Its applications span from general relativity to geometric analysis, showcasing its importance in modern mathematics and physics.

### Definition of Levi-Civita connection

- The Levi-Civita connection is a fundamental concept in Riemannian geometry that provides a way to define parallel transport and covariant derivatives on a Riemannian manifold
- It is the unique connection that is both metric-compatible and torsion-free, making it an essential tool for studying the intrinsic geometry of a manifold
- The Levi-Civita connection allows for the generalization of calculus concepts, such as derivatives and integrals, to curved spaces

### Koszul formula for Levi-Civita connection

- The Koszul formula is an explicit expression for the Levi-Civita connection in terms of the metric tensor and its derivatives
- It states that for vector fields  $X$ ,  $Y$ , and  $Z$  on a Riemannian manifold, the Levi-Civita connection  $\nabla$  satisfies:



$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X)$$

- The Koszul formula allows for the computation of the Levi-Civita connection from the metric tensor, making it a practical tool for calculations

## Christoffel symbols of Levi-Civita connection

- Christoffel symbols are the components of the Levi-Civita connection with respect to a chosen coordinate system
- They are denoted by  $\Gamma_{ij}^k$  and are defined in terms of the metric tensor  $g_{ij}$  and its derivatives:  

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$
- Christoffel symbols play a crucial role in expressing the geodesic equation and other geometric quantities in local coordinates

## Uniqueness of Levi-Civita connection

- The Levi-Civita connection is unique among all possible connections on a Riemannian manifold that are both metric-compatible and torsion-free
- This uniqueness theorem ensures that the Levi-Civita connection is a well-defined and intrinsic object associated with the Riemannian metric
- The uniqueness property allows for the consistent definition of geometric concepts, such as parallel transport and curvature, across different coordinate systems

## Properties of Levi-Civita connection

- The Levi-Civita connection possesses several important properties that make it a fundamental tool in Riemannian geometry
- These properties include metric compatibility, symmetry, and the absence of torsion, which together characterize the Levi-Civita connection uniquely
- Understanding these properties is crucial for studying the geometry of Riemannian manifolds and their applications in various fields, such as general relativity and geometric analysis

## Metric compatibility of Levi-Civita connection

- The Levi-Civita connection is metric-compatible, meaning that the covariant derivative of the metric tensor with respect to the connection vanishes
- Formally, for any vector fields  $X$  and  $Y$ ,  $\nabla_X g(Y, Y) = 2g(\nabla_X Y, Y)$
- Metric compatibility ensures that parallel transport preserves the inner product between vectors, a crucial property for defining geometric concepts consistently

## Symmetry of Levi-Civita connection

- The Christoffel symbols of the Levi-Civita connection are symmetric in their lower indices, i.e.,  $\Gamma_{ij}^k = \Gamma_{ji}^k$
- This symmetry property is a consequence of the metric compatibility and the absence of torsion
- The symmetry of the Christoffel symbols simplifies many calculations and is essential for deriving various geometric identities

## Torsion-free nature of Levi-Civita connection

- The Levi-Civita connection is torsion-free, meaning that the torsion tensor associated with the connection vanishes identically
- The torsion tensor measures the failure of the parallelogram law for infinitesimal parallelograms and is defined as  $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$
- The absence of torsion is a key property that distinguishes the Levi-Civita connection from other possible connections on a Riemannian manifold

## Geodesics and Levi-Civita connection

- Geodesics are curves that generalize the concept of straight lines to curved spaces and are closely related to the Levi-Civita connection
- The Levi-Civita connection provides a natural way to define parallel transport along a curve, which is essential for studying geodesics and their properties
- Geodesics play a fundamental role in understanding the geometry of Riemannian manifolds and have important applications in various fields, such as general relativity and calculus of variations

## Geodesic equation using Levi-Civita connection

- The geodesic equation is a second-order differential equation that characterizes geodesics on a Riemannian manifold using the Levi-Civita connection
- In local coordinates, the geodesic equation is given by:  $\ddot{x}^k + \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0$  where  $x^k(t)$  are the coordinates of the geodesic curve and  $\dot{x}^k = \frac{dx^k}{dt}$
- The geodesic equation expresses the fact that the acceleration of a geodesic curve vanishes when measured with respect to the Levi-Civita connection

## Parallel transport using Levi-Civita connection

- Parallel transport is a way to move vectors along a curve while preserving their direction and magnitude, as defined by the Levi-Civita connection
- A vector field  $V(t)$  along a curve  $\gamma(t)$  is said to be parallel if its covariant derivative along the curve vanishes, i.e.,  $\nabla_{\dot{\gamma}} V = 0$
- Parallel transport is path-dependent in general, meaning that the result of transporting a vector along different curves between two points may differ

## Exponential map and Levi-Civita connection

- The exponential map is a function that maps tangent vectors to points on the manifold using the Levi-Civita connection and geodesics
- For a point  $p$  on the manifold and a tangent vector  $v \in T_p M$ , the exponential map is defined as  $\exp_p(v) = \gamma_v(1)$ , where  $\gamma_v(t)$  is the unique geodesic starting at  $p$  with initial velocity  $v$
- The exponential map provides a local diffeomorphism between a neighborhood of the origin in the tangent space and a neighborhood of the point on the manifold, allowing for the study of the local geometry

## Curvature and Levi-Civita connection

- Curvature is a fundamental concept in Riemannian geometry that measures the deviation of a manifold from being flat, and it is closely related to the Levi-Civita connection
- The Levi-Civita connection allows for the definition of various curvature tensors, such as the Riemann curvature tensor, Ricci curvature, and scalar curvature
- Studying curvature is essential for understanding the intrinsic geometry of a Riemannian manifold and has important applications in fields such as general relativity and geometric analysis

## Riemann curvature tensor and Levi-Civita connection

- The Riemann curvature tensor is a (1,3)-tensor that measures the curvature of a Riemannian manifold and is defined using the Levi-Civita connection
- In terms of the Levi-Civita connection, the Riemann curvature tensor is given by:  
$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$
 for vector fields  $X$ ,  $Y$ , and  $Z$
- The Riemann curvature tensor satisfies several symmetry properties and can be used to derive other curvature quantities, such as the Ricci curvature and scalar curvature

## Sectional curvature and Levi-Civita connection

- Sectional curvature is a scalar-valued function that assigns a real number to each 2-dimensional subspace (called a section) of the tangent space at a point on the manifold
- For a section spanned by orthonormal vectors  $X$  and  $Y$ , the sectional curvature is given by:  
$$K(X, Y) = \frac{R(X, Y, X, Y)}{|X \wedge Y|^2} = R(X, Y, X, Y)$$
- Sectional curvature provides a way to measure the curvature of a manifold in a given direction and is related to the behavior of geodesics and the growth of the volume of geodesic balls

## Ricci curvature and Levi-Civita connection

- Ricci curvature is a (0,2)-tensor obtained by contracting the Riemann curvature tensor
- In local coordinates, the Ricci curvature is given by:  $R_{ij} = R^k_{ikj} = g^{kl} R_{klj}$
- Ricci curvature plays a crucial role in the study of Einstein manifolds, which are Riemannian manifolds with constant Ricci curvature, and in the formulation of the Einstein field equations in general relativity

## Scalar curvature and Levi-Civita connection

- Scalar curvature is a scalar function on a Riemannian manifold obtained by further contracting the Ricci curvature tensor
- In local coordinates, the scalar curvature is given by:  $R = g^{ij} R_{ij}$
- Scalar curvature appears in the Hilbert-Einstein action in general relativity and plays a role in the study of the topology and geometry of manifolds through the Yamabe problem and other geometric variational problems

## Applications of Levi-Civita connection

- The Levi-Civita connection has numerous applications in various fields of mathematics and physics, where the geometry of curved spaces plays a fundamental role
- Some of the most prominent applications include general relativity, Riemannian geometry, and geometric analysis
- Understanding the Levi-Civita connection and its properties is essential for working with these applications and exploring the rich interplay between geometry and other areas of mathematics and physics

## Levi-Civita connection in general relativity

- In general relativity, spacetime is modeled as a 4-dimensional Lorentzian manifold, and the Levi-Civita connection is used to describe the geometry of spacetime
- The Einstein field equations, which relate the curvature of spacetime to the presence of matter and energy, are formulated using the Ricci curvature tensor and scalar curvature derived from the Levi-Civita connection
- The geodesic equation, expressed using the Levi-Civita connection, describes the motion of free-falling particles and the paths of light rays in curved spacetime

## Levi-Civita connection in Riemannian geometry

- Riemannian geometry is the study of Riemannian manifolds, which are smooth manifolds equipped with a positive-definite metric tensor
- The Levi-Civita connection is the fundamental object that allows for the definition of parallel transport, curvature, and other geometric concepts on Riemannian manifolds
- Many important results in Riemannian geometry, such as the Hopf-Rinow theorem, the Bonnet-Myers theorem, and the Cartan-Hadamard theorem, rely on the properties of the Levi-Civita connection

## Levi-Civita connection in geometric analysis

- Geometric analysis is a field that combines techniques from analysis, partial differential equations, and differential geometry to study geometric problems
- The Levi-Civita connection plays a crucial role in formulating and solving various geometric PDEs, such as the Laplace-Beltrami operator, the heat equation, and the Ricci flow
- In the study of harmonic maps between Riemannian manifolds, the Levi-Civita connection is used to define the tension field and the Euler-Lagrange equations that characterize critical points of the energy functional

## 7.3 Parallel transport

Parallel transport is a key concept in differential geometry that allows us to compare vectors at different points on a manifold. It's essential for understanding curvature and geometry, playing a crucial role in various mathematical and physical applications.

This topic explores parallel vector fields, covariant derivatives, and the Levi-Civita connection. We'll also dive into geodesics, holonomy groups, and parallel transport in fiber bundles, connecting these ideas to the broader study of manifold geometry.

## Parallel transport

- Fundamental concept in differential geometry that describes how to transport vectors along curves on a manifold while preserving certain geometric properties
- Allows for the comparison of vectors at different points on a manifold by defining a notion of parallelism
- Plays a crucial role in understanding the geometry and curvature of manifolds and has applications in various areas of mathematics and physics

## Parallel vector fields

- Vector fields on a manifold whose vectors remain parallel when transported along any curve
- Characterized by the property that their covariant derivative vanishes identically
- Existence of parallel vector fields is closely related to the curvature of the manifold (flat manifolds always admit parallel vector fields)

## Covariant derivative

- Generalization of the directional derivative that takes into account the geometry of the manifold
- Measures the rate of change of a vector field along a curve while accounting for the curvature of the manifold
- Fundamental tool in defining parallel transport and studying the geometry of manifolds

## Levi-Civita connection

- Unique torsion-free metric connection on a Riemannian manifold
- Preserves the metric tensor under parallel transport, ensuring that the inner product of vectors remains constant
- Allows for the definition of geodesics as curves whose tangent vectors remain parallel along the curve

## Geodesics and parallel transport

- Geodesics are curves on a manifold that minimize the distance between two points and represent the straightest possible paths
- Tangent vectors to a geodesic remain parallel when transported along the geodesic using the Levi-Civita connection
- Parallel transport along geodesics preserves the angle between vectors and the length of vectors

## Parallel transport and curvature

- Curvature of a manifold can be measured by the failure of parallel transport to return a vector to its initial state after being transported along a closed loop
- Flat manifolds have zero curvature and parallel transport along any closed loop returns vectors to their initial state

- Non-zero curvature leads to holonomy, where parallel transport along closed loops results in a rotation of vectors

## Holonomy groups

- Group of linear transformations that describe the effect of parallel transport along closed loops based at a point on a manifold
- Measure the curvature and geometric properties of the manifold
- Trivial holonomy group (consisting only of the identity transformation) indicates a flat manifold, while non-trivial holonomy groups correspond to curved manifolds

## Parallel transport in fiber bundles

- Fiber bundles are geometric structures that consist of a base space and a fiber space attached to each point of the base space
- Parallel transport in fiber bundles allows for the transport of elements from one fiber to another along curves in the base space
- Requires the definition of a connection on the fiber bundle, which specifies how to lift curves from the base space to the total space

## Ehresmann connection

- Connection on a fiber bundle that provides a way to lift curves from the base space to the total space
- Determines how to transport elements of the fiber along curves in the base space
- Fundamental concept in the study of parallel transport in fiber bundles and the geometry of fiber bundles

## Parallel sections of vector bundles

- Vector bundles are fiber bundles whose fibers are vector spaces
- Parallel sections of a vector bundle are sections (i.e., vector fields) that remain parallel when transported along curves in the base space using the connection
- Existence and properties of parallel sections provide information about the geometry and topology of the vector bundle

## Parallel transport and Riemannian submersions

- Riemannian submersions are maps between Riemannian manifolds that preserve certain geometric properties
- Parallel transport commutes with Riemannian submersions, meaning that transporting a vector along a curve in the domain manifold and then mapping it to the codomain manifold yields the same result as first mapping the vector and then transporting it along the corresponding curve in the codomain manifold
- Allows for the study of the relationship between the geometry of the domain and codomain manifolds

## Parallel transport and isometries

- Isometries are maps between Riemannian manifolds that preserve the metric tensor
- Parallel transport commutes with isometries, meaning that transporting a vector along a curve and then applying an isometry yields the same result as first applying the isometry and then transporting the resulting vector along the corresponding curve
- Reflects the fact that isometries preserve the geometric properties of the manifold, including the notion of parallelism

## Schild's ladder

- Approximation method for parallel transport on curved manifolds
- Constructs a sequence of geodesic segments that approximate the parallel transport of a vector along a curve
- Provides a way to compute parallel transport numerically when an explicit formula is not available

## Parallel transport on Lie groups

- Lie groups are smooth manifolds that also have a group structure compatible with the manifold structure
- Parallel transport on Lie groups can be studied using the left-invariant or right-invariant vector fields and the Maurer-Cartan form
- Properties of parallel transport reflect the geometric and algebraic structure of the Lie group

## Left-invariant vs right-invariant vector fields

- Left-invariant vector fields on a Lie group are vector fields that are invariant under left multiplication by group elements
- Right-invariant vector fields are invariant under right multiplication by group elements
- Parallel transport of left-invariant (resp. right-invariant) vector fields along curves on the Lie group yields left-invariant (resp. right-invariant) vector fields

## Maurer-Cartan form

- Canonical one-form on a Lie group that takes values in the Lie algebra of the group
- Encodes the infinitesimal properties of the Lie group and its left-invariant (or right-invariant) vector fields
- Plays a key role in studying parallel transport and the geometry of Lie groups

## Parallel transport in Euclidean space

- In Euclidean space, parallel transport is the usual notion of translation of vectors along straight lines
- Parallel transported vectors maintain their direction and magnitude
- Reflects the flat geometry of Euclidean space, where the curvature is identically zero

## Parallel transport on spheres

- Spheres are examples of curved manifolds with non-zero curvature
- Parallel transport on spheres can be visualized by moving a tangent vector along a great circle while keeping it tangent to the sphere
- Parallel transport along closed loops on spheres leads to holonomy, where the initial and final vectors differ by a rotation

## Parallel transport and development of curves

- Development of a curve on a manifold onto a tangent space is a way to study the properties of the curve using the flat geometry of the tangent space
- Parallel transport along a curve on the manifold corresponds to drawing a straight line in the tangent space under the development map
- Allows for the analysis of curves on curved manifolds using tools from Euclidean geometry

## Parallel transport and Jacobi fields

- Jacobi fields are vector fields along a geodesic that describe the behavior of nearby geodesics
- Jacobi fields satisfy a second-order differential equation that involves the curvature of the manifold
- Parallel transport of Jacobi fields along a geodesic yields information about the stability and focusing properties of the geodesic

## Parallel transport and Fermi-Walker transport

- Fermi-Walker transport is a generalization of parallel transport that is defined along non-geodesic curves
- Preserves the orthogonality of vectors to the tangent vector of the curve, but not necessarily their length
- Useful in the study of relativistic physics and the motion of particles in curved spacetimes

## 7.4 Holonomy groups

Holonomy groups are a key concept in differential geometry, describing how vectors change when parallel transported around loops on a manifold. They provide crucial information about a manifold's geometric structure and curvature, helping us understand connections and overall geometry.

These groups have important properties reflecting the manifold's structure. They're Lie subgroups related to curvature, classified by Berger's theorem. Special holonomy groups, like those of Calabi-Yau manifolds, are particularly interesting in physics and geometry research.

### Definition of holonomy groups

- Holonomy groups are a fundamental concept in differential geometry that describe the behavior of parallel transport around loops on a manifold
- They capture important information about the geometric structure and curvature of the manifold



- Understanding holonomy groups is crucial for studying the properties of connections and the geometry of manifolds in metric differential geometry

## Parallel transport and holonomy

- Parallel transport is a way of moving vectors along curves on a manifold while preserving their direction relative to the connection
- Holonomy refers to the change in a vector after parallel transporting it around a closed loop
- The holonomy of a loop depends on the curvature of the manifold and the choice of connection
- Example: On a sphere, parallel transporting a vector around a closed loop results in a rotation of the vector (parallel transport along a great circle)

## Holonomy group of a connection

- The holonomy group of a connection is the group of all possible parallel transports along closed loops starting and ending at a fixed point
- It is a subgroup of the general linear group  $GL(n, \mathbb{R})$  or  $GL(n, \mathbb{C})$ , depending on the type of manifold and connection
- The holonomy group captures the global behavior of the connection and its curvature
- Example: The holonomy group of the Levi-Civita connection on a Riemannian manifold is a subgroup of the orthogonal group  $O(n)$

## Restricted holonomy group

- The restricted holonomy group is a subgroup of the holonomy group obtained by considering only contractible loops
- It is independent of the base point and is a connected Lie subgroup of the holonomy group
- The restricted holonomy group often has a simpler structure and is more closely related to the local geometry of the manifold
- Example: For a simply connected manifold, the restricted holonomy group coincides with the full holonomy group

## Properties of holonomy groups

- Holonomy groups have several important properties that reflect the geometric structure of the manifold and the behavior of the connection
- Studying these properties helps to classify and understand the geometry of different types of manifolds in metric differential geometry

## Lie group structure

- Holonomy groups are Lie subgroups of the general linear group  $GL(n, \mathbb{R})$  or  $GL(n, \mathbb{C})$
- The Lie group structure of holonomy groups allows for the use of powerful tools from Lie theory in their study

- The dimension and structure of the holonomy group provide information about the geometry of the manifold
- Example: The holonomy group of a Riemannian manifold is a compact Lie subgroup of the orthogonal group  $O(n)$

## Relationship to curvature

- The holonomy group is closely related to the curvature of the connection and the manifold
- The Lie algebra of the holonomy group is generated by the curvature tensor evaluated at a point
- Vanishing of certain components of the curvature tensor can lead to a reduction of the holonomy group
- Example: A Riemannian manifold with constant sectional curvature has holonomy group  $SO(n)$  or a subgroup thereof

## Ambrose-Singer theorem

- The Ambrose-Singer theorem relates the Lie algebra of the holonomy group to the curvature tensor
- It states that the Lie algebra of the holonomy group is generated by the curvature tensor and its covariant derivatives evaluated at a point
- This theorem provides a way to compute the holonomy group from local curvature information
- Example: For a Riemannian manifold, the Lie algebra of the holonomy group is spanned by the curvature tensor and its covariant derivatives at any point

## Berger's classification

- Berger's classification is a complete list of possible restricted holonomy groups for irreducible, non-symmetric, simply connected Riemannian manifolds
- The list includes the groups  $SO(n)$ ,  $U(n)$ ,  $SU(n)$ ,  $Sp(n)$ ,  $Sp(n)Sp(1)$ ,  $G_2$ , and  $Spin(7)$
- Each group in the list corresponds to a specific type of geometric structure on the manifold
- Example: A manifold with  $SU(n)$  holonomy is a Calabi-Yau manifold, which has a complex structure and a Ricci-flat Kähler metric

## Special holonomy groups

- Special holonomy groups are those that appear in Berger's classification and correspond to manifolds with exceptional geometric structures
- Studying manifolds with special holonomy is an active area of research in metric differential geometry and has applications in physics and string theory

## Riemannian vs Lorentzian manifolds

- Holonomy groups can be defined for both Riemannian and Lorentzian manifolds
- Riemannian manifolds have a positive-definite metric, while Lorentzian manifolds have an indefinite metric with signature  $(-, +, \dots, +)$

- The holonomy groups of Lorentzian manifolds are subgroups of the indefinite orthogonal group  $O(1, n - 1)$
- Example: The holonomy group of a space-time in general relativity is a subgroup of the Lorentz group  $O(1, 3)$

## Calabi-Yau manifolds and $SU(n)$ holonomy

- Calabi-Yau manifolds are complex manifolds with a Ricci-flat Kähler metric and  $SU(n)$  holonomy
- They are of interest in string theory as compactification spaces for extra dimensions
- The special holonomy  $SU(n)$  implies the existence of a parallel spinor and a covariantly constant complex volume form
- Example: The quintic hypersurface in  $CP^4$  is a famous example of a Calabi-Yau threefold with  $SU(3)$  holonomy

## $G_2$ and $Spin(7)$ holonomy

- $G_2$  and  $Spin(7)$  are exceptional Lie groups that appear as holonomy groups of 7- and 8-dimensional Riemannian manifolds, respectively
- Manifolds with these holonomy groups have exceptional geometric structures and are of interest in physics and string theory
- $G_2$  manifolds have a parallel 3-form and a covariantly constant spinor, while  $Spin(7)$  manifolds have a parallel 4-form
- Example: The 7-dimensional round sphere  $S^7$  has  $Spin(7)$  holonomy when equipped with a non-standard homogeneous metric

## Quaternionic Kähler manifolds and $Sp(n)Sp(1)$ holonomy

- Quaternionic Kähler manifolds are  $4n$ -dimensional Riemannian manifolds with holonomy group contained in  $Sp(n)Sp(1)$
- They have a parallel 4-form and a quaternionic structure compatible with the metric
- Quaternionic Kähler manifolds are Einstein manifolds but not necessarily Ricci-flat
- Example: The quaternionic projective space  $HP^n$  has  $Sp(n)Sp(1)$  holonomy and is a symmetric space

## Applications of holonomy groups

- Holonomy groups have numerous applications in differential geometry, physics, and related fields
- They provide a way to classify and study manifolds with special geometric structures and symmetries

## Holonomy and Einstein manifolds

- Einstein manifolds are Riemannian or Lorentzian manifolds whose Ricci tensor is proportional to the metric tensor
- The holonomy group of an Einstein manifold is closely related to its geometry and topology
- Manifolds with special holonomy, such as Calabi-Yau or  $G_2$  manifolds, are often Einstein manifolds

- Example: The 4-dimensional round sphere  $S^4$  is an Einstein manifold with constant positive curvature and  $SO(4)$  holonomy

## Holonomy and special geometries

- Holonomy groups are used to define and classify various types of special geometries on manifolds
- Examples include Kähler geometry, hyperKähler geometry, and quaternionic Kähler geometry
- The holonomy group determines the parallel tensors and spinors on the manifold, which define the special geometric structure
- Example: A Kähler manifold has holonomy contained in  $U(n)$  and admits a parallel complex structure and symplectic form

## Holonomy in physics and string theory

- Holonomy groups play a significant role in theoretical physics, particularly in string theory and M-theory
- Manifolds with special holonomy, such as Calabi-Yau or  $G_2$  manifolds, are used as compactification spaces for extra dimensions
- The holonomy group determines the amount of supersymmetry preserved by the compactification
- Example: In M-theory, compactification on a  $G_2$  manifold leads to a 4-dimensional theory with  $\mathcal{N} = 1$  supersymmetry

## Holonomy and geometric structures

- Holonomy groups are closely related to the existence of various geometric structures on manifolds
- Examples include complex structures, symplectic structures, and calibrations
- The holonomy group determines which parallel tensors and spinors exist on the manifold, which in turn define the geometric structures
- Example: A manifold with  $SU(3)$  holonomy admits a parallel complex 3-form, which is a calibration for special Lagrangian submanifolds

## Computations and examples

- Computing holonomy groups and understanding their implications for specific manifolds is an important aspect of metric differential geometry
- Various techniques and tools are used to calculate holonomy groups and study their properties

## Holonomy of symmetric spaces

- Symmetric spaces are manifolds with a high degree of symmetry, characterized by the existence of an involutive isometry at each point
- The holonomy group of a symmetric space is closely related to its isometry group and can often be determined explicitly
- Examples of symmetric spaces include Euclidean spaces, spheres, hyperbolic spaces, and Grassmannians

- Example: The complex projective space  $\mathbb{CP}^n$  is a symmetric space with holonomy  $U(n)$

## Holonomy of Kähler manifolds

- Kähler manifolds are complex manifolds with a compatible Riemannian metric and a closed, parallel symplectic form
- The holonomy group of a Kähler manifold is contained in the unitary group  $U(n)$
- The Levi-Civita connection of a Kähler manifold can be decomposed into its complex linear and antilinear parts, which simplifies the computation of the holonomy group
- Example: The complex Grassmannian  $Gr(k, \mathbb{C}^n)$  is a Kähler manifold with holonomy  $U(k) \times U(n - k)$

## Holonomy of homogeneous spaces

- Homogeneous spaces are manifolds that admit a transitive action by a Lie group
- The holonomy group of a homogeneous space can often be determined from the Lie algebra of the symmetry group and its isotropy representation
- Many examples of manifolds with special holonomy, such as symmetric spaces and 3-Sasakian manifolds, are homogeneous spaces
- Example: The 7-dimensional Aloff-Wallach space  $SU(3)/U(1)$  has  $G_2$  holonomy for a specific embedding of  $U(1)$  in  $SU(3)$

## Explicit computations for specific manifolds

- In some cases, the holonomy group of a specific manifold can be computed explicitly using various techniques from differential geometry
- These computations often involve analyzing the curvature tensor, solving the parallel transport equations, or studying the symmetries of the manifold
- Explicit examples help to illustrate the properties of holonomy groups and their relationship to the geometry of the manifold
- Example: The holonomy group of the Eguchi-Hanson metric, a self-dual Ricci-flat metric on the cotangent bundle of  $\mathbb{CP}^1$ , can be shown to be  $SU(2)$  by explicitly solving the parallel transport equations

## 7.5 Geodesic deviation and Jacobi fields

Geodesic deviation and Jacobi fields are key concepts in understanding the behavior of geodesics in curved spaces. They provide insights into how nearby geodesics diverge or converge, revealing the underlying geometry of the manifold.

These concepts are crucial in general relativity, where they describe the relative motion of freely falling particles in spacetime. By analyzing geodesic deviation and Jacobi fields, we can better grasp the effects of curvature on the paths of light and matter.

### Definition of geodesic deviation

- Geodesic deviation describes how nearby geodesics in a curved space diverge or converge relative to each other

- Provides a way to measure the curvature of a Riemannian manifold by analyzing the behavior of geodesics
- Plays a crucial role in understanding the geometry of spacetime in general relativity and its physical implications

## Geodesic variation vector fields

- A geodesic variation is a smooth one-parameter family of geodesics  $\gamma_s(t)$  with  $\gamma_0(t)$  being the reference geodesic
- The variation vector field  $J(t) = \left. \frac{\partial}{\partial s} \right|_{s=0} \gamma_s(t)$  measures the infinitesimal displacement between nearby geodesics
- $J(t)$  is a vector field along the reference geodesic  $\gamma_0(t)$  and satisfies the Jacobi equation

## Geodesic deviation equation

- The geodesic deviation equation describes the evolution of the variation vector field  $J(t)$  along the reference geodesic
- It is given by  $\frac{D^2}{dt^2} J(t) + R(J(t), \dot{\gamma}_0(t)) \dot{\gamma}_0(t) = 0$ , where  $\frac{D}{dt}$  is the covariant derivative and  $R$  is the Riemann curvature tensor
- The equation relates the acceleration of the deviation vector to the curvature of the manifold

## Physical interpretation of geodesic deviation

- In general relativity, geodesic deviation describes the relative acceleration of freely falling test particles due to spacetime curvature
- The equation shows how the presence of matter and energy (encoded in the Riemann tensor) causes the trajectories of nearby particles to converge or diverge
- Geodesic deviation is a manifestation of tidal forces in curved spacetime (e.g., the stretching and squeezing of objects in a gravitational field)

## Jacobi fields

- Jacobi fields are vector fields along a geodesic that satisfy the Jacobi equation, a generalization of the geodesic deviation equation
- They describe the behavior of infinitesimal geodesic variations and encode information about the curvature of the manifold
- Jacobi fields play a central role in the study of geodesic focusing, conjugate points, and the stability of geodesics

## Definition and properties of Jacobi fields

- A Jacobi field  $J(t)$  along a geodesic  $\gamma(t)$  is a vector field that satisfies the Jacobi equation  $\frac{D^2}{dt^2} J(t) + R(J(t), \dot{\gamma}(t)) \dot{\gamma}(t) = 0$
- Jacobi fields are uniquely determined by their initial conditions  $J(0)$  and  $\frac{D}{dt} J(0)$  at a point on the geodesic

- The set of all Jacobi fields along a geodesic forms a vector space of dimension  $2n$ , where  $n$  is the dimension of the manifold

## Jacobi equation

- The Jacobi equation is a second-order linear differential equation that describes the evolution of Jacobi fields along a geodesic
- It generalizes the geodesic deviation equation by allowing for arbitrary initial conditions for the Jacobi field and its derivative
- The Jacobi equation encodes information about the stability of geodesics and the occurrence of conjugate points

## Solutions to the Jacobi equation

- The solutions to the Jacobi equation form a  $2n$ -dimensional vector space, where  $n$  is the dimension of the manifold
- The solution space can be decomposed into two  $n$ -dimensional subspaces: the space of Jacobi fields that vanish at the initial point (satisfying  $J(0) = 0$ ) and the space of Jacobi fields that are orthogonal to the geodesic at the initial point (satisfying  $\langle J(0), \dot{\gamma}(0) \rangle = 0$ )
- The behavior of Jacobi fields can be analyzed using techniques from linear algebra and differential equations, such as the matrix Riccati equation and the Raychaudhuri equation

## Conjugate points and Jacobi fields

- A conjugate point along a geodesic  $\gamma(t)$  is a point  $\gamma(t_1)$  where a non-zero Jacobi field  $J(t)$  vanishes, i.e.,  $J(t_1) = 0$
- The existence of conjugate points is related to the stability of geodesics and the focusing of nearby geodesics
- Conjugate points play a crucial role in Morse theory and the singularity theorems of general relativity, as they indicate the presence of caustics or focal points in the geodesic flow

## Relationship between geodesic deviation and Jacobi fields

- Geodesic deviation and Jacobi fields are closely related concepts in the study of geodesics on Riemannian manifolds
- Jacobi fields provide a generalization of the geodesic deviation equation, allowing for more general initial conditions and a deeper analysis of geodesic behavior
- The relationship between these concepts is crucial for understanding the geometry of curved spaces and their physical implications

## Jacobi fields as solutions to geodesic deviation equation

- The geodesic deviation equation can be seen as a special case of the Jacobi equation, where the initial conditions for the Jacobi field are given by the variation vector field of a geodesic variation
- Every solution to the geodesic deviation equation is a Jacobi field, but not every Jacobi field arises from a geodesic variation

- The study of Jacobi fields provides a more general framework for analyzing the behavior of geodesics and their deviations

## Geodesic deviation in terms of Jacobi fields

- The geodesic deviation vector field  $J(t)$  can be expressed as a linear combination of Jacobi fields along the reference geodesic
- The coefficients in this linear combination are determined by the initial conditions of the geodesic variation and the corresponding Jacobi fields
- This relationship allows for the study of geodesic deviation using the powerful tools and techniques developed for Jacobi fields, such as the analysis of conjugate points and the Raychaudhuri equation

## Applications of geodesic deviation and Jacobi fields

- Geodesic deviation and Jacobi fields have numerous applications in differential geometry, general relativity, and other areas of mathematics and physics
- These concepts provide a framework for understanding the behavior of geodesics, the focusing of light rays, and the occurrence of singularities in curved spaces
- The study of geodesic deviation and Jacobi fields has led to important results in Riemannian geometry, cosmology, and the theory of black holes

## Geodesic focusing and defocusing

- Geodesic focusing occurs when nearby geodesics converge towards each other, leading to the formation of caustics or focal points
- Geodesic defocusing, on the other hand, describes the divergence of nearby geodesics and the spreading of geodesic congruences
- The behavior of Jacobi fields and the Raychaudhuri equation provide a mathematical framework for analyzing geodesic focusing and defocusing in curved spaces

## Raychaudhuri equation

- The Raychaudhuri equation is a differential equation that describes the evolution of the expansion, shear, and rotation of a geodesic congruence
- It relates the rate of change of these quantities to the Ricci curvature of the manifold and the energy-momentum tensor of the matter content
- The Raychaudhuri equation plays a crucial role in the singularity theorems of general relativity and the study of gravitational collapse

## Singularity theorems in general relativity

- The singularity theorems, developed by Penrose and Hawking, are a set of results that establish the inevitability of singularities in spacetimes satisfying certain conditions
- These theorems rely on the concepts of geodesic incompleteness, trapped surfaces, and the focusing of geodesic congruences
- Jacobi fields and the Raychaudhuri equation are essential tools in the proof of the singularity theorems, as they provide a way to analyze the focusing of geodesics and the occurrence of conjugate



points

## Morse theory and conjugate points

- Morse theory is a branch of differential topology that studies the relationship between the topology of a manifold and the critical points of smooth functions defined on it
- In the context of geodesics, Morse theory relates the topology of the space of geodesics to the conjugate points along them
- The index of a conjugate point, defined as the number of independent Jacobi fields that vanish at that point, provides information about the topology of the space of geodesics and the global structure of the manifold

## Computational aspects

- The study of geodesic deviation and Jacobi fields often involves computational methods for solving the Jacobi equation, finding conjugate points, and analyzing the behavior of geodesic congruences
- Numerical techniques and software packages have been developed to facilitate the computation and visualization of these geometric objects
- Computational methods play a crucial role in applying the theory of geodesic deviation and Jacobi fields to practical problems in physics, engineering, and computer graphics

## Numerical methods for solving Jacobi equation

- Various numerical schemes have been proposed for solving the Jacobi equation, including finite difference methods, Runge-Kutta methods, and symplectic integrators
- These methods discretize the Jacobi equation and approximate its solutions using numerical techniques
- The choice of numerical method depends on factors such as the desired accuracy, stability, and computational efficiency

## Algorithms for finding conjugate points

- Several algorithms have been developed for detecting and locating conjugate points along geodesics
- These algorithms often rely on the properties of Jacobi fields and the behavior of the geodesic flow
- Examples include the Jacobi field method, the Riccati equation method, and the Morse index theorem approach

## Software packages for geodesic deviation and Jacobi fields

- There are several software packages and libraries that provide tools for computing geodesics, Jacobi fields, and related geometric quantities
- These packages often integrate with popular scientific computing environments such as MATLAB, Python, and C++
- Examples include the Geometer's Toolkit (GTK), the Riemannian Geometry & Geometric Mechanics (RGGM) library, and the Geodesic Deviation Equation Solver (GDES) package

## 7.6 Riemannian submersions

Riemannian submersions connect two Riemannian manifolds through a smooth map that preserves certain geometric properties. They allow us to study relationships between metrics and curvatures of different manifolds, providing insights into their structure.

These maps split the tangent space of the domain into vertical and horizontal spaces, with the submersion acting as an isometry on the horizontal space. This framework enables us to analyze complex geometric relationships and construct new manifolds with specific properties.

### Definition of Riemannian submersions

- Riemannian submersions are a fundamental concept in Riemannian geometry that relate the geometry of two Riemannian manifolds via a smooth map
- They provide a way to study the relationship between the metrics and curvatures of the domain and codomain manifolds

### Smooth surjective maps

- A Riemannian submersion is a smooth map  $\pi : (M, g_M) \rightarrow (N, g_N)$  between Riemannian manifolds that is surjective
- The map  $\pi$  is required to be a submersion, meaning its differential  $d\pi$  is surjective at every point
- Surjectivity ensures that every point in the codomain  $N$  is the image of some point in the domain  $M$

### Riemannian metrics on domain and codomain

- The domain manifold  $M$  is equipped with a Riemannian metric  $g_M$ , which defines an inner product on each tangent space
- Similarly, the codomain manifold  $N$  has a Riemannian metric  $g_N$
- The metrics allow for the measurement of lengths, angles, and volumes on the respective manifolds

### Isometric on horizontal space

- At each point  $p \in M$ , the tangent space  $T_p M$  splits into the vertical space  $\mathcal{V}_p$  and the horizontal space  $\mathcal{H}_p$
- The Riemannian submersion  $\pi$  is required to be an isometry when restricted to the horizontal space
- This means that for any horizontal vectors  $X, Y \in \mathcal{H}_p$ , we have  $g_M(X, Y) = g_N(d\pi(X), d\pi(Y))$

### Horizontal and vertical distributions

- The vertical distribution  $\mathcal{V}$  consists of the kernel of  $d\pi$  at each point, representing the tangent spaces to the fibers of  $\pi$
- The horizontal distribution  $\mathcal{H}$  is the orthogonal complement of  $\mathcal{V}$  with respect to the metric  $g_M$
- The distributions  $\mathcal{V}$  and  $\mathcal{H}$  are smooth and provide a decomposition of the tangent bundle  $TM$

## Properties of Riemannian submersions

- Riemannian submersions exhibit several important properties that relate the geometry of the domain and codomain manifolds
- These properties involve the rank and nullity of the submersion, curvature relations, and the geometry of the fibers

## Rank and nullity

- The rank of a Riemannian submersion  $\pi$  at a point  $p \in M$  is the dimension of the horizontal space  $\mathcal{H}_p$
- The nullity of  $\pi$  at  $p$  is the dimension of the vertical space  $\mathcal{V}_p$ , which equals the dimension of the fiber  $\pi^{-1}(\pi(p))$
- The rank and nullity satisfy the relation  $\text{rank}(\pi) + \text{nullity}(\pi) = \dim(M)$

## O'Neill's tensors

- O'Neill's tensors, denoted by  $A$  and  $T$ , measure the integrability of the horizontal and vertical distributions
- The tensor  $A$  is the obstruction to the integrability of the horizontal distribution  $\mathcal{H}$
- The tensor  $T$  is the second fundamental form of the fibers and measures their extrinsic curvature

## Fundamental equations

- The fundamental equations relate the curvatures of the domain, codomain, and fibers of a Riemannian submersion
- The O'Neill formulas express the sectional curvatures of  $M$  and  $N$  in terms of the  $A$  and  $T$  tensors
- These equations provide insights into how the geometry of the domain and codomain are related

## Geometry of fibers

- The fibers of a Riemannian submersion  $\pi$ , given by  $\pi^{-1}(y)$  for  $y \in N$ , are submanifolds of  $M$
- Each fiber inherits a Riemannian metric from the restriction of  $\mathcal{G}_M$  to the vertical space
- The geometry of the fibers, such as their curvature and topology, plays a crucial role in understanding the overall structure of the submersion

## Totally geodesic fibers

- A fiber of a Riemannian submersion is called totally geodesic if every geodesic in the fiber is also a geodesic in the total space  $M$
- Totally geodesic fibers have vanishing second fundamental form, meaning  $T \equiv 0$
- Riemannian submersions with totally geodesic fibers have particularly nice properties and are often studied in geometry and physics

## Riemannian submersions vs Riemannian isometries

- Riemannian submersions and Riemannian isometries are both important maps between Riemannian manifolds
- While they share some similarities, there are key differences between the two concepts

### Similarities in preserving metrics

- Both Riemannian submersions and isometries preserve certain metric properties
- Riemannian submersions are isometric on the horizontal space, meaning they preserve the length of horizontal curves
- Riemannian isometries preserve the Riemannian metric on the entire tangent space, not just the horizontal space

### Differences in surjectivity and injectivity

- Riemannian submersions are surjective maps, ensuring that every point in the codomain is the image of some point in the domain
- Riemannian isometries, on the other hand, are not necessarily surjective and can be injective (one-to-one)
- Isometries preserve the Riemannian distance between points, while submersions may not preserve distances between points in different fibers

### Constructions using Riemannian submersions

- Riemannian submersions are a versatile tool for constructing new Riemannian manifolds with desired properties
- They provide a way to build manifolds with prescribed geometry and symmetries

### Warped product metrics

- Warped product metrics are a generalization of product metrics, where the metric on one factor is scaled by a function
- Riemannian submersions can be used to construct warped product metrics, with the base manifold being the codomain and the fibers being the other factor
- Warped products have applications in general relativity, such as in the study of spacetime geometries

### Projective spaces

- Projective spaces, such as real and complex projective spaces, can be constructed using Riemannian submersions
- The Hopf fibration, which is a Riemannian submersion from a sphere to a projective space, provides a geometric realization of these spaces
- Projective spaces are important in algebraic geometry and have connections to gauge theory and physics

## Hopf fibrations

- The Hopf fibrations are a family of Riemannian submersions between spheres of different dimensions
- The most well-known Hopf fibration is the submersion from the 3-sphere to the 2-sphere, with fibers being circles
- Hopf fibrations have important applications in topology, geometry, and mathematical physics

## Homogeneous spaces

- Homogeneous spaces are manifolds that admit a transitive action by a Lie group
- Riemannian submersions can be used to construct homogeneous spaces as quotients of Lie groups
- Many important manifolds in geometry and physics, such as Grassmannians and Stiefel manifolds, are homogeneous spaces

## Applications of Riemannian submersions

- Riemannian submersions have found numerous applications in various areas of mathematics and physics
- They provide a powerful framework for studying the geometry and topology of manifolds and their relationships

## Submanifold geometry

- Riemannian submersions are closely related to the study of submanifolds in Riemannian geometry
- The fibers of a Riemannian submersion are submanifolds of the domain manifold
- Properties of the submersion, such as the O'Neill tensors and curvature relations, provide information about the geometry of the submanifolds

## Quotient manifolds

- Riemannian submersions can be used to construct quotient manifolds by identifying points in the fibers
- The metric on the quotient manifold is induced by the submersion, making it a Riemannian manifold
- Quotient manifolds arise naturally in the study of symmetries and group actions on manifolds

## Kaluza-Klein theory

- Kaluza-Klein theory is a geometric approach to unifying gravity and electromagnetism by considering higher-dimensional spacetimes
- Riemannian submersions play a crucial role in Kaluza-Klein theory, with the extra dimensions being the fibers of the submersion
- The geometry of the fibers and the base manifold determine the properties of the unified theory

## Yang-Mills theory

- Yang-Mills theory is a gauge theory that describes the dynamics of non-Abelian gauge fields

- Riemannian submersions provide a geometric framework for studying Yang-Mills theory
- The principal bundles and connections in Yang-Mills theory can be described using Riemannian submersions and their associated structures

## Examples of Riemannian submersions

- There are many important and illustrative examples of Riemannian submersions in geometry and physics
- These examples demonstrate the wide range of applications and the richness of the theory

## Canonical projection of a product manifold

- The canonical projection from a product manifold  $M \times N$  onto one of its factors, such as  $\pi : M \times N \rightarrow M$ , is a Riemannian submersion
- The fibers of this submersion are the copies of the other factor  $N$
- This example serves as a prototype for more general constructions of Riemannian submersions

## Hopf map between spheres

- The Hopf map is a Riemannian submersion from the unit sphere  $S^{2n+1}$  to the complex projective space  $\mathbb{CP}^n$
- The fibers of the Hopf map are great circles, which are totally geodesic submanifolds of the sphere
- The Hopf map has important applications in topology, such as in the study of homotopy groups and characteristic classes

## Twistor fibrations

- Twistor theory, introduced by Roger Penrose, provides a geometric approach to studying solutions of certain differential equations
- Twistor fibrations are Riemannian submersions that relate twistor spaces to base manifolds, such as the fibration from the twistor space of a 4-manifold to the manifold itself
- Twistor fibrations have applications in mathematical physics, particularly in the study of Yang-Mills equations and integrable systems

## Riemannian submersions in physics

- Riemannian submersions have found numerous applications in theoretical physics
- In general relativity, submersions are used to construct models of spacetime with extra dimensions, such as in Kaluza-Klein theory
- In gauge theory, submersions provide a geometric framework for studying the geometry of gauge fields and their associated bundles
- Riemannian submersions also appear in the study of supergravity, string theory, and other areas of mathematical physics

# Unit 8 – Submanifolds and Hypersurfaces in Geometry

## 8.1 Embedded and immersed submanifolds

Embedded and immersed submanifolds are key concepts in differential geometry. They describe how one manifold can be contained within another of higher dimension, with embeddings being more restrictive and immersions allowing for self-intersections.

These concepts are crucial for understanding the geometry of curves and surfaces in higher-dimensional spaces. They provide tools for analyzing constraints in mechanics, modeling spacetime in physics, and creating complex shapes in computer graphics.

### Definition of embedded submanifolds

- Embedded submanifolds are a fundamental concept in differential geometry that describe how a manifold can be contained within another manifold of higher dimension
- Embeddedness is a stronger condition than immersion, requiring the submanifold to inherit its topology and smooth structure from the ambient manifold

### Topological embedding

- A topological embedding is a homeomorphism from a topological space  $M$  to a subspace of another topological space  $N$
- The image of  $M$  under the embedding is an embedded submanifold of  $N$
- The embedding preserves the topological properties of  $M$ , such as continuity and convergence of sequences

### Smooth embedding

- A smooth embedding is a smooth map  $f: M \rightarrow N$  between smooth manifolds that is also a topological embedding
- The differential  $df_p: T_p M \rightarrow T_{f(p)} N$  is injective at every point  $p \in M$
- Smooth embeddings preserve the smooth structure of  $M$ , allowing calculus and differential geometry techniques to be applied to the submanifold

### Codimension of embedded submanifolds

- The codimension of an embedded submanifold  $M$  in an ambient manifold  $N$  is the difference between their dimensions:  $\text{codim}(M) = \dim(N) - \dim(M)$
- Codimension measures the number of dimensions "lost" when embedding  $M$  into  $N$
- Submanifolds of codimension 1 are called hypersurfaces (curves in surfaces, surfaces in 3-manifolds)

## Properties of embedded submanifolds

- Embedded submanifolds inherit various properties from their ambient manifolds, which simplifies their study and allows for the application of results from the ambient space
- The induced structures on embedded submanifolds are compatible with those of the ambient manifold, ensuring consistency in calculations and theorems

## Induced topology from ambient manifold

- The subspace topology on an embedded submanifold  $M \subset N$  is induced by the topology of the ambient manifold  $N$
- Open sets in  $M$  are precisely the intersections of open sets in  $N$  with  $M$
- The induced topology makes the inclusion map  $i : M \rightarrow N$  continuous

## Induced smooth structure

- The smooth structure on an embedded submanifold  $M \subset N$  is induced by the smooth structure of the ambient manifold  $N$
- Smooth functions on  $M$  are precisely the restrictions of smooth functions on  $N$  to  $M$
- The induced smooth structure makes the inclusion map  $i : M \rightarrow N$  smooth

## Tangent spaces of embedded submanifolds

- The tangent space  $T_p M$  at a point  $P$  of an embedded submanifold  $M \subset N$  is a linear subspace of the tangent space  $T_p N$  of the ambient manifold
- $T_p M$  consists of the tangent vectors to curves in  $M$  passing through  $P$
- The differential of the inclusion map  $di_p : T_p M \rightarrow T_p N$  is an injective linear map

## Examples of embedded submanifolds

- Concrete examples help to illustrate the abstract concepts of embedded submanifolds and their properties
- Studying these examples provides insight into the geometry and topology of embedded submanifolds in various contexts

## Spheres in Euclidean space

- The  $n$ -sphere  $S^n$  is an embedded submanifold of the Euclidean space  $\mathbb{R}^{n+1}$
- $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$  is the set of points at unit distance from the origin
- The inclusion map  $i : S^n \rightarrow \mathbb{R}^{n+1}$  is a smooth embedding of codimension 1

## Graphs of smooth functions

- The graph of a smooth function  $f : M \rightarrow N$  between manifolds is an embedded submanifold of the product manifold  $M \times N$
- $\text{Graph}(f) = \{(x, f(x)) : x \in M\}$  is the set of points  $(x, y)$  with  $y = f(x)$



- The graph embedding  $\Gamma_f: M \rightarrow M \times N, x \mapsto (x, f(x))$  is a smooth embedding of codimension  $\dim(N)$

## Level sets of smooth maps

- The level set of a smooth map  $f: M \rightarrow N$  at a regular value  $y \in N$  is an embedded submanifold of  $M$
- $f^{-1}(y) = \{x \in M : f(x) = y\}$  is the set of points in  $M$  mapped to  $y$  by  $f$
- The codimension of the level set is equal to the dimension of  $N$ , and the inclusion map  $i: f^{-1}(y) \rightarrow M$  is a smooth embedding

## Definition of immersed submanifolds

- Immersed submanifolds are a generalization of embedded submanifolds that allow for self-intersections and non-injective maps
- Immersions capture the local geometry of a manifold while allowing for more flexible global behavior

## Immersion as a smooth map

- An immersion is a smooth map  $f: M \rightarrow N$  between manifolds whose differential  $df_p: T_p M \rightarrow T_{f(p)} N$  is injective at every point  $p \in M$
- Immersions preserve the local geometry of  $M$ , mapping tangent spaces injectively
- Unlike embeddings, immersions need not be injective globally, allowing for self-intersections

## Local embedding property

- Every immersion  $f: M \rightarrow N$  is locally an embedding
- For each point  $p \in M$ , there exists a neighborhood  $U$  of  $p$  such that the restriction  $f|_U: U \rightarrow N$  is an embedding
- Immersions are locally indistinguishable from embeddings, but their global structure may differ

## Codimension of immersed submanifolds

- The codimension of an immersed submanifold  $M$  in an ambient manifold  $N$  is the difference between their dimensions:  $\text{codim}(M) = \dim(N) - \dim(M)$
- As with embedded submanifolds, codimension measures the number of dimensions "lost" when immersing  $M$  into  $N$
- Immersed submanifolds of codimension 0 are local diffeomorphisms (open maps)

## Properties of immersed submanifolds

- Immersed submanifolds share some properties with embedded submanifolds but may exhibit different global behavior
- Understanding the properties of immersed submanifolds is crucial for studying their geometry and topology

## Induced topology vs ambient topology

- Unlike embedded submanifolds, immersed submanifolds may not inherit their topology from the ambient manifold
- The induced topology on an immersed submanifold  $M$  is generally finer than the subspace topology from the ambient manifold  $N$
- Continuous functions on  $M$  with respect to the induced topology may not extend continuously to  $N$

## Tangent spaces of immersed submanifolds

- The tangent space  $T_p M$  at a point  $P$  of an immersed submanifold  $M \subset N$  is a linear subspace of the tangent space  $T_{f(p)} N$  of the ambient manifold
- $T_p M$  consists of the tangent vectors to curves in  $M$  passing through  $P$ , mapped injectively by the differential  $df_p$
- The tangent spaces of an immersed submanifold may intersect non-trivially at self-intersection points

## Self-intersections in immersed submanifolds

- Immersed submanifolds may exhibit self-intersections, where distinct points in the domain manifold are mapped to the same point in the ambient manifold
- Self-intersections occur when the immersion  $f: M \rightarrow N$  is not injective
- The local geometry of an immersed submanifold at a self-intersection point is determined by the tangent spaces of the intersecting branches

## Examples of immersed submanifolds

- Studying concrete examples of immersed submanifolds helps to develop intuition for their properties and behavior
- These examples showcase the flexibility and variety of immersed submanifolds in different settings

## Lemniscate curve in the plane

- The lemniscate of Bernoulli is an immersed curve in the plane  $\mathbb{R}^2$  defined by the equation  $(x^2 + y^2)^2 = a^2(x^2 - y^2)$
- The lemniscate has a self-intersection at the origin and is symmetric about both the  $X$  and  $Y$  axes
- The immersion  $f: \mathbb{R} \rightarrow \mathbb{R}^2$  given by  $f(t) = (a \cos t / (1 + \sin^2 t), a \sin t \cos t / (1 + \sin^2 t))$  parametrizes the lemniscate

## Klein bottle in 4-dimensional space

- The Klein bottle is a non-orientable surface that can be immersed in 4-dimensional Euclidean space  $\mathbb{R}^4$
- It is obtained by gluing the edges of a rectangle with a twist, resulting in a single self-intersection
- The Klein bottle cannot be embedded in  $\mathbb{R}^3$ , but it admits an immersion in  $\mathbb{R}^4$  given by  $f(u, v) = ((r + \cos u/2) \cos v, (r + \cos u/2) \sin v, \sin u/2 \cos v/2, \sin u/2 \sin v/2)$

## Immersion of the circle in the plane

- The circle  $S^1$  admits various immersions into the plane  $\mathbb{R}^2$ , showcasing different self-intersection patterns
- The figure-eight immersion  $f: S^1 \rightarrow \mathbb{R}^2$  given by  $f(t) = (\sin t, \sin 2t)$  has a single self-intersection at the origin
- The immersion  $f(t) = (\cos^3 t, \sin^3 t)$  creates a trifolium with three self-intersections, resembling a three-leaf clover

## Comparison of embedded vs immersed submanifolds

- Understanding the differences between embedded and immersed submanifolds is essential for choosing the appropriate concept for a given problem
- The comparison highlights the trade-offs between the stricter conditions of embeddedness and the flexibility of immersions

## Topological differences

- Embedded submanifolds inherit their topology from the ambient manifold, while immersed submanifolds may have a finer topology
- Embedded submanifolds are homeomorphic to their image in the ambient manifold, whereas immersed submanifolds may not be
- Continuous functions on embedded submanifolds extend continuously to the ambient manifold, which may not be true for immersed submanifolds

## Smooth structure differences

- Both embedded and immersed submanifolds inherit their smooth structure from the ambient manifold
- Smooth functions on embedded submanifolds are restrictions of smooth functions on the ambient manifold
- Smooth functions on immersed submanifolds may not extend smoothly to the ambient manifold due to self-intersections

## Tangent space properties

- Tangent spaces of embedded submanifolds are linear subspaces of the tangent spaces of the ambient manifold
- Tangent spaces of immersed submanifolds are mapped injectively into the tangent spaces of the ambient manifold
- At self-intersection points of immersed submanifolds, the tangent spaces of the intersecting branches may intersect non-trivially

## Applications of embedded and immersed submanifolds

- Embedded and immersed submanifolds have numerous applications in various fields, demonstrating their practical importance

- These applications rely on the geometric and topological properties of submanifolds to model and analyze complex systems

## Constraint systems in mechanics

- Embedded submanifolds can be used to model constraint systems in classical mechanics
- The configuration space of a constrained system is an embedded submanifold of the unconstrained configuration space
- Lagrangian and Hamiltonian mechanics can be formulated on embedded submanifolds, taking the constraints into account

## Submanifold geometry in physics

- Submanifold geometry plays a crucial role in various areas of physics, such as general relativity and string theory
- In general relativity, spacetime is modeled as a 4-dimensional Lorentzian manifold, and physical objects are represented by embedded submanifolds (worldlines, worldsheets)
- String theory describes fundamental particles as vibrating strings, which can be viewed as embedded or immersed submanifolds of a higher-dimensional spacetime

## Modeling surfaces in computer graphics

- Immersed submanifolds, particularly surfaces, are widely used in computer graphics for modeling and rendering 3D objects
- Parametric surfaces, such as Bézier surfaces and NURBS, are immersions of rectangular domains into 3D space
- Subdivision surfaces, obtained by recursively refining a coarse mesh, provide a flexible way to model complex shapes with self-intersections and non-manifold features

## 8.2 Second fundamental form and mean curvature

The second fundamental form and mean curvature are essential concepts in differential geometry. They provide insights into the extrinsic curvature of surfaces embedded in higher-dimensional spaces, measuring how surfaces bend and curve in their ambient environment.

These concepts play a crucial role in understanding surface geometry, minimal surfaces, and geometric flows. They have wide-ranging applications in physics, computer graphics, and general relativity, connecting abstract mathematical ideas to real-world phenomena and technological advancements.

### Definition of second fundamental form

- The second fundamental form is a quadratic form defined on the tangent space of a surface embedded in a higher-dimensional space
- It measures how the surface curves in the ambient space and provides information about the extrinsic geometry of the surface
- The second fundamental form is closely related to the shape operator, which is a self-adjoint linear operator on the tangent space

## Quadratic form on tangent space

- The second fundamental form assigns a scalar value to each pair of tangent vectors at a point on the surface
- It is a symmetric bilinear form, meaning that it is linear in each argument and symmetric under the exchange of arguments
- The second fundamental form can be represented by a symmetric matrix in local coordinates

## Normal component of directional derivative

- The second fundamental form measures the normal component of the directional derivative of a tangent vector field along another tangent vector
- It captures how the surface normal changes as one moves along the surface in different directions
- The second fundamental form can be expressed in terms of the Christoffel symbols and the coefficients of the first fundamental form

## Relation to shape operator

- The shape operator, also known as the Weingarten map, is a self-adjoint linear operator on the tangent space of a surface
- It maps each tangent vector to the directional derivative of the unit normal vector along that tangent vector
- The matrix representation of the shape operator with respect to an orthonormal basis of the tangent space is the second fundamental form

## Properties of second fundamental form

- The second fundamental form satisfies several important properties that reflect the geometric nature of the surface
- These properties are crucial for understanding the behavior of the surface and its relation to other geometric quantities
- The properties of the second fundamental form are closely tied to the intrinsic and extrinsic geometry of the surface

## Symmetry and bilinearity

- The second fundamental form is symmetric, meaning that  $II(X, Y) = II(Y, X)$  for any tangent vectors  $X$  and  $Y$
- It is also bilinear, which means that it is linear in each argument separately
- $II(aX_1 + bX_2, Y) = aII(X_1, Y) + bII(X_2, Y)$  and  $II(X, cY_1 + dY_2) = cII(X, Y_1) + dII(X, Y_2)$  for any scalars  $a, b, c$ , and  $d$

## Invariance under isometries

- The second fundamental form is invariant under isometries, which are distance-preserving maps between Riemannian manifolds

- If  $f: M \rightarrow N$  is an isometry between surfaces  $M$  and  $N$ , then  $||_M(X, Y) = ||_N(df(X), df(Y))$  for any tangent vectors  $X$  and  $Y$  on  $M$
- This property reflects the fact that isometries preserve the extrinsic geometry of the surface

## Relation to Gaussian curvature

- The determinant of the second fundamental form at a point is equal to the Gaussian curvature of the surface at that point
- Gaussian curvature is an intrinsic quantity that measures the product of the principal curvatures
- The sign of the Gaussian curvature determines the local shape of the surface (elliptic, hyperbolic, or parabolic)

## Computation of second fundamental form

- The second fundamental form can be computed using various formulas depending on the representation of the surface
- In local coordinates, the second fundamental form can be expressed in terms of the Christoffel symbols and the coefficients of the first fundamental form
- For surfaces defined explicitly or parametrically in  $\mathbb{R}^3$ , there are specific formulas for computing the second fundamental form

## Formula in local coordinates

- In local coordinates  $(u, v)$  on a surface  $M$ , the second fundamental form can be written as  $II = h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2$
- The coefficients  $h_{ij}$  are given by  $h_{ij} = N \cdot \frac{\partial^2 \mathbf{r}}{\partial u^i \partial u^j}$ , where  $N$  is the unit normal vector and  $\mathbf{r}(u, v)$  is the position vector of the surface
- The coefficients  $h_{ij}$  can also be expressed in terms of the Christoffel symbols  $\Gamma_{ij}^k$  and the coefficients  $g_{ij}$  of the first fundamental form

## Examples for surfaces in $\mathbb{R}^3$

- For a graph of a function  $z = f(x, y)$ , the second fundamental form is given by  $II = \frac{f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2}{(1 + f_x^2 + f_y^2)^{1/2}}$
- For a parametric surface  $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ , the coefficients of the second fundamental form are  $h_{ij} = (\mathbf{r}_u \times \mathbf{r}_v) \cdot \frac{\partial^2 \mathbf{r}}{\partial u^i \partial u^j}$
- For an implicit surface  $F(x, y, z) = 0$ , the second fundamental form can be expressed in terms of the gradient and Hessian of  $F$

## Geometric interpretation

- The second fundamental form has a rich geometric interpretation that relates to the curvature and shape of the surface
- The eigenvalues and eigenvectors of the second fundamental form provide information about the principal curvatures and principal directions

- The sign of the Gaussian curvature, which is the determinant of the second fundamental form, characterizes the local shape of the surface

## Normal curvature and principal curvatures

- The normal curvature  $\kappa_n(X)$  in the direction of a unit tangent vector  $X$  is given by  $\kappa_n(X) = II(X, X)$
- The principal curvatures  $\kappa_1$  and  $\kappa_2$  are the maximum and minimum values of the normal curvature
- The principal curvatures are the eigenvalues of the shape operator, and their corresponding eigenvectors are the principal directions

## Relation to principal directions

- The principal directions are the eigenvectors of the shape operator, which correspond to the directions of maximum and minimum normal curvature
- At each point on the surface, the principal directions are orthogonal to each other
- The principal directions determine the directions in which the surface bends the most and the least

## Elliptic, hyperbolic, and parabolic points

- The sign of the Gaussian curvature determines the local shape of the surface at a point
- Elliptic points have positive Gaussian curvature, and the surface is locally shaped like a bowl (both principal curvatures have the same sign)
- Hyperbolic points have negative Gaussian curvature, and the surface is locally shaped like a saddle (principal curvatures have opposite signs)
- Parabolic points have zero Gaussian curvature, and the surface is locally shaped like a cylinder or a plane (at least one principal curvature is zero)

## Definition of mean curvature

- The mean curvature is an extrinsic measure of curvature that quantifies the average bending of a surface at a point
- It is defined as the average of the principal curvatures and is a fundamental concept in differential geometry
- The mean curvature is closely related to the shape operator and the second fundamental form

## Average of principal curvatures

- The mean curvature  $H$  at a point on a surface is defined as the arithmetic mean of the principal curvatures  $\kappa_1$  and  $\kappa_2$
- $$H = \frac{\kappa_1 + \kappa_2}{2}$$
- The mean curvature measures the average bending of the surface in all directions

## Trace of shape operator

- The mean curvature can also be defined as half the trace of the shape operator
- $H = \frac{1}{2}\text{tr}(S)$ , where  $S$  is the shape operator

- This definition highlights the connection between the mean curvature and the extrinsic geometry of the surface

## Extrinsic measure of curvature

- The mean curvature is an extrinsic measure of curvature, as it depends on the embedding of the surface in the ambient space
- Unlike Gaussian curvature, which is intrinsic, the mean curvature can change under isometric deformations of the surface
- The mean curvature is sensitive to the orientation of the surface and can be positive, negative, or zero

## Properties of mean curvature

- The mean curvature satisfies several important properties that reflect its geometric and variational nature
- These properties are essential for understanding the behavior of surfaces and their relation to other geometric quantities
- The properties of mean curvature play a crucial role in various applications, such as minimal surface theory and geometric flows

## Invariance under isometries

- The mean curvature is invariant under isometries of the ambient space
- If  $f: M \rightarrow N$  is an isometry between surfaces  $M$  and  $N$ , then  $H_M(p) = H_N(f(p))$  for any point  $p$  on  $M$
- This property reflects the fact that isometries preserve the extrinsic geometry of the surface

## Relation to area-minimizing surfaces

- Surfaces with zero mean curvature, called minimal surfaces, locally minimize the surface area for a given boundary
- The mean curvature is the gradient of the area functional, and its vanishing is a necessary condition for a surface to be area-minimizing
- Minimal surfaces have important applications in physics, architecture, and computer graphics

## Mean curvature flow

- The mean curvature flow is a geometric evolution equation that deforms a surface in the direction of its mean curvature vector
- The mean curvature vector is given by  $\mathbf{H} = -2H\mathbf{N}$ , where  $\mathbf{N}$  is the unit normal vector
- The mean curvature flow has been studied extensively in geometric analysis and has applications in image processing and surface fairing

## Computation of mean curvature

- The mean curvature can be computed using various formulas depending on the representation of the surface



- In local coordinates, the mean curvature can be expressed in terms of the coefficients of the first and second fundamental forms
- For surfaces defined explicitly or parametrically in  $\mathbb{R}^3$ , there are specific formulas for computing the mean curvature

## Formula in local coordinates

- In local coordinates  $(u, v)$  on a surface  $M$ , the mean curvature can be written as  $H = \frac{1}{2} \frac{Eg - 2Ff + Ge}{EG - F^2}$
- Here,  $E$ ,  $F$ , and  $G$  are the coefficients of the first fundamental form, and  $e$ ,  $f$ , and  $g$  are the coefficients of the second fundamental form
- This formula expresses the mean curvature in terms of the intrinsic and extrinsic geometry of the surface

## Examples for surfaces in $\mathbb{R}^3$

- For a graph of a function  $z = f(x, y)$ , the mean curvature is given by  $H = \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{2(1 + f_x^2 + f_y^2)^{3/2}}$
- For a parametric surface  $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ , the mean curvature can be computed using the coefficients of the first and second fundamental forms
- For an implicit surface  $F(x, y, z) = 0$ , the mean curvature can be expressed in terms of the gradient and Hessian of  $F$

## Surfaces with constant mean curvature

- Surfaces with constant mean curvature (CMC surfaces) are of particular interest in differential geometry and have important applications
- CMC surfaces are characterized by having the same mean curvature value at every point on the surface
- Examples of CMC surfaces include minimal surfaces, spheres, cylinders, and Delaunay surfaces

## Minimal surfaces (zero mean curvature)

- Minimal surfaces are surfaces with zero mean curvature at every point
- They locally minimize the surface area for a given boundary and have important applications in physics and geometry
- Examples of minimal surfaces include the catenoid, the helicoid, and the Enneper surface

## Spheres and cylinders

- Spheres have constant positive mean curvature, with  $H = \frac{1}{R}$ , where  $R$  is the radius of the sphere
- Cylinders have constant zero mean curvature along the rulings and constant positive mean curvature along the cross-sections
- Spheres and cylinders are examples of CMC surfaces with simple geometry

## Delaunay surfaces

- Delaunay surfaces are surfaces of revolution with constant mean curvature
- They are generated by rotating a curve called the meridian curve around an axis
- Examples of Delaunay surfaces include the unduloid, the nodoid, and the catenoid (a minimal surface)

## Applications and further topics

- The study of the second fundamental form and mean curvature has numerous applications and connections to other areas of mathematics and physics
- These concepts play a crucial role in understanding the geometry and topology of surfaces, as well as in various physical phenomena
- Some notable applications and further topics include the Gauss-Bonnet theorem, Willmore energy, and the role of curvature in general relativity and cosmology

## Gauss-Bonnet theorem for surfaces with boundary

- The Gauss-Bonnet theorem relates the total Gaussian curvature of a surface to its Euler characteristic and the geodesic curvature of its boundary
- For a compact surface  $M$  with boundary  $\partial M$ , the theorem states that  $\int_M K dA + \int_{\partial M} K_g ds = 2\pi\chi(M)$
- Here,  $K$  is the Gaussian curvature,  $K_g$  is the geodesic curvature of the boundary, and  $\chi(M)$  is the Euler characteristic of the surface

## Willmore energy and Willmore surfaces

- The Willmore energy of a surface is a functional that measures the total squared mean curvature of the surface
- It is defined as  $W(M) = \int_M H^2 dA$ , where  $H$  is the mean curvature and  $dA$  is the area element
- Willmore surfaces are critical points of the Willmore energy and have important applications in conformal geometry and biological membranes

## Role in general relativity and cosmology

- In general relativity, the curvature of spacetime plays a fundamental role in describing gravitational phenomena
- The Einstein field equations relate the curvature of spacetime to the distribution of matter and energy
- The extrinsic curvature of spacelike hypersurfaces, analogous to the second fundamental form, is crucial in the formulation of the initial value problem in general relativity
- In cosmology, the curvature of space is related to the density and expansion of the universe, and different curvature scenarios (flat, spherical, or hyperbolic) have different implications for the fate of the universe

## 8.3 Gauss and Codazzi equations

The Gauss and Codazzi equations are crucial in differential geometry, linking intrinsic and extrinsic surface properties. They relate Gaussian curvature to principal curvatures and establish compatibility conditions for the second fundamental form.

These equations are key to the fundamental theorem of surfaces, which states that a surface is uniquely determined by its first and second fundamental forms. They also play a vital role in isometric embeddings and the study of special surface classes.

### Fundamental theorem of surfaces

- The fundamental theorem of surfaces states that a surface in Euclidean space is uniquely determined, up to rigid motion, by its first and second fundamental forms
- Provides a link between the intrinsic and extrinsic geometry of a surface
- Crucial result in the study of surfaces in differential geometry, as it allows for the characterization of surfaces based on their metric properties

### Gauss equation

#### Intrinsic curvature

- Intrinsic curvature is a measure of curvature that depends only on the metric of the surface and not on how it is embedded in the ambient space
- Can be computed using the first fundamental form and its derivatives
- Examples include Gaussian curvature and scalar curvature

#### Extrinsic curvature

- Extrinsic curvature measures how a surface curves with respect to the ambient space in which it is embedded
- Depends on the second fundamental form of the surface
- Mean curvature and principal curvatures are examples of extrinsic curvature quantities

### Relationship between intrinsic and extrinsic curvature

- The Gauss equation establishes a relationship between the intrinsic and extrinsic curvature of a surface
- States that the Gaussian curvature (intrinsic) can be expressed in terms of the principal curvatures (extrinsic)
- Specifically, the Gaussian curvature is equal to the product of the principal curvatures

### Gaussian curvature in terms of first and second fundamental forms

- The Gauss equation allows for the computation of Gaussian curvature using the coefficients of the first and second fundamental forms
- Involves the determinants of the first and second fundamental form matrices

- Provides a way to calculate intrinsic curvature from the extrinsic data of the surface

## Codazzi equation

### Compatibility condition for second fundamental form

- The Codazzi equation serves as a compatibility condition for the second fundamental form
- Ensures that the second fundamental form is well-defined and consistent across the surface
- Necessary condition for the existence of an isometric embedding of a surface into Euclidean space

### Relationship between Christoffel symbols and second fundamental form

- The Codazzi equation relates the Christoffel symbols (connection coefficients) of the surface to the coefficients of the second fundamental form
- Involves the covariant derivatives of the second fundamental form coefficients
- Establishes a link between the intrinsic and extrinsic geometry of the surface

### Symmetry of second fundamental form

- The Codazzi equation implies the symmetry of the second fundamental form
- Specifically, it states that the mixed partial derivatives of the second fundamental form coefficients are equal
- This symmetry property is crucial for the consistency and well-definedness of the second fundamental form

## Derivation of Gauss and Codazzi equations

### Gauss equation derivation

- The Gauss equation can be derived by considering the compatibility of the first and second fundamental forms
- Involves computing the Riemann curvature tensor of the surface and comparing it with the ambient Riemann curvature tensor
- Uses the Gauss formula, which relates the covariant derivatives of the surface and the ambient space

### Codazzi equation derivation

- The Codazzi equation can be derived by considering the compatibility of the second fundamental form
- Involves computing the covariant derivatives of the second fundamental form coefficients and using the symmetry of the Riemann curvature tensor
- Utilizes the properties of the Levi-Civita connection and the Gauss formula

## Applications of Gauss and Codazzi equations

### Local isometric embedding theorem

- The Gauss and Codazzi equations play a crucial role in the local isometric embedding theorem

- States that a surface with a given first and second fundamental form can be locally isometrically embedded into Euclidean space if and only if the Gauss and Codazzi equations are satisfied
- Provides a necessary and sufficient condition for the existence of a local isometric embedding

## Rigidity of isometric embeddings

- The Gauss and Codazzi equations are used to study the rigidity of isometric embeddings
- An isometric embedding is rigid if it is uniquely determined up to rigid motions by the intrinsic geometry of the surface
- The Gauss and Codazzi equations provide constraints on the possible isometric embeddings of a surface

## Bonnet theorem for surfaces of constant mean curvature

- The Bonnet theorem states that a surface with constant mean curvature is uniquely determined, up to rigid motion, by its first and second fundamental forms
- The proof of the Bonnet theorem relies on the Gauss and Codazzi equations
- Demonstrates the importance of these equations in the study of special classes of surfaces

## Geometric interpretation

### Gauss equation and Theorema Egregium

- The Gauss equation is closely related to the Theorema Egregium, a fundamental result in differential geometry
- Theorema Egregium states that the Gaussian curvature of a surface is an intrinsic property, i.e., it depends only on the first fundamental form
- The Gauss equation provides a geometric interpretation of this result by relating the intrinsic Gaussian curvature to the extrinsic principal curvatures

### Codazzi equation and integrability of second fundamental form

- The Codazzi equation has a geometric interpretation in terms of the integrability of the second fundamental form
- Integrability refers to the existence of a surface with a given second fundamental form
- The Codazzi equation ensures that the second fundamental form is integrable, i.e., it arises from an actual surface

## Generalizations

### Higher-dimensional analogs

- The Gauss and Codazzi equations have higher-dimensional analogs for hypersurfaces and submanifolds
- In higher dimensions, the Gauss equation relates the intrinsic curvature of the submanifold to its extrinsic curvature and the curvature of the ambient space

- The Codazzi equation generalizes to the Codazzi-Mainardi equations, which provide compatibility conditions for the second fundamental form of a submanifold

## Gauss and Codazzi equations in Riemannian geometry

- The Gauss and Codazzi equations can be formulated in the context of Riemannian geometry
- In this setting, the equations relate the intrinsic curvature of a submanifold to its extrinsic curvature and the curvature of the ambient Riemannian manifold
- The equations play a fundamental role in the study of submanifolds and their geometric properties

## Gauss and Codazzi equations for hypersurfaces

- Hypersurfaces are submanifolds of codimension one, i.e., their dimension is one less than the dimension of the ambient space
- The Gauss and Codazzi equations take a particularly simple form for hypersurfaces
- In this case, the equations involve the Gaussian curvature, mean curvature, and principal curvatures of the hypersurface
- The study of hypersurfaces using the Gauss and Codazzi equations has numerous applications in geometry and physics

## 8.4 Minimal surfaces

Minimal surfaces are fascinating mathematical objects that minimize area while constrained by boundaries or topology. They have zero mean curvature at every point, making them as flat as possible given their constraints. Examples include the catenoid and helicoid.

The study of minimal surfaces combines differential geometry, complex analysis, and calculus of variations. Key concepts include the Weierstrass-Enneper parameterization, stability analysis, and Plateau's problem. Applications range from architecture to materials science and computer graphics.

### Definition of minimal surfaces

- Minimal surfaces are surfaces that locally minimize area while being constrained by a boundary or topology
- Characterized by having zero mean curvature at every point, meaning the surface has no intrinsic curvature and is as "flat" as possible given its constraints
- Can be defined variationally as critical points of the area functional, or as surfaces whose mean curvature vanishes identically

### Examples of minimal surfaces

#### Catenoid

- Minimal surface formed by rotating a catenary curve around an axis
- Only minimal surface of revolution besides the plane
- Can be physically realized by dipping a wire frame into soap solution (soap films naturally minimize surface area due to surface tension)

## Helicoid

- Minimal surface formed by twisting a straight line as it moves along an axis perpendicular to the line
- Conjugate to the catenoid, meaning it can be isometrically deformed into a catenoid
- Appears in nature in certain seashells and in the structure of some proteins

## Enneper's surface

- Minimal surface discovered by Alfred Enneper in the 19th century
- Parameterized by  $x(u, v) = u - \frac{u^3}{3} + uv^2$ ,  $y(u, v) = -v + \frac{v^3}{3} - u^2v$ ,  $z(u, v) = u^2 - v^2$
- Has self-intersections and a unique "Enneper point" where the surface normal is vertical

## Weierstrass-Enneper parameterization

### Isothermal parameters

- Parameterization of a surface is isothermal if the parameter curves are orthogonal and the metric is conformal (angle-preserving)
- Minimal surfaces locally admit isothermal parameterizations, which greatly simplifies their study
- Isothermal parameters can be found by solving the Beltrami equation  $\Delta u = 0$ , where  $\Delta$  is the Laplace-Beltrami operator on the surface

### Holomorphic functions

- Functions that are complex differentiable at every point in their domain
- Play a key role in the Weierstrass-Enneper representation of minimal surfaces
- Real and imaginary parts of holomorphic functions are harmonic (satisfy Laplace's equation), which is related to the minimality condition

### Weierstrass data

- Minimal surface can be constructed from a pair of holomorphic functions  $f(z)$  and  $g(z)$ , known as the Weierstrass data
- Parametric equations are given by  $x = \Re \int f(1 - g^2)dz$ ,  $y = \Re \int if(1 + g^2)dz$ ,  $z = \Re \int 2fgdz$
- Different choices of  $f$  and  $g$  generate different minimal surfaces, providing a powerful tool for studying their properties and constructing examples

### First variation formula

### Area functional

- Assigns to each surface its total area, defined as the integral of the area element  $dA = \sqrt{EG - F^2} du dv$  (where  $E, F, G$  are coefficients of the first fundamental form)
- Minimal surfaces are critical points of the area functional, meaning small perturbations do not change the total area to first order

- Leads to the Euler-Lagrange equation for minimal surfaces, which characterizes them as having zero mean curvature

## Euler-Lagrange equation

- Differential equation that expresses the minimality condition for surfaces
- For minimal surfaces, takes the form  $H = \frac{1}{2}(k_1 + k_2) = 0$ , where  $H$  is the mean curvature and  $k_1, k_2$  are the principal curvatures
- Solutions to the Euler-Lagrange equation are minimal surfaces, and conversely, every regular minimal surface satisfies this equation

## Mean curvature

- Average of the two principal curvatures at a point on a surface
- Measures the local deviation of a surface from being flat (a plane has zero mean curvature)
- Vanishes identically for minimal surfaces, providing a simple characterization of minimality
- Can be expressed in terms of the Laplacian of the position vector:  $H = -\frac{1}{2}\Delta \mathbf{x}$ , where  $\mathbf{x}$  is the position vector of the surface

## Second variation formula

### Jacobi operator

- Linear differential operator that arises in the second variation formula for minimal surfaces
- Defined as  $L = \Delta + |A|^2$ , where  $\Delta$  is the Laplace-Beltrami operator and  $|A|$  is the norm of the second fundamental form
- Eigenvalues and eigenfunctions of the Jacobi operator determine the stability properties of minimal surfaces

## Stability of minimal surfaces

- A minimal surface is stable if the second variation of area is non-negative for all normal variations with compact support
- Equivalent to the condition that the first eigenvalue of the Jacobi operator is non-negative
- Stable minimal surfaces minimize area among nearby surfaces with the same boundary or topology
- Examples of stable minimal surfaces include the plane, the catenoid, and the helicoid

## Bernstein's theorem

### Statement and proof

- Theorem stating that the only complete, simply-connected minimal surfaces in  $\mathbb{R}^3$  are planes
- Proved by Serge Bernstein in 1915 using techniques from complex analysis and PDE theory
- Key steps include showing that the Gauss map of a minimal surface is holomorphic, and that a bounded holomorphic function must be constant (Liouville's theorem)



- Demonstrates the rigidity of minimal surfaces and the strong constraints imposed by the minimality condition

## Generalizations

- Bernstein's theorem has been extended to higher dimensions, with the result that complete, simply-connected minimal hypersurfaces in  $\mathbb{R}^n$  are hyperplanes for  $n \leq 7$
- In higher dimensions, there exist non-trivial minimal hypersurfaces, such as the Simons cone in  $\mathbb{R}^8$
- Bernstein-type theorems have also been proven for other ambient spaces, such as minimal surfaces in the sphere or in hyperbolic space

## Plateau's problem

### Formulation and existence

- Problem of finding a minimal surface with a given boundary curve, named after Joseph Plateau
- Can be formulated variationally as minimizing the area functional among all surfaces spanning the given boundary
- Existence of solutions was first proven by Jesse Douglas and Tibor Radó independently in the 1930s, using methods from calculus of variations and geometric measure theory
- Solutions to Plateau's problem may have branch points or self-intersections, as in the case of the Enneper-Riemann minimal surface

### Douglas-Rado solution

- Approach to solving Plateau's problem developed by Jesse Douglas and Tibor Radó
- Key idea is to minimize the Dirichlet energy  $\int |\nabla u|^2 dA$  among all maps  $u$  from the unit disk to  $\mathbb{R}^3$  with the given boundary values
- Minimizers of the Dirichlet energy are harmonic maps, which are conformally parameterized minimal surfaces
- Douglas-Rado solution provides a conformally parameterized minimal surface spanning the given boundary, which may be immersed or have branch points

## Minimal surfaces in Riemannian manifolds

### Isometric immersions

- An isometric immersion is a map between Riemannian manifolds that preserves the metric (i.e., distances and angles)
- Minimal surfaces in Riemannian manifolds are isometric immersions that are also critical points of the area functional
- The condition for a surface to be minimal in a Riemannian manifold is that the mean curvature vector  $\mathbf{H}$  vanishes identically

## Gauss map

- The Gauss map of a surface in a Riemannian manifold assigns to each point the unit normal vector to the surface at that point
- For minimal surfaces, the Gauss map is a harmonic map into the unit sphere in the tangent space
- The Gauss map of a minimal surface in a Riemannian manifold satisfies certain differential equations that constrain its behavior and relate it to the ambient geometry

## Monotonicity formula

- A key tool in the study of minimal surfaces in Riemannian manifolds, expressing the rate of change of area of a minimal surface under dilation
- States that the quantity  $r^{-n} \int_{B_r(x)} dA$  is increasing in  $r$  for any point  $x$  on a minimal surface (where  $B_r(x)$  is the ball of radius  $r$  centered at  $x$ )
- Monotonicity formula implies density estimates for minimal surfaces and is used to study their local behavior and regularity properties

## Regularity theory

### Interior regularity

- Concerns the smoothness properties of minimal surfaces away from their boundary
- Minimal surfaces are analytic in their interior, meaning they can be locally expressed as convergent power series
- Interior regularity follows from the ellipticity of the minimal surface equation and standard PDE techniques, such as the maximum principle and Schauder estimates

### Boundary regularity

- Studies the behavior of minimal surfaces near their boundary
- Boundary regularity is more delicate than interior regularity, as minimal surfaces may develop singularities or branch points at the boundary
- Key results include the boundary maximum principle, which constrains the behavior of minimal surfaces near their boundary, and the Allard regularity theorem, which gives conditions for a minimal surface to be smooth up to the boundary

## Removable singularities

- Isolated points where a minimal surface fails to be smooth, but the singularity can be "removed" by extending the surface smoothly across the point
- Removable singularities are characterized by the condition that the area of the surface is finite in a neighborhood of the singular point
- Removability of singularities is a consequence of the monotonicity formula and density estimates for minimal surfaces

# Applications of minimal surfaces

## Architecture and design

- Minimal surfaces appear in many architectural and design contexts due to their aesthetic appeal and structural efficiency
- Examples include tensile structures, such as the Munich Olympic Stadium roof, and freeform facades, like the Guggenheim Museum Bilbao
- Minimal surfaces are also used in the design of lightweight and strong structures, such as space frames and geodesic domes

## Physics and materials science

- Minimal surfaces arise naturally in many physical systems, such as soap films, interfaces between fluids, and crystalline structures
- The shape of soap films is determined by surface tension, which seeks to minimize the total surface area subject to boundary constraints
- Triply periodic minimal surfaces, such as the Schwarz P and D surfaces, are used to model the structure of certain block copolymers and other self-assembling materials

## Computer graphics and geometry processing

- Minimal surfaces are used in computer graphics and geometry processing as a tool for surface modeling, fairing, and deformation
- Algorithms for computing discrete minimal surfaces, such as the mean curvature flow and the discrete Plateau problem, are used for smoothing and denoising 3D meshes
- Minimal surfaces are also used for parameterization and texture mapping, as they provide low-distortion and conformal mappings between surfaces

## 8.5 Hypersurfaces and the Gauss-Bonnet theorem

Hypersurfaces are submanifolds with one dimension less than their ambient space. They're key to understanding geometry and topology in higher dimensions. Studying hypersurfaces helps us grasp concepts like curvature, geodesics, and minimal surfaces.

The Gauss-Bonnet theorem connects a surface's curvature to its topology. It's a powerful tool that links local geometry with global properties, revealing deep insights about the nature of surfaces and their shapes.

### Definition of hypersurfaces

- Hypersurfaces are submanifolds of codimension one embedded in a higher-dimensional space
- They generalize the concept of surfaces in three-dimensional space to higher dimensions
- Studying hypersurfaces is essential for understanding the geometry and topology of manifolds in Metric Differential Geometry

## Hypersurfaces in Euclidean space

- In Euclidean space  $\mathbb{R}^{n+1}$ , a hypersurface is a subspace of dimension  $n$
- Can be defined as the level set of a smooth function  $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  (implicit form)
- Examples include spheres, hyperplanes, and hypercylinders

## Hypersurfaces in Riemannian manifolds

- Hypersurfaces can be defined in the context of Riemannian manifolds
- A hypersurface in an  $(n+1)$ -dimensional Riemannian manifold  $(M, g)$  is a submanifold of dimension  $n$
- The induced metric on the hypersurface is obtained by restricting the metric  $g$  to the tangent space of the hypersurface

## Regular and singular points

- A point on a hypersurface is called regular if the gradient of the defining function is non-zero at that point
- Regular points form an open dense subset of the hypersurface
- Singular points are points where the gradient vanishes, leading to non-smooth or degenerate behavior

## Induced metric on hypersurfaces

- The induced metric on a hypersurface is a fundamental object that captures its intrinsic geometry
- It allows us to study properties such as length, area, and curvature on the hypersurface
- The induced metric is crucial for understanding the relationship between the geometry of the hypersurface and the ambient space

## First fundamental form

- The first fundamental form is the induced metric on a hypersurface
- It is obtained by restricting the metric of the ambient space to the tangent space of the hypersurface
- In local coordinates, the first fundamental form is represented by a symmetric positive definite matrix  $g_{ij}$

## Intrinsic geometry of hypersurfaces

- The intrinsic geometry of a hypersurface is determined by the first fundamental form
- Intrinsic properties depend only on the induced metric and not on the embedding in the ambient space
- Examples of intrinsic properties include geodesics, Gaussian curvature, and the Laplace-Beltrami operator

## Length, area, and volume

- The induced metric allows us to define length, area, and volume on a hypersurface

- The length of a curve on a hypersurface is computed by integrating the square root of the first fundamental form along the curve
- The area of a region on a hypersurface is obtained by integrating the volume form associated with the induced metric
- The volume of a hypersurface can be defined using the induced volume form

## Second fundamental form

- The second fundamental form captures the extrinsic geometry of a hypersurface
- It measures how the hypersurface bends and twists in the ambient space
- The second fundamental form is a symmetric bilinear form on the tangent space of the hypersurface

## Definition and properties

- The second fundamental form  $II(X, Y)$  is defined as the normal component of the covariant derivative of the tangent vector field  $Y$  along  $X$
- It is a symmetric bilinear form that takes two tangent vectors and returns a normal vector
- The second fundamental form is independent of the choice of the unit normal vector field

## Principal curvatures and directions

- The principal curvatures are the eigenvalues of the second fundamental form with respect to the first fundamental form
- The corresponding eigenvectors are called principal directions
- The principal curvatures measure the maximum and minimum curvatures of the hypersurface at a point

## Gaussian and mean curvature

- The Gaussian curvature  $K$  is the product of the principal curvatures
- It is an intrinsic quantity that depends only on the first fundamental form
- The mean curvature  $H$  is the average of the principal curvatures and is an extrinsic quantity

## Gauss map and Weingarten map

- The Gauss map and Weingarten map are important tools for studying the geometry of hypersurfaces
- They provide a connection between the tangent space and the normal space of the hypersurface
- These maps are used to derive various curvature properties and relations

## Definition and properties

- The Gauss map  $N : M \rightarrow S^n$  assigns to each point on the hypersurface  $M$  its unit normal vector
- The Weingarten map  $L : TM \rightarrow TM$  is the negative differential of the Gauss map

- The eigenvalues of the Weingarten map are the principal curvatures, and the eigenvectors are the principal directions

## Relation to second fundamental form

- The second fundamental form can be expressed in terms of the Weingarten map as  $II(X, Y) = \langle L(X), Y \rangle$
- The matrix representation of the Weingarten map with respect to an orthonormal basis is the shape operator
- The Gaussian and mean curvatures can be computed from the trace and determinant of the shape operator

## Applications in geometry

- The Gauss map and Weingarten map are used to study various geometric properties of hypersurfaces
- They play a crucial role in the formulation and proof of theorems such as the Gauss-Bonnet theorem
- These maps are also used in the study of minimal surfaces and constant mean curvature surfaces

## Gauss' theorema egregium

- Gauss' theorema egregium, or the "remarkable theorem," is a fundamental result in differential geometry
- It states that the Gaussian curvature of a surface is an intrinsic property
- The theorem has far-reaching consequences in the study of surfaces and their geometry

## Statement and significance

- The theorem states that the Gaussian curvature of a surface can be computed solely from the first fundamental form
- It implies that the Gaussian curvature is invariant under local isometries (preserving the metric)
- The theorem highlights the intrinsic nature of Gaussian curvature and its independence from the embedding

## Intrinsic nature of Gaussian curvature

- Gaussian curvature is a measure of the intrinsic curvature of a surface
- It depends only on the induced metric and not on the way the surface is embedded in the ambient space
- Surfaces with the same Gaussian curvature can be locally isometric, even if they have different embeddings

## Proof and consequences

- The proof of Gauss' theorema egregium involves expressing the Gaussian curvature in terms of the Christoffel symbols

- The Christoffel symbols can be computed from the first fundamental form, establishing the intrinsic nature of Gaussian curvature
- The theorem has important consequences, such as the existence of non-isometric surfaces with the same Gaussian curvature (pseudosphere and tractricoid)

## Geodesics on hypersurfaces

- Geodesics are the shortest paths between points on a hypersurface
- They are crucial for understanding the intrinsic geometry of the hypersurface
- Geodesics are characterized by specific properties and equations

## Definition and properties

- A geodesic on a hypersurface is a curve whose acceleration vector is normal to the surface at every point
- Geodesics are locally length-minimizing curves and have zero geodesic curvature
- The equations of geodesics can be derived from the Christoffel symbols of the induced metric

## Geodesic curvature

- The geodesic curvature measures how much a curve on a hypersurface deviates from being a geodesic
- It is the component of the curvature vector that lies in the tangent plane of the hypersurface
- Geodesics have zero geodesic curvature, while great circles on a sphere have constant geodesic curvature

## Clairaut's relation

- Clairaut's relation is a conservation law for geodesics on surfaces of revolution
- It states that the product of the distance from the axis of revolution and the sine of the angle between the geodesic and the meridian is constant along the geodesic
- Clairaut's relation is useful for studying the behavior of geodesics on surfaces like spheres, cones, and cylinders

## Gauss-Bonnet theorem for surfaces

- The Gauss-Bonnet theorem is a fundamental result that relates the geometry and topology of surfaces
- It connects the integral of the Gaussian curvature over a surface to its Euler characteristic
- The theorem has important applications in the study of surfaces and their properties

## Statement and significance

- The Gauss-Bonnet theorem states that the integral of the Gaussian curvature over a closed surface, plus the integral of the geodesic curvature along the boundary, is equal to  $2\pi$  times the Euler characteristic of the surface

- The theorem provides a deep connection between the local geometry (curvature) and the global topology (Euler characteristic) of a surface
- It has significant implications in the classification and study of surfaces

## Relation to Euler characteristic

- The Euler characteristic is a topological invariant that measures the number of vertices, edges, and faces of a polyhedron
- For a closed surface, the Euler characteristic is related to the genus (number of holes) of the surface
- The Gauss-Bonnet theorem relates the Euler characteristic to the integral of the Gaussian curvature, providing a geometric interpretation

## Applications and examples

- The Gauss-Bonnet theorem can be used to compute the Euler characteristic of surfaces (sphere, torus)
- It provides a way to determine the total curvature of a surface by integrating the Gaussian curvature
- The theorem has applications in the study of geodesic triangles, the Poincaré-Hopf theorem, and the Chern-Gauss-Bonnet theorem

## Generalized Gauss-Bonnet theorem

- The generalized Gauss-Bonnet theorem extends the classical Gauss-Bonnet theorem to higher dimensions
- It relates the curvature of a Riemannian manifold to its Euler characteristic and other topological invariants
- The theorem has deep connections to topology, geometry, and differential equations

## Statement for higher dimensions

- For a compact, oriented Riemannian manifold  $M$  of even dimension  $n$ , the generalized Gauss-Bonnet theorem states that the integral of the Pfaffian of the curvature form over  $M$  is equal to the Euler characteristic of  $M$  multiplied by a constant factor
- The Pfaffian is a polynomial in the components of the curvature tensor that generalizes the Gaussian curvature
- The theorem provides a higher-dimensional analog of the classical Gauss-Bonnet theorem

## Chern-Gauss-Bonnet theorem

- The Chern-Gauss-Bonnet theorem is a further generalization of the Gauss-Bonnet theorem to complex manifolds
- It relates the Chern classes of a complex manifold to its Euler characteristic
- The theorem has important applications in algebraic geometry and the study of characteristic classes



## Applications in topology and geometry

- The generalized Gauss-Bonnet theorem is used to compute topological invariants of manifolds (Euler characteristic, signature)
- It provides a connection between the geometry of a manifold (curvature) and its topology
- The theorem has applications in the study of characteristic classes, index theorems, and the Atiyah-Singer index theorem

## Minimal surfaces

- Minimal surfaces are surfaces that locally minimize their area subject to certain boundary conditions
- They have zero mean curvature at every point and are characterized by specific equations and properties
- Minimal surfaces have important applications in mathematics, physics, and engineering

## Definition and examples

- A minimal surface is a surface whose mean curvature vanishes identically
- Examples of minimal surfaces include the plane, the catenoid, the helicoid, and the Enneper surface
- Soap films and bubbles often form minimal surfaces due to surface tension

## Minimal surface equation

- The minimal surface equation is a partial differential equation that characterizes minimal surfaces
- In local coordinates, the equation is given by  $\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = 0$ , where  $f$  is the height function of the surface
- Solutions to the minimal surface equation are minimal surfaces, and vice versa

## Bernstein's theorem and generalizations

- Bernstein's theorem states that any complete minimal surface in  $\mathbb{R}^3$  that is also a graph over a plane must be a plane
- The theorem highlights the rigidity of minimal surfaces and the constraints on their geometry
- Generalizations of Bernstein's theorem have been studied for higher dimensions and different ambient spaces

## Constant mean curvature surfaces

- Constant mean curvature (CMC) surfaces are surfaces with a constant mean curvature at every point
- They generalize minimal surfaces and have important physical and geometric properties
- CMC surfaces are studied in differential geometry, physics, and applied mathematics

## Definition and examples

- A surface has constant mean curvature if its mean curvature is a non-zero constant value

- Examples of CMC surfaces include spheres, cylinders, and Delaunay surfaces (unduloids, nodoids)
- CMC surfaces can be viewed as critical points of the area functional with a volume constraint

## Relation to soap films and bubbles

- Soap films and bubbles often form CMC surfaces due to surface tension and pressure differences
- The constant mean curvature is related to the pressure difference across the surface by the Young-Laplace equation
- CMC surfaces provide mathematical models for the behavior of soap films and bubbles

## Alexandrov's theorem and uniqueness results

- Alexandrov's theorem states that any compact, embedded CMC surface in  $\mathbb{R}^3$  must be a sphere
- The theorem establishes the uniqueness of spheres among compact CMC surfaces
- Generalizations of Alexandrov's theorem have been studied for higher dimensions and different ambient spaces, leading to various uniqueness results for CMC surfaces

## 8.6 Foliations

Foliations are a key concept in differential geometry, describing how manifolds can be divided into submanifolds called leaves. They provide insight into the structure and properties of manifolds, helping us analyze geometric objects and their symmetries.

Studying foliations involves examining leaf dimensions, foliation charts, and tangent distributions. The codimension of a foliation is crucial, with codimension one foliations being particularly well-studied. Holonomy measures how leaves intertwine, providing important insights into foliation structure.

### Definition of foliations

- Foliations are a fundamental concept in differential geometry that describe how a manifold can be decomposed into a collection of submanifolds called leaves
- The study of foliations is important in understanding the structure and properties of manifolds, which is a central theme in metric differential geometry
- Foliations provide a way to analyze the local and global behavior of geometric objects and their symmetries

### Foliation of manifolds

- A foliation of a manifold  $M$  is a partition of  $M$  into a disjoint union of connected submanifolds called leaves
- Each leaf of the foliation has the same dimension, and the leaves are locally parallel to each other
- The leaves of a foliation are required to vary smoothly, meaning that the tangent spaces of nearby leaves are close to each other

### Foliation charts

- Foliation charts are local coordinate systems that are adapted to the foliation structure

- A foliation chart consists of a local coordinate system  $(U, \phi)$  where  $U$  is an open subset of the manifold and  $\phi : U \rightarrow \mathbb{R}^n$  is a homeomorphism
- In a foliation chart, the leaves of the foliation are given by the level sets of the last  $k$  coordinates, where  $k$  is the codimension of the foliation

## Tangent distributions

- A tangent distribution on a manifold  $M$  is a smooth assignment of a subspace of the tangent space at each point of  $M$
- Given a foliation  $\mathcal{F}$  of  $M$ , the tangent distribution  $T\mathcal{F}$  assigns to each point  $p \in M$  the tangent space to the leaf through  $p$
- Conversely, the Frobenius theorem states that every involutive tangent distribution on a manifold determines a unique foliation

## Codimension of foliations

- The codimension of a foliation is the difference between the dimension of the ambient manifold and the dimension of the leaves
- Codimension is a key invariant of a foliation that determines many of its properties and behaviors
- The study of foliations of different codimensions reveals interesting geometric and topological phenomena

## Codimension one foliations

- Codimension one foliations are foliations where the leaves have codimension one, meaning they are hypersurfaces in the ambient manifold
- Examples of codimension one foliations include the foliation of  $\mathbb{R}^3$  by parallel planes and the Reeb foliation of the solid torus
- Codimension one foliations have been extensively studied and have connections to contact geometry and dynamical systems

## Higher codimension foliations

- Higher codimension foliations are foliations where the leaves have codimension greater than one
- Examples of higher codimension foliations include the foliation of  $\mathbb{R}^4$  by parallel 2-planes and the Hopf fibration of  $S^3$
- Higher codimension foliations exhibit more complex behavior and have been studied using techniques from algebraic topology and homotopy theory

## Holonomy of foliations

- The holonomy of a foliation measures how the leaves of the foliation are "twisted" or "intertwined" with each other
- Holonomy captures the global structure of a foliation and is an important invariant in the study of foliations

- The holonomy of a foliation can be described using various algebraic and geometric objects, such as pseudogroups and groupoids

## Holonomy pseudogroup

- The holonomy pseudogroup of a foliation is a collection of local diffeomorphisms between transversals to the foliation that preserve the foliation structure
- The holonomy pseudogroup encodes the local symmetries of the foliation and can be used to study the dynamics of the foliation
- The holonomy pseudogroup is an important tool in the study of foliation invariants and the classification of foliations

## Holonomy groupoid

- The holonomy groupoid of a foliation is a categorical generalization of the holonomy pseudogroup that captures the global structure of the foliation
- The objects of the holonomy groupoid are points on the manifold, and the morphisms are equivalence classes of paths between points that are tangent to the leaves of the foliation
- The holonomy groupoid provides a unified framework for studying the geometry and topology of foliations

## Reeb stability theorem

- The Reeb stability theorem is a fundamental result in foliation theory that describes the local structure of codimension one foliations
- The theorem states that if a codimension one foliation has no compact leaves, then it is locally trivial, meaning it looks like a product of a leaf and an interval
- The Reeb stability theorem has important applications in the study of dynamical systems and the classification of 3-manifolds

## Foliations vs fibrations

- Foliation and fibrations are two important classes of geometric structures that describe how a manifold can be decomposed into submanifolds
- While foliations and fibrations share some similarities, they also have important differences that distinguish them as distinct concepts
- Understanding the relationship between foliations and fibrations is crucial for developing a deeper understanding of the geometry and topology of manifolds

## Similarities between foliations and fibrations

- Both foliations and fibrations partition a manifold into a disjoint union of submanifolds called leaves (for foliations) or fibers (for fibrations)
- The leaves of a foliation and the fibers of a fibration are required to vary smoothly, meaning that nearby leaves or fibers are close to each other in a suitable sense
- Foliation and fibrations can both be described locally using adapted coordinate systems, such as foliation charts or fibered coordinate charts

## Differences between foliations and fibrations

- The main difference between foliations and fibrations is that the fibers of a fibration are required to be diffeomorphic to each other, while the leaves of a foliation may have different topologies
- Fibrations have a global projection map that sends each point in the total space to its corresponding fiber, while foliations do not necessarily have a global projection map
- The holonomy of a fibration is always trivial, meaning that parallel transport along loops in the base space always brings a point back to its starting fiber, while the holonomy of a foliation can be nontrivial

## Riemannian foliations

- Riemannian foliations are a special class of foliations that are equipped with a Riemannian metric that is compatible with the foliation structure
- The study of Riemannian foliations combines ideas from foliation theory and Riemannian geometry to understand the interplay between the intrinsic geometry of the leaves and the extrinsic geometry of the ambient manifold
- Riemannian foliations have important applications in various areas of mathematics, such as dynamical systems, geometric group theory, and mathematical physics

## Definition of Riemannian foliations

- A Riemannian foliation is a foliation  $\mathcal{F}$  of a Riemannian manifold  $(M, g)$  such that the metric  $g$  is bundle-like with respect to  $\mathcal{F}$
- The bundle-like condition means that the metric  $g$  induces a Riemannian metric on the normal bundle of the foliation, which is the quotient of the tangent bundle of  $M$  by the tangent bundle of the foliation
- Equivalently, a Riemannian foliation can be defined as a foliation whose holonomy pseudogroup acts by isometries on the transverse metric

## Geometry of Riemannian foliations

- The geometry of Riemannian foliations is characterized by the interplay between the intrinsic geometry of the leaves and the extrinsic geometry of the ambient manifold
- The leaves of a Riemannian foliation are locally equidistant, meaning that the distance between nearby leaves is constant along the leaves
- The mean curvature of the leaves of a Riemannian foliation is a basic function, meaning it is constant along the leaves and descends to a function on the leaf space

## Molino's structure theory

- Molino's structure theory is a powerful tool for studying the geometry and topology of Riemannian foliations
- The theory describes the structure of the closures of the leaves of a Riemannian foliation, which form a singular Riemannian foliation with compact leaves
- Molino's theory provides a structure theorem for the leaf closures, showing that they are the fibers of a Riemannian submersion onto a Riemannian orbifold called the basic manifold

## Foliation cycles

- Foliation cycles are a generalization of the notion of cycles in homology theory to the setting of foliations
- The study of foliation cycles is important for understanding the global structure of foliations and their relationship to the topology of the ambient manifold
- Foliation cycles can be defined using various approaches, such as transverse measures, invariant measures, and geometric currents

## Transverse measures on foliations

- A transverse measure on a foliation is a measure on the space of leaves that is invariant under the holonomy pseudogroup
- Transverse measures can be used to define foliation cycles by integrating differential forms over the leaves of the foliation
- The space of transverse measures on a foliation is a convex cone that encodes important information about the dynamics and topology of the foliation

## Invariant measures and foliation cycles

- An invariant measure on a foliation is a measure on the ambient manifold that is invariant under the holonomy pseudogroup and supported on the leaves of the foliation
- Invariant measures can be used to define foliation cycles by pushing forward the measure to the space of leaves and integrating differential forms over the resulting measure
- The study of invariant measures on foliations has connections to ergodic theory and dynamical systems

## Geometric currents and foliation cycles

- A geometric current on a manifold is a continuous linear functional on the space of differential forms that generalizes the notion of a submanifold
- Foliation cycles can be defined as geometric currents that are tangent to the leaves of a foliation and closed under the exterior derivative
- The study of geometric currents and foliation cycles has applications in geometric measure theory and the study of minimal submanifolds

## Characteristic classes of foliations

- Characteristic classes are cohomological invariants that measure the global topological properties of vector bundles and foliations
- The study of characteristic classes of foliations is important for understanding the obstruction theory of foliations and their relationship to the topology of the ambient manifold
- Characteristic classes of foliations can be defined using various approaches, such as the Bott connection, the normal bundle, and the Godbillon-Vey class

## Bott connection and curvature

- The Bott connection is a natural connection on the normal bundle of a foliation that is defined using the Lie bracket of vector fields
- The curvature of the Bott connection is a basic form that measures the integrability of the normal bundle and the obstruction to the existence of a complementary foliation
- The Bott connection and its curvature can be used to define characteristic classes of foliations, such as the Pontryagin classes and the Godbillon-Vey class

## Characteristic classes of the normal bundle

- The normal bundle of a foliation is a vector bundle over the manifold whose fibers are the quotient spaces of the tangent spaces by the tangent spaces to the leaves
- The characteristic classes of the normal bundle, such as the Pontryagin classes and the Euler class, provide important information about the topology of the foliation and the ambient manifold
- The vanishing of certain characteristic classes of the normal bundle can imply the existence of complementary foliations or the triviality of the foliation

## Godbillon-Vey class

- The Godbillon-Vey class is a secondary characteristic class of codimension one foliations that measures the helical wobble of the foliation
- The Godbillon-Vey class is defined using the Bott connection and a 1-form that represents the foliation, and it can be interpreted as the volume of the holonomy pseudogroup
- The non-vanishing of the Godbillon-Vey class implies the existence of resilient leaves and has applications in the study of dynamical systems and the topology of 3-manifolds

## Foliations of low codimension

- Foliations of low codimension, such as codimension one, two, and three foliations, have been extensively studied and have special properties and classification results
- The study of foliations of low codimension is important for understanding the local and global structure of foliations and their relationship to the topology of the ambient manifold
- Foliations of low codimension often have connections to other areas of mathematics, such as contact geometry, symplectic topology, and algebraic topology

## Codimension one foliations

- Codimension one foliations are the simplest and most well-understood class of foliations, and they have been completely classified in dimensions up to three
- The Reeb stability theorem implies that codimension one foliations are locally trivial in the absence of compact leaves, and the Novikov compact leaf theorem gives a criterion for the existence of compact leaves
- Codimension one foliations have connections to contact geometry and the study of tight and overtwisted contact structures

## Codimension two foliations

- Codimension two foliations are more complex than codimension one foliations and have a rich theory that involves both geometry and topology
- The study of codimension two foliations often involves the use of techniques from symplectic topology, such as the study of symplectic fibrations and Lefschetz pencils
- The classification of codimension two foliations is an active area of research, with important results such as the Thurston stability theorem and the Eliashberg-Thurston theorem

## Codimension three foliations

- Codimension three foliations are even more complex than codimension two foliations and have been studied using techniques from homotopy theory and algebraic topology
- The study of codimension three foliations often involves the use of homotopy theoretic invariants, such as the Gelfand-Fuks cohomology and the Godbillon-Vey class
- The classification of codimension three foliations is largely open, with only partial results known in special cases such as homogeneous foliations and foliations with certain geometric structures

## Foliations of Lie groups

- Foliation of Lie groups are an important class of foliations that arise naturally in the study of homogeneous spaces and geometric structures
- The study of foliations of Lie groups involves the use of techniques from Lie theory, representation theory, and harmonic analysis
- Foliation of Lie groups often have connections to other areas of mathematics, such as number theory, dynamical systems, and mathematical physics

## Left-invariant foliations

- A left-invariant foliation of a Lie group  $G$  is a foliation that is invariant under the left action of  $G$  on itself
- Left-invariant foliations are determined by Lie subalgebras of the Lie algebra of  $G$ , and they can be studied using techniques from Lie theory and representation theory
- Examples of left-invariant foliations include the foliation of  $SL(2, \mathbb{R})$  by hyperbolic surfaces and the foliation of the Heisenberg group by parallel planes

## Right-invariant foliations

- A right-invariant foliation of a Lie group  $G$  is a foliation that is invariant under the right action of  $G$  on itself
- Right-invariant foliations are determined by Lie subgroups of  $G$ , and they can be studied using techniques from group theory and homogeneous geometry
- Examples of right-invariant foliations include the foliation of  $SO(3)$  by cosets of  $SO(2)$  and the foliation of the Euclidean group by parallel lines



## Bi-invariant foliations

- A bi-invariant foliation of a Lie group  $G$  is a foliation that is invariant under both the left and right actions of  $G$  on itself
- Bi-invariant foliations are determined by ideals in the Lie algebra of  $G$ , and they can be studied using techniques from the structure theory of Lie algebras
- Examples of bi-invariant foliations include the foliation of  $SU(2)$  by cosets of  $U(1)$  and the foliation of the Heisenberg group by cosets of its center

## Applications of foliations

- Foliation have numerous applications in various areas of mathematics, including dynamical systems, topology, and mathematical physics
- The study of foliations provides a unifying framework for understanding the geometry and topology of manifolds and their submanifolds
- The applications of foliations often involve the use of techniques from other areas of mathematics, such as analysis, algebra, and geometry

## Foliations in dynamical systems

- Foliation arise naturally in the study of dynamical systems, where they can be used to describe the global structure of invariant sets and the stability of trajectories
- The study of foliations in dynamical systems often involves the use of techniques from hyperbolic geometry, ergodic theory, and smooth ergodic theory
- Examples of foliations in dynamical systems include the stable and unstable foliations of

# Unit 9 – Variational Principles & Geodesic Equations

## 9.1 Calculus of variations

Calculus of variations is a powerful mathematical tool for optimizing functionals. It finds the function that extremizes a given functional, subject to constraints. This field has wide-ranging applications in physics, engineering, and economics.

The fundamental problem is finding a function that minimizes or maximizes a functional. Key concepts include functionals, function spaces, extrema, variations, and the Euler-Lagrange equation. These tools allow us to solve complex optimization problems in many fields.

### Fundamental concepts of calculus of variations

- Calculus of variations is a field of mathematical analysis that deals with optimizing functionals, which are mappings from a set of functions to the real numbers
- It has wide-ranging applications in physics, engineering, and economics, where it is used to find the function that extremizes a given functional subject to certain constraints
- The fundamental problem in calculus of variations is to find a function  $u(x)$  that minimizes or maximizes a given functional  $J[u]$ , often subject to boundary conditions or other constraints

### Functionals and function spaces

- A functional  $J[u]$  is a mapping that assigns a real number to each function  $u$  in a certain function space
- Function spaces are sets of functions that share certain properties, such as continuity, differentiability, or integrability
- Examples of functionals include the length of a curve  $L[u] = \int_a^b \sqrt{1 + u'(x)^2} dx$  and the energy of a system  $E[u] = \int_a^b (u'(x)^2 - u(x)^2) dx$

### Extrema of functionals

- An extremum of a functional is a function  $u^*$  that minimizes or maximizes the functional among all functions in the function space
- A necessary condition for  $u^*$  to be an extremum is that the first variation of the functional vanishes at  $u^*$ , i.e.,  $\delta J[u^*; v] = 0$  for all admissible variations  $v$
- A sufficient condition for  $u^*$  to be a minimum is that the second variation of the functional is non-negative at  $u^*$ , i.e.,  $\delta^2 J[u^*; v, v] \geq 0$  for all admissible variations  $v$

### Weak and strong extrema

- A function  $u^*$  is a weak extremum of a functional if it satisfies the necessary condition  $\delta J[u^*; v] = 0$  for all admissible variations  $v$

- A function  $u^*$  is a strong extremum of a functional if it satisfies both the necessary and sufficient conditions for an extremum
- Weak extrema are not necessarily global extrema, while strong extrema are always global extrema in their function space

## Variations and variational derivatives

- A variation of a function  $u$  is a small perturbation  $v$  added to  $u$ , i.e.,  $u_\varepsilon = u + \varepsilon v$ , where  $\varepsilon$  is a small parameter
- The first variation of a functional  $J[u]$  at  $u$  in the direction  $v$  is defined as  $\delta J[u; v] = \lim_{\varepsilon \rightarrow 0} \frac{J[u + \varepsilon v] - J[u]}{\varepsilon}$
- The variational derivative of a functional  $J[u]$  is a function  $\frac{\delta J}{\delta u}$  such that  $\delta J[u; v] = \int_a^b \frac{\delta J}{\delta u}(x) v(x) dx$  for all admissible variations  $v$

## Euler-Lagrange equation

- The Euler-Lagrange equation is a necessary condition for a function to be an extremum of a functional
- It is a second-order differential equation that the extremizing function must satisfy, along with any boundary conditions or constraints
- The Euler-Lagrange equation arises from setting the first variation of the functional equal to zero and applying the fundamental lemma of calculus of variations

## Derivation of Euler-Lagrange equation

- Consider a functional of the form  $J[u] = \int_a^b L(x, u(x), u'(x)) dx$ , where  $L$  is the Lagrangian of the system
- The first variation of  $J[u]$  at  $u$  in the direction  $v$  is given by  $\delta J[u; v] = \int_a^b \left( \frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u'} v' \right) dx$
- Setting  $\delta J[u; v] = 0$  for all admissible variations  $v$  and applying integration by parts leads to the Euler-Lagrange equation  $\frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u'} = 0$

## First variation and its properties

- The first variation  $\delta J[u; v]$  measures the rate of change of the functional  $J[u]$  in the direction of the variation  $v$
- The first variation is a linear functional in  $v$ , i.e.,  $\delta J[u; \alpha v_1 + \beta v_2] = \alpha \delta J[u; v_1] + \beta \delta J[u; v_2]$  for all scalars  $\alpha, \beta$  and variations  $v_1, v_2$
- The first variation satisfies the product rule  $\delta(J_1[u]J_2[u]) = J_1[u]\delta J_2[u] + J_2[u]\delta J_1[u]$  and the chain rule  $\delta(J[G[u]]) = \frac{\delta J}{\delta G} \delta G[u]$

## Second variation and sufficient conditions

- The second variation  $\delta^2 J[u; v, w]$  measures the second-order change of the functional  $J[u]$  in the directions of the variations  $v$  and  $w$
- The second variation is a bilinear functional in  $v$  and  $w$ , i.e.,  $\delta^2 J[u; \alpha v_1 + \beta v_2, w] = \alpha \delta^2 J[u; v_1, w] + \beta \delta^2 J[u; v_2, w]$  for all scalars  $\alpha, \beta$  and variations  $v_1, v_2, w$

- A sufficient condition for a function  $u^*$  to minimize the functional  $J[u]$  is that  $\delta J[u^*; v] = 0$  for all admissible variations  $v$  and  $\delta^2 J[u^*; v, v] \geq 0$  for all non-zero admissible variations  $v$

## Generalizations of Euler-Lagrange equation

- The Euler-Lagrange equation can be generalized to functionals involving higher-order derivatives, such as  $J[u] = \int_a^b L(x, u(x), u'(x), u''(x)) dx$
- In this case, the Euler-Lagrange equation becomes  $\frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u'} + \frac{d^2}{dx^2} \frac{\partial L}{\partial u''} = 0$
- The Euler-Lagrange equation can also be generalized to functionals involving multiple functions, such as  $J[u_1, \dots, u_n] = \int_a^b L(x, u_1(x), \dots, u_n(x), u_1'(x), \dots, u_n'(x)) dx$ , leading to a system of coupled Euler-Lagrange equations

## Variational problems with constraints

- Many variational problems involve finding the extrema of a functional subject to certain constraints on the admissible functions
- Constraints can arise from physical or geometrical considerations, such as fixed boundary conditions, prescribed arc length, or conservation laws
- The presence of constraints modifies the Euler-Lagrange equation and introduces additional terms, such as Lagrange multipliers or transversality conditions

## Isoperimetric problems and Lagrange multipliers

- An isoperimetric problem is a variational problem where the functional is extremized subject to a constraint of the form  $G[u] = \int_a^b g(x, u(x), u'(x)) dx = c$ , where  $c$  is a constant
- The isoperimetric constraint can be incorporated into the functional using a Lagrange multiplier  $\lambda$ , leading to the modified functional  $J_\lambda[u] = J[u] + \lambda G[u]$
- The Euler-Lagrange equation for the modified functional is  $\frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial u'} + \lambda \left( \frac{\partial g}{\partial u} - \frac{d}{dx} \frac{\partial g}{\partial u'} \right) = 0$ , along with the isoperimetric constraint  $G[u] = c$

## Holonomic and non-holonomic constraints

- Holonomic constraints are constraints that can be expressed as equations involving the functions and their derivatives, such as  $u(a) = u_a$ ,  $u(b) = u_b$ , or  $G[u] = c$
- Non-holonomic constraints are constraints that cannot be expressed as equations, such as inequality constraints  $u(x) \geq 0$  or integral constraints  $\int_a^b u(x) dx = c$
- Holonomic constraints can be incorporated into the functional using Lagrange multipliers, while non-holonomic constraints require more advanced techniques, such as the Karush-Kuhn-Tucker conditions or the Pontryagin maximum principle

## Transversality conditions and natural boundary conditions

- Transversality conditions are additional conditions that the extremizing function must satisfy at the boundary points where the function or its derivatives are not prescribed

- Transversality conditions arise from the boundary terms in the first variation of the functional and ensure that the extremizing function is "perpendicular" to the boundary in a suitable sense
- Natural boundary conditions are boundary conditions that are not prescribed a priori but follow from the transversality conditions, such as  $\frac{\partial L}{\partial u'}(a) = 0$  or  $\frac{\partial L}{\partial u'}(b) = 0$

## Hamilton's principle and variational mechanics

- Hamilton's principle is a variational principle in classical mechanics that states that the motion of a system between two fixed points in configuration space is such that the action integral  $S[q] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt$  is stationary, where  $L$  is the Lagrangian of the system
- The Euler-Lagrange equation for Hamilton's principle yields the equations of motion for the system, which are second-order differential equations in the generalized coordinates  $q(t)$
- Hamilton's principle provides a unified framework for deriving the equations of motion in various branches of mechanics, such as particle dynamics, rigid body dynamics, and continuum mechanics

## Direct methods in calculus of variations

- Direct methods are techniques for proving the existence of minimizers of functionals without explicitly solving the Euler-Lagrange equation
- Direct methods rely on functional analytic tools, such as compactness, lower semicontinuity, and coercivity, to establish the existence of minimizers in suitable function spaces
- Direct methods are particularly useful for functionals that lack smoothness or convexity properties, or for problems with constraints or singular behavior

## Existence of minimizers and lower semicontinuity

- A functional  $J[u]$  is said to have a minimizer in a function space  $X$  if there exists a function  $u^* \in X$  such that  $J[u^*] \leq J[u]$  for all  $u \in X$
- A sufficient condition for the existence of a minimizer is the sequential lower semicontinuity of the functional, which means that  $J[u] \leq \liminf_{n \rightarrow \infty} J[u_n]$  for any sequence  $u_n$  converging to  $u$  in  $X$
- Lower semicontinuity is often established using convexity or compactness arguments, such as the Banach-Alaoglu theorem or the Rellich-Kondrachov compactness theorem

## Tonelli's theorem and its applications

- Tonelli's theorem provides sufficient conditions for the existence of a minimizer of a functional of the form  $J[u] = \int_a^b L(x, u(x), u'(x)) dx$  in the Sobolev space  $W^{1,p}(a, b)$
- The conditions are: (1)  $L(x, u, p)$  is convex in  $p$  for almost all  $x$  and all  $u$ ; (2)  $L(x, u, p) \geq \alpha(x) + \beta|p|^p$  for some  $\alpha \in L^1(a, b)$ ,  $\beta > 0$ , and  $p > 1$ ; (3)  $L(x, u, p)$  is measurable in  $x$  for all  $(u, p)$  and continuous in  $(u, p)$  for almost all  $x$
- Tonelli's theorem has applications in various areas, such as nonlinear elasticity, image processing, and optimal control theory

## Regularity of minimizers and Hilbert's invariant integral

- Regularity theory studies the smoothness properties of minimizers of functionals, such as their continuity, differentiability, or higher regularity

- A key tool in regularity theory is the Euler-Lagrange equation, which provides a necessary condition for minimizers in the form of a differential equation
- Hilbert's invariant integral is a functional of the form  $J[u] = \int_a^b L(x, u(x), u'(x)) dx$  that is invariant under a certain group of transformations, such as translations or rotations
- The invariance of the functional leads to conservation laws for the minimizers, which can be used to establish their regularity or symmetry properties

## Lavrentiev phenomenon and relaxation methods

- The Lavrentiev phenomenon refers to the situation where the infimum of a functional over a certain function space is strictly less than the infimum over a dense subspace
- The Lavrentiev phenomenon occurs when the functional lacks lower semicontinuity or coercivity, and it indicates a gap between the original problem and its relaxed version
- Relaxation methods are techniques for extending the functional to a larger space where the infimum is attained, often by convexifying the functional or by adding lower semicontinuous terms
- Relaxation methods provide a way to approximate the original problem by a sequence of more tractable problems, and to establish the existence of generalized minimizers

## Applications of calculus of variations

- Calculus of variations has numerous applications in geometry, physics, engineering, and other fields where optimization problems arise
- Many physical principles, such as the principle of least action, the principle of minimum potential energy, or the maximum entropy principle, can be formulated as variational problems
- The solutions to variational problems often have important geometric or physical interpretations, such as geodesics, minimal surfaces, or equilibrium configurations

## Geodesics and shortest paths in Riemannian geometry

- A geodesic on a Riemannian manifold is a curve that locally minimizes the length functional  $L[\gamma] = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt$ , where  $g$  is the Riemannian metric and  $\gamma(t)$  is a parametrized curve
- Geodesics are the generalization of straight lines in Euclidean space to curved spaces, and they play a fundamental role in Riemannian geometry
- The Euler-Lagrange equation for the length functional yields the geodesic equation  $\ddot{\gamma}^k + \Gamma_{ij}^k \dot{\gamma}^i \dot{\gamma}^j = 0$ , where  $\Gamma_{ij}^k$  are the Christoffel symbols of the metric
- Geodesics have applications in general relativity, where they describe the motion of free-falling particles in curved spacetime, and in computer graphics, where they are used for shortest path planning and mesh parameterization

## Minimal surfaces and soap films

- A minimal surface is a surface that locally minimizes the area functional  $A[S] = \int_S dA$ , where  $dA$  is the area element of the surface
- Minimal surfaces are characterized by having zero mean curvature, which means that their principal curvatures are equal and opposite at each point

- Soap films are physical realizations of minimal surfaces, as they minimize the surface tension energy subject to the constraint of enclosing a fixed volume
- The Euler-Lagrange equation for the area functional yields the minimal surface equation  $\Delta_S X = 0$ , where  $\Delta_S$  is the Laplace-Beltrami operator on the surface and  $X$  is the position vector of the surface
- Minimal surfaces have applications in architecture, where they are used for designing efficient and aesthetically pleasing structures, and in materials science, where they arise in the study of crystal growth and self-assembly

## Elastica and rod theory

- An elastica is a curve that minimizes the bending energy functional  $E[\gamma] = \int_a^b \kappa(t)^2 ds$

## 9.2 Euler-Lagrange equations

The Euler-Lagrange equations are a cornerstone of variational calculus, providing a way to find extrema of functionals. These equations bridge the gap between calculus and physics, offering a powerful tool for describing the behavior of physical systems.

In metric differential geometry, Euler-Lagrange equations find applications in determining geodesics, minimal surfaces, and harmonic maps. They provide a mathematical framework for understanding the geometry of curved spaces and optimizing paths on manifolds.

### Derivation of Euler-Lagrange equations

- The Euler-Lagrange equations are a set of differential equations that provide a necessary condition for a function to be an extremum of a functional
- They are derived by considering small variations of the function and setting the first variation of the functional to zero
- The derivation involves techniques from calculus of variations, a branch of mathematical analysis dealing with functionals and their extrema

## Functional variations in calculus

### Concept of functional derivatives

- Functional derivatives extend the concept of partial derivatives to the realm of functionals, which are mappings from a space of functions to real numbers
- The functional derivative  $\frac{\delta F}{\delta f(x)}$  measures the rate of change of the functional  $F[f]$  with respect to variations of the function  $f(x)$  at a specific point  $x$
- Functional derivatives play a crucial role in the calculus of variations and the derivation of the Euler-Lagrange equations

### Extrema of functionals

- Functionals can have extrema (maxima or minima) just like ordinary functions
- Finding the extrema of functionals is a central problem in the calculus of variations and has applications in physics, geometry, and optimization

- The Euler-Lagrange equations provide a necessary condition for a function to be an extremum of a functional, analogous to the vanishing of partial derivatives for ordinary functions

## Lagrangian formulation

### Configuration space and generalized coordinates

- The configuration space is the set of all possible positions or configurations of a system, often described by generalized coordinates  $q_i$
- Generalized coordinates are a set of independent variables that uniquely specify the configuration of the system (e.g., Cartesian coordinates, polar coordinates, or angles)
- The choice of generalized coordinates can simplify the description of the system and the form of the Euler-Lagrange equations

### Kinetic and potential energy in Lagrangian

- The Lagrangian  $L(q, \dot{q}, t)$  is a function that describes the dynamics of a system in terms of its kinetic and potential energy
- The kinetic energy  $T(\dot{q})$  depends on the generalized velocities  $\dot{q}_i$  and represents the energy associated with the motion of the system
- The potential energy  $V(q)$  depends on the generalized coordinates  $q_i$  and represents the energy associated with the position or configuration of the system

### Principle of least action

- The principle of least action states that the path taken by a system between two points in configuration space is the one that minimizes the action functional  $S[q] = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$
- The action functional is a scalar quantity that depends on the entire path  $q(t)$  and its derivatives  $\dot{q}(t)$
- The principle of least action is a fundamental variational principle in physics and leads to the Euler-Lagrange equations

## Euler-Lagrange equations

### General form of Euler-Lagrange equations

- The Euler-Lagrange equations are a set of second-order differential equations that describe the dynamics of a system in the Lagrangian formulation
- For a system with  $n$  generalized coordinates  $q_i$ , the Euler-Lagrange equations are given by  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$  for  $i = 1, \dots, n$
- The Euler-Lagrange equations are obtained by requiring that the action functional  $S[q]$  is stationary with respect to small variations of the path  $q(t)$

### Coordinate-free representation

- The Euler-Lagrange equations can be written in a coordinate-free form using the language of differential geometry



- In this representation, the Lagrangian is a function on the tangent bundle  $TQ$  of the configuration space  $Q$ , and the Euler-Lagrange equations are expressed using the exterior derivative and the vertical endomorphism
- The coordinate-free representation emphasizes the geometric structure underlying the Euler-Lagrange equations and facilitates their generalization to more abstract settings

## Euler-Lagrange equations vs Newton's laws

- The Euler-Lagrange equations provide an alternative formulation of classical mechanics, equivalent to Newton's laws of motion
- While Newton's laws are expressed in terms of forces and accelerations, the Euler-Lagrange equations are expressed in terms of the Lagrangian and generalized coordinates
- The Lagrangian formulation has advantages in dealing with constrained systems, symmetries, and conservation laws, and is more easily extended to quantum mechanics and field theory

## Generalized momenta and Hamiltonian

### Legendre transformation

- The Legendre transformation is a mathematical technique that relates the Lagrangian formulation to the Hamiltonian formulation of classical mechanics
- It defines the generalized momenta  $p_i = \frac{\partial L}{\partial \dot{q}_i}$  as conjugate variables to the generalized coordinates  $q_i$
- The Hamiltonian  $H(q, p, t)$  is obtained from the Lagrangian  $L(q, \dot{q}, t)$  by the Legendre transformation

$$H = \sum_{i=1}^n p_i \dot{q}_i - L$$

## Hamiltonian formulation vs Lagrangian formulation

- The Hamiltonian formulation of classical mechanics is based on the Hamiltonian function  $H(q, p, t)$  and the generalized coordinates  $q_i$  and momenta  $p_i$
- Hamilton's equations  $\dot{q}_i = \frac{\partial H}{\partial p_i}$  and  $\dot{p}_i = -\frac{\partial H}{\partial q_i}$  describe the dynamics of the system in the Hamiltonian formulation
- The Hamiltonian formulation is equivalent to the Lagrangian formulation but emphasizes the role of generalized momenta and has a more symmetric structure

## Conservation laws from Euler-Lagrange equations

### Noether's theorem

- Noether's theorem establishes a profound connection between symmetries and conservation laws in physical systems described by the Euler-Lagrange equations
- It states that for every continuous symmetry of the Lagrangian (i.e., a transformation that leaves the Lagrangian invariant), there is a corresponding conserved quantity
- Examples of symmetries include time translation invariance (energy conservation), spatial translation invariance (momentum conservation), and rotational invariance (angular momentum conservation)

## Conservation of energy and momentum

- The conservation of energy and momentum are two fundamental conservation laws that follow from the Euler-Lagrange equations and Noether's theorem
- Energy conservation arises from the time translation invariance of the Lagrangian and is expressed by the constancy of the Hamiltonian  $H(q, p, t)$  along the solutions of the Euler-Lagrange equations
- Momentum conservation arises from the spatial translation invariance of the Lagrangian and is expressed by the constancy of the generalized momenta  $P_i$  conjugate to cyclic coordinates (i.e., coordinates that do not appear in the Lagrangian)

## Applications in metric differential geometry

### Geodesic equations from variational principles

- Geodesics are the shortest paths between points on a Riemannian manifold (a manifold with a Riemannian metric) and can be obtained as solutions of the Euler-Lagrange equations
- The Lagrangian for geodesics is given by the kinetic energy  $L = \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$ , where  $g_{ij}(x)$  are the components of the Riemannian metric
- The resulting Euler-Lagrange equations are the geodesic equations, which describe the motion of particles in curved spaces and the geometry of Riemannian manifolds

### Minimal surfaces and Plateau's problem

- Minimal surfaces are surfaces with zero mean curvature that minimize the area functional  $A[S] = \int_S dA$  among all surfaces with a given boundary
- Plateau's problem is the variational problem of finding a minimal surface spanning a given closed curve in space
- The Euler-Lagrange equations for the area functional lead to the minimal surface equation, a nonlinear partial differential equation that describes the shape of minimal surfaces

### Harmonic maps between Riemannian manifolds

- Harmonic maps are a generalization of geodesics and minimal surfaces to mappings between Riemannian manifolds
- They are defined as critical points of the energy functional  $E[\phi] = \frac{1}{2} \int_M |d\phi|^2 dV_M$ , where  $\phi : M \rightarrow N$  is a map between Riemannian manifolds  $M$  and  $N$
- The Euler-Lagrange equations for the energy functional are the harmonic map equations, a system of nonlinear partial differential equations that describe the geometry of harmonic maps and their applications in physics and geometry

## Numerical methods for Euler-Lagrange equations

### Finite difference schemes

- Finite difference schemes are numerical methods for approximating the solutions of the Euler-Lagrange equations by discretizing the equations on a grid

- They replace derivatives with finite difference approximations and lead to a system of algebraic equations that can be solved numerically
- Finite difference schemes are simple to implement but may suffer from numerical instabilities and loss of accuracy for long-time simulations

## Variational integrators

- Variational integrators are numerical methods that preserve the variational structure of the Euler-Lagrange equations and lead to symplectic and momentum-preserving schemes
- They are based on discretizing the action functional  $S[q]$  rather than the Euler-Lagrange equations directly and lead to a discrete version of the least action principle
- Variational integrators have excellent long-time stability and conservation properties but may be more complex to implement than finite difference schemes

## Convergence and stability analysis

- Convergence and stability analysis are essential for assessing the accuracy and reliability of numerical methods for the Euler-Lagrange equations
- Convergence refers to the property that the numerical solution approaches the exact solution as the discretization parameters (e.g., time step or grid size) tend to zero
- Stability refers to the property that small perturbations in the initial conditions or numerical errors do not grow exponentially in time
- Rigorous convergence and stability analysis can be performed using techniques from numerical analysis and backward error analysis

## 9.3 Geodesic equations

Geodesic equations describe the shortest paths between points on curved spaces. They're fundamental to Riemannian geometry, helping us analyze the intrinsic geometry of surfaces and manifolds. These equations have wide-ranging applications in physics, engineering, and computer graphics.

Understanding geodesics is crucial for solving optimization problems involving shortest paths. They're characterized as both the shortest and straightest paths on curved spaces, parallel transporting their own tangent vectors. This concept is key to grasping particle and light behavior in general relativity.

### Definition of geodesic equations

- Geodesic equations are a system of second-order differential equations that describe the shortest paths between points on a curved space, known as geodesics
- These equations are fundamental to the study of Riemannian geometry and have applications in various fields, including physics, engineering, and computer graphics
- Understanding geodesic equations is essential for analyzing the intrinsic geometry of surfaces and manifolds, as well as for solving optimization problems involving shortest paths

### Geodesic as shortest path

- A geodesic is defined as the shortest path between two points on a curved space, such as a surface or a manifold

- In the context of Riemannian geometry, the length of a curve is measured using the metric tensor, which assigns a positive definite inner product to each tangent space
- The geodesic equation minimizes the length functional, ensuring that the resulting curve is the shortest path between its endpoints (provided it exists)

## Geodesic as straightest path

- Geodesics can also be characterized as the "straightest" paths on a curved space, in the sense that they parallel transport their own tangent vectors
- This means that the tangent vector to a geodesic remains parallel to itself when transported along the curve using the Levi-Civita connection
- The straightness property of geodesics is related to the notion of acceleration in curved spaces and is crucial for understanding the behavior of particles and light in general relativity

## Parameterization by arc length

- Geodesics are often parameterized by arc length, which is a natural choice for measuring distances along curves
- The arc length parameter, usually denoted by  $s$ , is defined such that the magnitude of the tangent vector  $\frac{dy}{ds}$  is always equal to one
- Parameterizing geodesics by arc length simplifies the geodesic equations and makes it easier to analyze their properties, such as the conservation of energy and momentum

## Derivation of geodesic equations

- The derivation of geodesic equations relies on the calculus of variations, which seeks to find curves that minimize or maximize a given functional
- In the case of geodesics, the functional to be minimized is the length functional, which measures the total length of a curve between two fixed points
- The minimization problem leads to a system of second-order differential equations known as the Euler-Lagrange equations, which characterize the geodesics

## Euler-Lagrange equations

- The Euler-Lagrange equations are a set of partial differential equations that arise in the calculus of variations when seeking to minimize or maximize a functional
- For the length functional, the Euler-Lagrange equations take the form  $\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0$ , where  $L$  is the Lagrangian function and  $x^i$  are the coordinates of the curve
- Solving the Euler-Lagrange equations leads to the geodesic equations, which describe the shortest paths on a curved space

## Christoffel symbols

- Christoffel symbols, denoted by  $\Gamma_{jk}^i$ , are the components of the Levi-Civita connection, which is a unique metric-compatible and torsion-free connection on a Riemannian manifold

- They quantify how the basis vectors of the tangent space change as one moves along the manifold, and they appear in the expression of the covariant derivative
- Christoffel symbols play a crucial role in the geodesic equations, as they encode the information about the curvature of the space and determine how geodesics behave

## Geodesic equations in local coordinates

- In local coordinates, the geodesic equations take the form  $\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$ , where  $x^i$  are the coordinates of the curve and  $s$  is the arc length parameter
- This system of second-order differential equations describes how the coordinates of a geodesic change as a function of the arc length, given the initial position and velocity
- The presence of the Christoffel symbols in the equations reflects the fact that geodesics are affected by the curvature of the space, which is encoded in the metric tensor

## Properties of geodesics

- Geodesics possess several important properties that distinguish them from arbitrary curves on a curved space
- These properties are related to the minimization principle that defines geodesics, as well as to the geometric structure of the underlying Riemannian manifold
- Understanding the properties of geodesics is crucial for analyzing their behavior and for applying them to various problems in physics, engineering, and other fields

## Existence and uniqueness

- The existence and uniqueness of geodesics are fundamental questions in Riemannian geometry, as they determine whether shortest paths between points always exist and whether they are unique
- The Hopf-Rinow theorem states that if a Riemannian manifold is complete (i.e., all geodesics can be extended indefinitely), then any two points can be connected by a minimal geodesic
- However, uniqueness is not always guaranteed, as there may be multiple geodesics with the same length connecting two points (e.g., on a sphere, where antipodal points are connected by infinitely many great circles)

## Affine parameter

- An affine parameter is a parametrization of a geodesic that is independent of the choice of coordinates and preserves the affine structure of the manifold
- In general relativity, the affine parameter is often denoted by  $\lambda$  and is defined such that the tangent vector  $\frac{dx^\mu}{d\lambda}$  satisfies the geodesic equation  $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$
- The affine parameter is particularly useful for describing the motion of particles and light in curved spacetime, as it simplifies the equations of motion and makes their physical interpretation more transparent

## Geodesic deviation

- Geodesic deviation is the phenomenon whereby initially parallel geodesics can diverge or converge due to the curvature of the space

- It is quantified by the Jacobi equation, which describes how the separation vector between two nearby geodesics evolves along the curves
- The behavior of geodesic deviation is determined by the Riemann curvature tensor, which measures the non-commutativity of the covariant derivative and encodes the intrinsic curvature of the manifold

## Jacobi fields

- Jacobi fields are vector fields along a geodesic that describe the infinitesimal separation between the given geodesic and nearby geodesics
- They satisfy the Jacobi equation, which is a second-order linear differential equation involving the Riemann curvature tensor
- Jacobi fields play a crucial role in the study of geodesic deviation, as they provide a way to analyze the stability of geodesics and to compute the focusing and defocusing of geodesic congruences

## Geodesics on surfaces

- The study of geodesics on surfaces is a classical topic in differential geometry, with a rich history dating back to the works of Euler, Gauss, and other mathematicians
- Surfaces are two-dimensional Riemannian manifolds, and their geodesics exhibit special properties that are not present in higher dimensions
- Understanding geodesics on surfaces is important for various applications, such as computer graphics, robotics, and the design of curved structures

## Geodesic curvature

- The geodesic curvature of a curve on a surface measures how much the curve deviates from being a geodesic
- It is defined as the magnitude of the projection of the curve's acceleration vector onto the tangent plane of the surface
- Geodesics on a surface have zero geodesic curvature, which means that they are the "straightest" possible curves on the surface

## Clairaut's relation

- Clairaut's relation is a classic result in differential geometry that relates the geodesic curvature of a curve on a surface of revolution to its distance from the axis of revolution
- It states that the product of the distance and the sine of the angle between the curve and the meridian is constant along the curve
- Clairaut's relation is particularly useful for analyzing geodesics on surfaces of revolution, such as spheres, cylinders, and cones

## Geodesic triangles

- A geodesic triangle on a surface is a triangle whose sides are geodesics
- The geometry of geodesic triangles is governed by the Gauss-Bonnet theorem, which relates the total angle defect of the triangle (i.e., the difference between the sum of its angles and  $\pi$ ) to the integral of the Gaussian curvature over the triangle's interior

- Geodesic triangles play a crucial role in the development of non-Euclidean geometries, such as spherical and hyperbolic geometry, where the angle sum of a triangle can be greater or less than  $\pi$

## Gauss-Bonnet theorem

- The Gauss-Bonnet theorem is a fundamental result in differential geometry that relates the geometry of a surface to its topology
- It states that the integral of the Gaussian curvature over a compact surface, plus the integral of the geodesic curvature along the boundary, is equal to  $2\pi$  times the Euler characteristic of the surface
- The Gauss-Bonnet theorem has numerous applications in geometry, topology, and physics, including the study of geodesic triangles, the classification of surfaces, and the analysis of gravitational lensing in general relativity

## Geodesics in general relativity

- In general relativity, geodesics play a central role in describing the motion of particles and the propagation of light in curved spacetime
- The geometry of spacetime is determined by the Einstein field equations, which relate the curvature of spacetime to the presence of matter and energy
- Geodesics in general relativity exhibit unique features that are not present in Riemannian geometry, such as the distinction between timelike, spacelike, and null geodesics

## Timelike vs spacelike geodesics

- In general relativity, geodesics can be classified into three types based on the sign of their tangent vector's norm: timelike, spacelike, and null
- Timelike geodesics represent the world lines of massive particles and have a tangent vector with a negative norm (i.e.,  $g(\dot{\gamma}, \dot{\gamma}) < 0$ )
- Spacelike geodesics have a tangent vector with a positive norm (i.e.,  $g(\dot{\gamma}, \dot{\gamma}) > 0$ ) and do not correspond to the motion of physical particles, but they can be used to analyze the spatial geometry of spacetime

## Null geodesics

- Null geodesics, also known as lightlike geodesics, have a tangent vector with a zero norm (i.e.,  $g(\dot{\gamma}, \dot{\gamma}) = 0$ ) and represent the paths of light rays in curved spacetime
- They play a crucial role in the study of gravitational lensing, the formation of black hole shadows, and the propagation of gravitational waves
- Null geodesics have special properties, such as the absence of proper time and the invariance of the affine parameter under rescaling

## Geodesic motion in spacetime

- The motion of particles in general relativity is governed by the geodesic equation, which describes how the particle's four-velocity (tangent vector) changes along its world line
- In the presence of gravitational fields, particles follow geodesics that are "straight" in the sense of parallel transport, but curved in the sense of the embedding space

- The geodesic equation in general relativity takes into account the effects of spacetime curvature, such as the precession of orbits and the deflection of light

## Geodesic completeness

- Geodesic completeness is a property of spacetimes that ensures the existence of geodesics between any two points, as well as the possibility of extending geodesics indefinitely
- In general relativity, not all spacetimes are geodesically complete, as they may contain singularities (e.g., black holes) where geodesics cannot be extended beyond a certain point
- The study of geodesic completeness is important for understanding the global structure of spacetime and for formulating singularity theorems, which establish the conditions under which singularities are inevitable

## Numerical methods for geodesics

- In many practical applications, such as computer graphics, robotics, and numerical relativity, it is necessary to compute geodesics numerically
- Numerical methods for geodesics aim to approximate the solutions of the geodesic equations on discrete domains, such as triangulated surfaces or computational grids
- The development of efficient and accurate numerical methods for geodesics is an active area of research, with various approaches tailored to specific problem settings

## Geodesic shooting

- Geodesic shooting is a numerical method for computing geodesics on Riemannian manifolds by solving an initial value problem
- Given an initial point and a tangent vector, the method "shoots" a geodesic by numerically integrating the geodesic equations using schemes such as Runge-Kutta methods
- Geodesic shooting is particularly useful for computing geodesics on implicitly defined manifolds, such as level sets of functions or solutions of partial differential equations

## Relaxation methods

- Relaxation methods are iterative techniques for computing geodesics by minimizing an energy functional that measures the deviation from the true geodesic
- These methods start with an initial guess for the geodesic (e.g., a straight line) and progressively refine it by moving the curve's points in the direction of steepest descent of the energy functional
- Examples of relaxation methods include the geodesic active contours algorithm and the fast marching method, which are widely used in image processing and computer vision

## Geodesic tracing on meshes

- Geodesic tracing is a technique for computing geodesics on triangulated surfaces by propagating a wavefront from a source point using a discrete version of the geodesic equations
- The method maintains a priority queue of vertices based on their distance from the source and updates the distances using a local stencil that approximates the exponential map



- Geodesic tracing is efficient and robust, making it suitable for applications in computer graphics, such as texture mapping, remeshing, and shape analysis

## Discrete geodesic calculus

- Discrete geodesic calculus is a framework for formulating and solving geodesic problems on discrete domains, such as graphs and simplicial complexes
- It extends the concepts of differential geometry, such as parallel transport, curvature, and the Laplace-Beltrami operator, to discrete settings using tools from algebraic topology and finite element analysis
- Discrete geodesic calculus provides a unified approach to various problems involving shortest paths, such as routing in networks, sampling on manifolds, and shape matching

## Applications of geodesic equations

- Geodesic equations have numerous applications across various fields, including physics, engineering, computer science, and applied mathematics
- These applications leverage the properties of geodesics as shortest paths and straightest curves to solve optimization problems, analyze dynamical systems, and study the geometry of curved spaces
- The versatility of geodesic equations stems from their ability to capture the intrinsic geometry of manifolds and to provide a natural framework for formulating and solving geometric problems

## Shortest path problems

- One of the most common applications of geodesic equations is in solving shortest path problems on curved spaces, such as surfaces and manifolds
- Examples include finding the shortest route between two points on the Earth's surface (e.g., in navigation systems), optimizing the layout of transportation networks, and computing minimal paths in robotic motion planning
- Geodesic equations provide a principled way to formulate and solve these problems, taking into account the intrinsic geometry of the space and the constraints imposed by the application

## Mechanics on curved spaces

- Geodesic equations play a fundamental role in the study of mechanics on curved spaces, such as the motion of particles and rigid bodies on surfaces and in curved spacetime
- In this context, geodesics represent the paths of least action, which minimize the integral of the Lagrangian function over the path
- The geodesic equations can be derived from the principle of least action and provide a natural framework for formulating the equations of motion in the presence of curvature and constraints

## Geodesic flows and dynamical systems

- Geodesic flows are dynamical systems that describe the motion of particles along geodesics on a Riemannian manifold
- They are defined on the tangent bundle of the manifold and are governed by the geodesic equations, which can be written as a system of first-order differential equations

- The study of geodesic flows is central to the field of dynamical systems, as it provides insights into the global geometry of the manifold and the long-term behavior of trajectories (e.g., ergodicity, mixing, and chaos)

## Geodesic-based shape analysis

- Geodesic equations have found numerous applications in the field of shape analysis, which aims to develop mathematical and computational tools for comparing, classifying, and processing shapes
- Geodesic distances and geodesic paths provide a natural way to measure the similarity between shapes and to define shape descriptors that are invariant under isometric transformations
- Examples of geodesic-based shape analysis techniques include the computation of Gaussian curvature, the detection of symmetries and self-similarities, and the registration of deformable shapes using geodesic shooting and diffeomorphic matching

## 9.4 Energy functionals and harmonic maps

Energy functionals are key tools in studying harmonic maps between Riemannian manifolds. They provide a way to measure the "energy" of a map and allow us to use calculus of variations to analyze properties of maps that minimize energy.

Harmonic maps are critical points of these energy functionals. They generalize concepts like harmonic functions and geodesics to maps between manifolds. Studying harmonic maps reveals deep connections between the geometry of different spaces and has applications across math and physics.

### Definition of energy functionals

- Energy functionals play a central role in the study of harmonic maps between Riemannian manifolds
- Provide a variational formulation for finding maps that minimize a certain notion of energy
- Enable the use of powerful techniques from calculus of variations and partial differential equations to study the existence, regularity, and properties of harmonic maps

### Energy of a map between manifolds

- Let  $f: (M, g) \rightarrow (N, h)$  be a smooth map between Riemannian manifolds  $(M, g)$  and  $(N, h)$
- The energy density of  $f$  at a point  $x \in M$  is defined as  $e(f)(x) = \frac{1}{2}|df(x)|^2$ , where  $|df(x)|$  denotes the Hilbert-Schmidt norm of the differential  $df(x): T_x M \rightarrow T_{f(x)} N$
- The energy of  $f$  is the integral of the energy density over the domain manifold  $M$ :  $E(f) = \int_M e(f) dV_g$ , where  $dV_g$  is the volume form of  $(M, g)$

### Dependence on metric tensors

- The energy of a map depends on the choice of Riemannian metrics  $g$  and  $h$  on the domain and target manifolds, respectively
- Changing the metrics will generally result in a different energy value for the same map
- The dependence on metrics allows for the study of how geometric properties of the manifolds influence the behavior of harmonic maps

## Variational formulation

- Harmonic maps are defined as critical points of the energy functional  $E(f)$  with respect to variations of the map  $f$
- A map  $f: (M, g) \rightarrow (N, h)$  is called harmonic if it satisfies the Euler-Lagrange equation for the energy functional, which is a system of nonlinear partial differential equations
- The variational formulation provides a powerful tool for studying the existence and properties of harmonic maps using techniques from calculus of variations

## Harmonic maps

- Harmonic maps are a central object of study in geometric analysis and have deep connections to various areas of mathematics and physics
- They provide a natural generalization of harmonic functions and geodesics to the setting of maps between Riemannian manifolds
- Harmonic maps have applications in minimal surface theory, topology, complex analysis, and mathematical physics

## Definition and motivation

- A map  $f: (M, g) \rightarrow (N, h)$  between Riemannian manifolds is called harmonic if it is a critical point of the energy functional  $E(f)$
- Harmonic maps are a natural generalization of harmonic functions (maps from a Riemannian manifold to  $\mathbb{R}$ ) and geodesics (harmonic maps from an interval to a Riemannian manifold)
- The study of harmonic maps is motivated by their role as a bridge between the geometry of the domain and target manifolds, as well as their applications in various fields

## Harmonic map equation

- A map  $f: (M, g) \rightarrow (N, h)$  is harmonic if and only if it satisfies the harmonic map equation:  $\tau(f) = \text{tr}_g \nabla df = 0$ , where  $\tau(f)$  is the tension field of  $f$
- The tension field is a vector field along the map  $f$  that measures the deviation of  $f$  from being a harmonic map
- The harmonic map equation is a nonlinear system of partial differential equations that generalizes the Laplace equation for harmonic functions

## Relationship to energy functionals

- Harmonic maps are critical points of the energy functional  $E(f)$ , which means they satisfy the Euler-Lagrange equation for this functional
- The Euler-Lagrange equation for the energy functional is equivalent to the harmonic map equation  $\tau(f) = 0$
- The energy functional provides a variational characterization of harmonic maps and enables the use of techniques from calculus of variations to study their existence and properties

## Examples of harmonic maps

- Harmonic functions: Maps  $f: (M, g) \rightarrow \mathbb{R}$  are harmonic if and only if they satisfy the Laplace equation  $\Delta_g f = 0$
- Geodesics: A curve  $\gamma: I \rightarrow (N, h)$  is a geodesic if and only if it is a harmonic map from the interval  $I$  with its standard metric to  $(N, h)$
- Conformal maps: A conformal map between Riemannian surfaces is harmonic with respect to the induced metrics
- Minimal immersions: An isometric immersion  $f: (M, g) \rightarrow (N, h)$  is minimal (has zero mean curvature) if and only if it is harmonic

## Existence and regularity of harmonic maps

- The existence and regularity theory for harmonic maps is a central topic in geometric analysis
- It involves the study of conditions under which harmonic maps exist, as well as their smoothness properties
- The development of this theory has led to significant advances in our understanding of the structure of harmonic maps and their applications

## Existence via minimizing energy

- One approach to establishing the existence of harmonic maps is by minimizing the energy functional within a suitable class of maps
- The direct method in the calculus of variations can be used to show the existence of energy-minimizing maps under appropriate assumptions on the domain and target manifolds
- Energy-minimizing maps are harmonic, as they are critical points of the energy functional

## Regularity theory for harmonic maps

- The regularity theory for harmonic maps concerns the smoothness properties of weak solutions to the harmonic map equation
- Under suitable assumptions, it can be shown that weakly harmonic maps are smooth ( $C^\infty$ ) or have a prescribed degree of Hölder continuity
- The regularity theory often relies on techniques from nonlinear partial differential equations, such as the use of maximum principles, Moser iteration, and Schauder estimates

## Unique continuation property

- Harmonic maps satisfy a unique continuation property, which states that if two harmonic maps agree on an open subset of the domain manifold, then they must agree everywhere
- This property is a consequence of the elliptic nature of the harmonic map equation and the maximum principle
- The unique continuation property is useful in the study of the structure and properties of harmonic maps

## Removable singularity theorem

- The removable singularity theorem states that if a harmonic map has a singular set of sufficiently small Hausdorff dimension, then the map can be extended to a harmonic map on the entire domain manifold
- This theorem is an important tool in the study of the structure of the singular set of harmonic maps
- The proof of the removable singularity theorem typically involves a careful analysis of the behavior of the map near the singular set and the use of the unique continuation property

## Harmonic map heat flow

- The harmonic map heat flow is a parabolic evolution equation that provides a natural approach to the study of harmonic maps
- It can be used to construct harmonic maps by evolving an initial map in the direction of steepest descent for the energy functional
- The study of the harmonic map heat flow has led to significant advances in our understanding of the existence, regularity, and properties of harmonic maps

## Heat flow equation for harmonic maps

- The harmonic map heat flow equation is given by  $\frac{\partial f}{\partial t} = \tau(f)$ , where  $f: M \times [0, T) \rightarrow N$  is a time-dependent map and  $\tau(f)$  is the tension field of  $f$
- The equation describes the evolution of a map  $f$  in the direction of steepest descent for the energy functional  $E(f)$
- The harmonic map heat flow can be viewed as a gradient flow for the energy functional in the space of maps between Riemannian manifolds

## Short-time existence and uniqueness

- The short-time existence and uniqueness of solutions to the harmonic map heat flow equation can be established under suitable assumptions on the initial map and the geometry of the domain and target manifolds
- The proof typically involves the use of the contraction mapping principle in appropriate function spaces
- The short-time existence and uniqueness result provides a foundation for the study of the long-time behavior of the harmonic map heat flow

## Long-time existence vs finite-time blowup

- The long-time behavior of the harmonic map heat flow depends on the geometry of the domain and target manifolds, as well as the properties of the initial map
- In some cases, the harmonic map heat flow may exist for all time and converge to a harmonic map as  $t \rightarrow \infty$
- In other cases, the solution may develop singularities in finite time, a phenomenon known as finite-time blowup
- The study of the long-time behavior of the harmonic map heat flow is an active area of research, with many open questions and conjectures

## Convergence to harmonic maps

- When the harmonic map heat flow exists for all time, it is natural to ask whether the solution converges to a harmonic map as  $t \rightarrow \infty$
- Under suitable assumptions, it can be shown that the harmonic map heat flow converges to a harmonic map, either in the smooth topology or in a weaker sense
- The convergence of the harmonic map heat flow to harmonic maps provides a constructive approach to the existence of harmonic maps and has applications in various settings

## Applications of harmonic maps

- Harmonic maps have found numerous applications in various areas of mathematics and physics
- These applications often rely on the interplay between the geometric and analytic properties of harmonic maps and the structure of the domain and target manifolds
- The study of harmonic maps has led to significant advances in our understanding of minimal surfaces, topology, complex analysis, and mathematical physics

## Minimal surfaces and minimal immersions

- Harmonic maps play a central role in the theory of minimal surfaces and minimal immersions
- An isometric immersion  $f: (M, g) \rightarrow (N, h)$  is minimal (has zero mean curvature) if and only if it is harmonic
- The study of harmonic maps has led to the discovery of new examples of minimal surfaces and has provided insights into their structure and properties

## Harmonic map representation of homotopy classes

- Harmonic maps can be used to represent homotopy classes of maps between Riemannian manifolds
- Given a homotopy class of maps  $[f]: M \rightarrow N$ , one can seek a harmonic representative of this class by minimizing the energy functional within the class
- The existence and uniqueness of harmonic representatives of homotopy classes depend on the topology and geometry of the manifolds involved

## Harmonic map approach to Teichmüller theory

- Harmonic maps have found important applications in Teichmüller theory, which studies the deformation spaces of Riemann surfaces
- The harmonic map approach to Teichmüller theory, developed by Eells and Sampson, provides a geometric and analytic framework for understanding the structure of Teichmüller spaces
- Harmonic maps between Riemann surfaces are closely related to quadratic differentials and have been used to study the geometry and topology of moduli spaces

## Harmonic maps in mathematical physics

- Harmonic maps arise naturally in various areas of mathematical physics, such as gauge theory, string theory, and general relativity

- In gauge theory, harmonic maps between Riemannian manifolds are closely related to Yang-Mills connections and have been used to study the structure of moduli spaces of connections
- In string theory, harmonic maps from Riemann surfaces to target manifolds are used to model the dynamics of strings and have led to important insights into the geometry of string theory backgrounds

## Variants and generalizations

- The theory of harmonic maps has been extended and generalized in various directions to encompass a wider range of geometric and analytic settings
- These variants and generalizations often aim to capture additional geometric or physical properties of the maps or to extend the applicability of the theory to new contexts
- The study of these variants and generalizations has led to new insights and connections between different areas of mathematics

## Biharmonic and polyharmonic maps

- Biharmonic maps are a generalization of harmonic maps that are defined as critical points of a higher-order energy functional involving the square of the Laplacian
- Polyharmonic maps are a further generalization that involve higher powers of the Laplacian
- The study of biharmonic and polyharmonic maps is motivated by their connections to higher-order geometric and physical theories, such as the theory of elasticity and the study of higher-order curvature flows

## Harmonic maps with potential

- Harmonic maps with potential are a generalization of harmonic maps that include an additional potential function in the energy functional
- The potential function can be used to model external forces or constraints acting on the map
- The study of harmonic maps with potential has applications in various areas, such as the theory of liquid crystals and the study of magnetic skyrmions

## Harmonic maps between singular spaces

- The theory of harmonic maps has been extended to the setting of maps between singular spaces, such as metric spaces with curvature bounds or algebraic varieties
- The study of harmonic maps between singular spaces requires the development of new analytic and geometric techniques to handle the singularities and the lack of smooth structure
- Harmonic maps between singular spaces have found applications in the study of the geometry and topology of singular spaces and in the theory of minimal submanifolds

## Harmonic maps in sub-Riemannian geometry

- Sub-Riemannian geometry is a generalization of Riemannian geometry that studies spaces equipped with a distribution of admissible directions and a metric defined only on this distribution
- The theory of harmonic maps has been extended to the sub-Riemannian setting, where the energy functional and the harmonic map equation are formulated using the sub-Riemannian structure

- Harmonic maps in sub-Riemannian geometry have found applications in the study of hypoelliptic operators, control theory, and the geometry of Carnot groups

## 9.5 Variation of arc length and minimal surfaces

Variation of arc length and minimal surfaces are key concepts in differential geometry. They explore how curves and surfaces can minimize length or area, respectively. These ideas connect to broader themes of optimization and variational problems in geometry.

Geodesics, which locally minimize arc length, and minimal surfaces, which locally minimize area, have wide-ranging applications. From physics to computer graphics, these mathematical objects help us understand natural phenomena and solve complex problems in various fields.

### Definition of arc length

- Arc length is a fundamental concept in differential geometry that measures the distance along a curve between two points
- It is a way to quantify the length of a curved path in a metric space, such as a Riemannian manifold or a Euclidean space
- Arc length is intrinsic to the curve itself and does not depend on the parametrization chosen to represent the curve

### Arc length formula

- The arc length of a curve  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  is given by the formula  $L(\gamma) = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt$
- This formula involves integrating the norm of the tangent vector  $\gamma'(t)$  with respect to the metric  $\langle \cdot, \cdot \rangle$  along the curve
- The arc length is invariant under reparametrization of the curve, meaning that different parametrizations of the same curve will yield the same arc length
- For a curve in Euclidean space, the arc length formula simplifies to

$$L(\gamma) = \int_a^b \sqrt{\sum_{i=1}^n \left(\frac{dy_i}{dt}\right)^2} dt$$

### Parametric curves

- Parametric curves are a way to represent curves in a metric space using a function  $\gamma : [a, b] \rightarrow M$  from an interval  $[a, b]$  to the space  $M$
- The parameter  $t \in [a, b]$  can be thought of as a "time" variable that traces out the curve as it varies from  $a$  to  $b$
- Parametric curves are useful for studying the geometry of curves, such as their length, curvature, and geodesic properties
- Examples of parametric curves include circles ( $x(t) = \cos t, y(t) = \sin t$ ), helices ( $x(t) = \cos t, y(t) = \sin t, z(t) = t$ ), and Bézier curves used in computer graphics

### Smooth curves

- A curve  $\gamma : [a, b] \rightarrow M$  is said to be smooth if its derivatives of all orders exist and are continuous



- Smooth curves have well-defined tangent vectors, curvature, and other geometric properties at every point along the curve
- The set of smooth curves forms an infinite-dimensional manifold, which is an important object of study in differential geometry
- Smooth curves can be approximated by piecewise linear curves, which are used in numerical computations and discrete differential geometry

## Variation of arc length

- Variation of arc length is a technique used to find curves that minimize or maximize the arc length functional  $L(\gamma) = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt$
- It involves considering small perturbations or variations of a curve and analyzing how the arc length changes under these variations
- The goal is to find curves that are critical points of the arc length functional, meaning that small perturbations do not change the arc length to first order

## Variational problem

- The problem of finding curves that minimize or maximize the arc length functional is an example of a variational problem
- Variational problems involve finding functions or curves that optimize a given functional, such as the arc length, energy, or action functional
- Variational problems arise in many areas of mathematics, physics, and engineering, such as classical mechanics, general relativity, and optimal control theory
- Techniques from calculus of variations, such as the Euler-Lagrange equation and the principle of least action, are used to solve variational problems

## Euler-Lagrange equation

- The Euler-Lagrange equation is a necessary condition for a curve  $\gamma$  to be a critical point of a functional  $J(\gamma) = \int_a^b L(t, \gamma(t), \gamma'(t)) dt$
- It states that the curve must satisfy the differential equation  $\frac{\partial L}{\partial \gamma} - \frac{d}{dt} \frac{\partial L}{\partial \gamma'} = 0$
- For the arc length functional, the Euler-Lagrange equation leads to the geodesic equation, which characterizes curves that locally minimize arc length
- The Euler-Lagrange equation is a powerful tool in variational calculus and has applications in classical mechanics (Hamilton's principle), general relativity (Einstein's equations), and quantum mechanics (Schrödinger's equation)

## Geodesics

- Geodesics are curves that locally minimize the arc length between two points in a metric space
- They are the analogues of straight lines in Euclidean space and great circles on a sphere
- Geodesics are characterized by the property that their acceleration vector is always perpendicular to their velocity vector, meaning that they have zero tangential acceleration

- Examples of geodesics include straight lines in Euclidean space, great circles on a sphere, and curves of shortest distance on a surface or Riemannian manifold

## Geodesic equations

- The geodesic equations are a system of second-order differential equations that characterize geodesics in a metric space
- For a curve  $\gamma(t) = (x^1(t), \dots, x^n(t))$  in local coordinates, the geodesic equations are  $\ddot{x}^k + \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0$ , where  $\Gamma_{ij}^k$  are the Christoffel symbols of the metric
- The Christoffel symbols encode information about the curvature of the space and how it affects the acceleration of geodesics
- The geodesic equations can be derived from the Euler-Lagrange equation applied to the arc length functional or from the principle of parallel transport along the curve

## Minimal surfaces

- Minimal surfaces are surfaces that locally minimize the area functional  $A(S) = \int_S dA$  among all surfaces with a given boundary curve
- They are the two-dimensional analogues of geodesics and have important applications in geometry, physics, and engineering
- Minimal surfaces have zero mean curvature at every point, meaning that their average curvature in all directions is zero

## Definition of minimal surface

- A surface  $S$  in a Riemannian manifold  $M$  is called minimal if its mean curvature vanishes identically, i.e.,  $H \equiv 0$  on  $S$
- Equivalently, a surface is minimal if and only if it is a critical point of the area functional under variations that fix the boundary of the surface
- Minimal surfaces can be characterized as surfaces whose coordinate functions are harmonic functions, satisfying the Laplace equation  $\Delta x^i = 0$
- Examples of minimal surfaces include planes, helicoids, catenoids, and the gyroid surface found in certain butterfly wing scales

## Mean curvature

- The mean curvature  $H$  of a surface at a point is the average of the principal curvatures  $K_1$  and  $K_2$  at that point, i.e.,  $H = \frac{1}{2}(K_1 + K_2)$
- It measures how much the surface deviates from being flat in a given direction
- The mean curvature is an extrinsic quantity that depends on the embedding of the surface in the ambient space
- Surfaces with positive mean curvature (such as spheres) are called convex, while those with negative mean curvature (such as hyperboloids) are called saddle surfaces

## Plateau's problem

- Plateau's problem is the problem of finding a minimal surface with a given boundary curve, named after the Belgian physicist Joseph Plateau
- It is a classic problem in geometric measure theory and calculus of variations, with important applications in soap films, capillary surfaces, and membrane physics
- The existence and regularity of solutions to Plateau's problem have been extensively studied, leading to important results such as the Douglas-Rado theorem and the Federer-Fleming regularity theorem
- Numerical methods, such as the finite element method and the level set method, have been developed to solve Plateau's problem computationally

## Soap films and bubbles

- Soap films and bubbles are physical realizations of minimal surfaces, formed by dipping a wire frame into a soap solution or blowing air into a soap film
- They minimize the surface area for a given enclosed volume, subject to the constraint of the wire frame or the pressure difference across the film
- The geometry of soap films and bubbles is governed by the Laplace-Young equation, which relates the mean curvature to the pressure difference and surface tension
- Soap films and bubbles exhibit fascinating geometric properties, such as the double bubble conjecture (the shape that encloses two given volumes with the least total surface area) and the Weaire-Phelan structure (a complex periodic minimal surface used in foam physics)

## Examples of minimal surfaces

- There are many interesting examples of minimal surfaces that arise in geometry, physics, and nature
- Some of the most well-known examples include the catenoid, helicoid, Enneper's surface, and Scherk's surface
- These surfaces exhibit a rich variety of geometric and topological properties, such as self-intersection, periodicity, and symmetry

## Catenoid

- The catenoid is the minimal surface formed by rotating a catenary curve (the shape of a hanging chain) around an axis
- It is the only minimal surface of revolution besides the plane, and has the parametrization  $x(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$
- The catenoid has negative Gaussian curvature everywhere and is a ruled surface, meaning that it contains straight lines through every point
- Catenoids appear in nature as the shapes of soap films spanning two circular rings, and have applications in architecture and engineering

## Helicoid

- The helicoid is the minimal surface formed by twisting a vertical strip of paper and joining its ends to form a spiral ramp

- It has the parametrization  $x(u, v) = (\sinh u \sin v, \sinh u \cos v, u)$  and is the only ruled minimal surface besides the plane
- The helicoid is a periodic surface with screw symmetry, meaning that it is invariant under a rotation and vertical translation
- Helicoids appear in nature as the shapes of certain seaweed and fungi, and have applications in propeller and screw design

## Enneper's surface

- Enneper's surface is a self-intersecting minimal surface discovered by the German mathematician Alfred Enneper in 1864
- It has the parametrization  $x(u, v) = (u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + vu^2, u^2 - v^2)$  and is the conjugate surface of the catenoid
- Enneper's surface has a unique triple point where three sheets of the surface intersect at a common point and tangent plane
- It is an example of a complete regular minimal surface with finite total curvature and is embeddable in Euclidean space

## Scherk's surface

- Scherk's surface is a doubly periodic minimal surface discovered by the German mathematician Heinrich Scherk in 1834
- It has the implicit equation  $e^z \cos x = \cos y$  and can be parametrized using the Weierstrass  $\wp$ -function
- Scherk's surface has a saddle shape with alternating positive and negative curvature, and contains two orthogonal families of straight lines
- It is an example of a triply periodic minimal surface, meaning that it has translational symmetry in three independent directions, and arises in the study of crystal structures and foam geometry

## Properties of minimal surfaces

- Minimal surfaces have many interesting geometric and analytic properties that make them a rich subject of study in differential geometry
- Some of the most important properties include the Gauss curvature, isothermal parameters, the Weierstrass-Enneper representation, and the existence of conjugate minimal surfaces

## Gauss curvature

- The Gauss curvature  $K$  of a surface at a point is the product of the principal curvatures  $K_1$  and  $K_2$  at that point, i.e.,  $K = K_1 K_2$
- It is an intrinsic quantity that measures how the surface curves in different directions and is related to the total curvature and topology of the surface by the Gauss-Bonnet theorem
- For minimal surfaces, the Gauss curvature is always non-positive ( $K \leq 0$ ), and vanishes identically ( $K \equiv 0$ ) only for planes

- The Gauss curvature of a minimal surface satisfies the Gauss equation  $\Delta \log \lambda = -2K\lambda^2$ , where  $\lambda$  is the conformal factor of the metric

## Isothermal parameters

- Isothermal parameters are local coordinates  $(u, v)$  on a surface that make the metric conformal to the Euclidean metric, i.e.,  $ds^2 = \lambda^2(du^2 + dv^2)$
- In isothermal parameters, the coordinate curves  $u = \text{const}$  and  $v = \text{const}$  intersect at right angles and have equal magnitude of the first fundamental form
- Every minimal surface admits isothermal parameters, which can be found by solving the Beltrami equation  $\Delta x = 2H \frac{\partial x}{\partial \bar{n}}$  for the coordinate functions  $x$
- Isothermal parameters are useful for studying the conformal geometry of minimal surfaces and for deriving the Weierstrass-Enneper representation formula

## Weierstrass-Enneper representation

- The Weierstrass-Enneper representation is a formula that generates all minimal surfaces in terms of two holomorphic functions  $\phi(z)$  and  $\psi(z)$ , called the Gauss map and the height differential
- It states that every minimal surface can be parametrized as  $x(z) = \text{Re} \int^z (\phi(z), \phi(z)^2, \psi(z)) dz$ , where  $z = u + iv$  is a complex parameter and  $\text{Re}$  denotes the real part
- The Weierstrass-Enneper representation provides a powerful tool for constructing and classifying minimal surfaces, and has led to the discovery of many new examples and families of minimal surfaces
- It also reveals the close connection between minimal surfaces and complex analysis, as many properties of minimal surfaces can be studied using techniques from complex function theory

## Conjugate minimal surfaces

- Every minimal surface  $S$  has a conjugate minimal surface  $S^*$ , which is obtained by rotating the principal directions of  $S$  by 90 degrees at each point
- The conjugate surface has the same first fundamental form as the original surface, but the opposite second fundamental form, i.e.,  $I^* = I$  and  $II^* = -II$
- The coordinate functions of  $S$  and  $S^*$  are harmonic conjugates, satisfying the Cauchy-Riemann equations  $\frac{\partial x^*}{\partial u} = \frac{\partial x}{\partial v}$  and  $\frac{\partial x^*}{\partial v} = -\frac{\partial x}{\partial u}$
- Examples of conjugate minimal surfaces include the catenoid and Enneper's surface, the helicoid and the conjugate helicoid, and the Scherk surfaces of different periods
- The existence of conjugate minimal surfaces is a manifestation of the integrable nature of the minimal surface equations and has applications in the study of constant mean curvature surfaces and harmonic maps

## Generalizations and applications

- The theory of minimal surfaces has been generalized in many directions, leading to the study of higher-dimensional minimal submanifolds, minimal surfaces in Riemannian manifolds, and other variational problems in geometry and physics

- Minimal surfaces have also found numerous applications in fields such as architecture, materials science, and computer graphics

## Higher-dimensional minimal submanifolds

- A minimal submanifold is a submanifold of a Riemannian manifold that locally minimizes the volume functional among all submanifolds with the same boundary
- Examples of minimal submanifolds include geodesics in Riemannian manifolds, minimal surfaces in Euclidean space, and special Lagrangian submanifolds in Calabi-Yau manifolds
- The theory of minimal submanifolds is a rich and active area of research, with connections to geometric measure theory, partial differential equations, and calibrated geometry
- Important results in the theory of minimal submanifolds include the monotonicity formula, the compactness theorem, and the regularity theory for area-minimizing currents

## Minimal surfaces in Riemannian manifolds

- The theory of minimal surfaces can be extended to surfaces in Riemannian manifolds, where the notion of area is defined using the intrinsic metric of the manifold
- Examples of minimal surfaces in Riemannian manifolds include geodesics in surfaces, minimal spheres in spheres, and minimal tori in flat tori
- The existence and regularity of minimal surfaces in Riemannian manifolds is a difficult problem that depends on the geometry and topology of the ambient manifold
- Important results in this area include the Douglas-Morrey solution

## 9.6 Morse theory on manifolds

Morse theory is a powerful tool for studying smooth manifolds' topology. It uses Morse functions, which have non-degenerate critical points, to gain insights into a manifold's structure. These functions allow us to decompose manifolds into simple pieces based on critical points.

The theory connects critical points of Morse functions to a manifold's Betti numbers through Morse inequalities. This link provides a way to analyze manifold topology using gradient vector fields and cell decompositions, leading to applications in minimal surfaces, manifold classification, and physics.

## Morse functions

- Morse functions are a key concept in Morse theory, which is a powerful tool for studying the topology of smooth manifolds
- A Morse function is a smooth real-valued function defined on a smooth manifold that satisfies certain non-degeneracy conditions at its critical points
- The behavior of a Morse function near its critical points provides valuable information about the topology of the underlying manifold

## Definition of Morse functions

- A smooth function  $f: M \rightarrow \mathbb{R}$  on a smooth manifold  $M$  is called a Morse function if all its critical points are non-degenerate

- A critical point  $p \in M$  is non-degenerate if the Hessian matrix  $(\frac{\partial^2 f}{\partial x_i \partial x_j}(p))$  is non-singular
- The Hessian matrix is the matrix of second partial derivatives of  $f$  at  $p$
- Morse functions are generic in the sense that a small perturbation of a smooth function typically yields a Morse function

## Examples of Morse functions

- The height function  $f(x, y) = y$  on the torus  $\mathbb{T}^2$  is a Morse function
- The critical points are the maximum, minimum, and two saddle points
- The function  $f(x, y) = x^2 - y^2$  on  $\mathbb{R}^2$  is a Morse function with a saddle point at the origin
- The function  $f(x, y, z) = x^2 + y^2 - z^2$  on  $\mathbb{R}^3$  is a Morse function with a critical point at the origin

## Properties of Morse functions

- Morse functions are dense in the space of smooth functions with the  $C^2$  topology
- This means that any smooth function can be approximated by a Morse function
- The critical points of a Morse function are isolated
- The level sets  $f^{-1}(c)$  of a Morse function change their topology only when passing through a critical point
- Morse functions provide a way to decompose a manifold into simple pieces (cells) based on the critical points

## Critical points

- Critical points are a fundamental concept in Morse theory and play a crucial role in understanding the topology of smooth manifolds
- The local behavior of a Morse function near its critical points determines the global topology of the manifold

## Definition of critical points

- A point  $p \in M$  is a critical point of a smooth function  $f: M \rightarrow \mathbb{R}$  if the differential  $df_p = 0$
- In local coordinates, this means that all partial derivatives of  $f$  vanish at  $p$
- The value  $f(p)$  is called a critical value of  $f$
- Points that are not critical are called regular points

## Types of critical points

- Non-degenerate critical points of a Morse function can be classified into three types based on the index: 1. Minimum (index 0): The Hessian matrix has all positive eigenvalues 2. Saddle (index 1 to  $n - 1$ ): The Hessian matrix has both positive and negative eigenvalues 3. Maximum (index  $n$ ): The Hessian matrix has all negative eigenvalues
- The index of a critical point is the number of negative eigenvalues of the Hessian matrix

## Morse lemma

- The Morse lemma states that near a non-degenerate critical point  $P$ , a Morse function  $f$  can be expressed in a canonical form using a suitable coordinate system
- In this canonical form,  $f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$ , where  $k$  is the index of the critical point
- The Morse lemma provides a local model for the behavior of a Morse function near its critical points

## Morse index

- The Morse index of a critical point  $P$  is the number of negative eigenvalues of the Hessian matrix at  $P$
- The Morse index characterizes the type of the critical point (minimum, saddle, or maximum)
- The Morse index is related to the number of descending directions of the function at the critical point
- The Morse index plays a crucial role in the Morse inequalities and the construction of the Morse-Smale complex

## Morse inequalities

- Morse inequalities are a set of relations between the critical points of a Morse function and the Betti numbers of the underlying manifold
- They provide a powerful tool for studying the topology of smooth manifolds using Morse theory

## Weak Morse inequalities

- Let  $M$  be a compact smooth manifold and  $f: M \rightarrow \mathbb{R}$  a Morse function
- Denote by  $c_k$  the number of critical points of  $f$  with index  $k$ , and by  $b_k$  the  $k$ -th Betti number of  $M$
- The weak Morse inequalities state that  $c_k \geq b_k$  for all  $k$
- In other words, the number of critical points of index  $k$  is always greater than or equal to the  $k$ -th Betti number

## Strong Morse inequalities

- The strong Morse inequalities provide a more refined relation between the critical points and the Betti numbers
- They state that

$$\sum_{i=0}^k (-1)^{k-i} c_i \geq \sum_{i=0}^k (-1)^{k-i} b_i$$

for all  $k$

- Equality holds for  $k = n$ , where  $n$  is the dimension of the manifold
- The strong Morse inequalities imply the weak Morse inequalities



## Morse inequalities vs Betti numbers

- The Morse inequalities relate the critical points of a Morse function to the Betti numbers of the manifold
- The Betti numbers  $b_k$  measure the number of independent  $k$ -dimensional holes in the manifold
- $b_0$  counts the number of connected components
- $b_1$  counts the number of independent loops (1D holes)
- $b_2$  counts the number of independent voids (2D holes), etc.
- The Morse inequalities provide lower bounds for the Betti numbers in terms of the critical points

## Applications of Morse inequalities

- The Morse inequalities can be used to prove the existence of critical points of a certain index
- They provide a way to estimate the homology groups of a manifold without explicitly computing them
- The Morse inequalities can be used to study the topology of level sets and sublevel sets of a Morse function
- They play a crucial role in the proof of the h-cobordism theorem and the surgery exact sequence in differential topology

## Gradient vector fields

- Gradient vector fields are a key concept in Morse theory and provide a way to study the flow lines between critical points of a Morse function
- The gradient vector field of a Morse function determines the dynamics of the associated gradient flow

## Definition of gradient vector fields

- Given a smooth function  $f: M \rightarrow \mathbb{R}$  on a Riemannian manifold  $(M, g)$ , the gradient vector field  $\nabla f$  is defined as the unique vector field satisfying  $g(\nabla f, X) = df(X)$  for all vector fields  $X$  on  $M$
- In local coordinates, the gradient vector field is given by  $\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}$ , where  $(g^{ij})$  is the inverse of the metric tensor  $(g_{ij})$
- The gradient vector field points in the direction of steepest ascent of the function  $f$

## Integral curves of gradient vector fields

- An integral curve of a gradient vector field  $\nabla f$  is a curve  $\gamma: I \rightarrow M$  satisfying  $\gamma'(t) = \nabla f(\gamma(t))$  for all  $t \in I$
- Integral curves are also called gradient flow lines or trajectories
- Integral curves of a gradient vector field always flow in the direction of increasing values of  $f$
- The critical points of  $f$  are the stationary points of the gradient flow
- Integral curves provide a way to understand the global behavior of a Morse function

## Stable vs unstable manifolds

- For a critical point  $p$  of a Morse function  $f$ , the stable manifold  $W^s(p)$  is the set of points that flow to  $p$  under the gradient flow of  $-\nabla f$  as  $t \rightarrow \infty$
- The unstable manifold  $W^u(p)$  is the set of points that flow to  $p$  under the gradient flow of  $\nabla f$  as  $t \rightarrow -\infty$
- The dimension of the stable manifold is equal to the Morse index of the critical point, while the dimension of the unstable manifold is equal to the codimension of the Morse index
- Stable and unstable manifolds play a crucial role in the construction of the Morse-Smale complex

## Morse-Smale condition

- A Morse function  $f$  on a compact manifold  $M$  is called Morse-Smale if the stable and unstable manifolds of its critical points intersect transversely
- The Morse-Smale condition ensures that the gradient flow of  $f$  has a well-behaved global structure
- Morse-Smale functions are generic in the sense that a small perturbation of a Morse function typically yields a Morse-Smale function
- The Morse-Smale condition is necessary for the construction of the Morse-Smale complex and the computation of Morse homology

## Cell decomposition

- Cell decomposition is a technique in Morse theory that allows one to decompose a manifold into simple pieces (cells) based on the critical points of a Morse function
- The cell decomposition provides a way to study the topology of the manifold using the information encoded in the Morse function

## Handlebody decomposition

- A handlebody decomposition of a manifold  $M$  is a decomposition of  $M$  into a union of handles (cells) of different dimensions
- Each handle is attached to the boundary of the previous handles along its boundary sphere
- The index of a handle is the dimension of its core (the part of the handle that is not part of the boundary sphere)
- The handles in a handlebody decomposition are in one-to-one correspondence with the critical points of a Morse function on  $M$

## Morse-Smale complex

- The Morse-Smale complex is a cellular decomposition of a manifold  $M$  associated with a Morse-Smale function  $f$
- The cells in the Morse-Smale complex are the intersections of the stable and unstable manifolds of the critical points of  $f$
- The dimension of a cell is equal to the Morse index of the corresponding critical point

- The Morse-Smale complex encodes the global structure of the gradient flow of  $f$

## Computation of homology groups

- The Morse-Smale complex can be used to compute the homology groups of the manifold  $M$
- Each cell in the Morse-Smale complex corresponds to a generator of a chain complex, called the Morse complex
- The boundary operator in the Morse complex is defined by counting the gradient flow lines between critical points of consecutive indices
- The homology of the Morse complex is isomorphic to the singular homology of the manifold  $M$

## Morse homology vs singular homology

- Morse homology is an alternative way to compute the homology groups of a manifold using the critical points of a Morse function
- Morse homology is defined using the Morse complex, which is a chain complex built from the critical points and gradient flow lines
- The Morse homology groups are isomorphic to the singular homology groups of the manifold
- Morse homology provides a more geometric and computational approach to homology theory compared to singular homology

## Morse theory applications

- Morse theory has numerous applications in various fields of mathematics and physics
- It provides a powerful tool for studying the topology of manifolds and understanding the behavior of functions on them

## Existence of minimal surfaces

- Morse theory can be used to prove the existence of minimal surfaces in Riemannian manifolds
- A minimal surface is a surface with zero mean curvature, which can be characterized as a critical point of the area functional
- The Morse index of a minimal surface is related to its stability properties
- Morse theory provides a way to construct minimal surfaces using the gradient flow of the area functional

## Topology of manifolds

- Morse theory is a fundamental tool in the study of the topology of smooth manifolds
- It allows one to analyze the homology groups, homotopy groups, and other topological invariants of manifolds using the critical points of Morse functions
- Morse theory can be used to prove the Poincaré conjecture in dimensions greater than 4
- It also plays a crucial role in the classification of surfaces and the study of cobordisms between manifolds

## Morse theory in physics

- Morse theory has important applications in various areas of physics, such as quantum mechanics, statistical mechanics, and general relativity
- In quantum mechanics, the critical points of the potential energy function correspond to the classical equilibrium states of the system
- The Morse index of a critical point is related to the stability of the corresponding equilibrium state
- In statistical mechanics, Morse theory can be used to study the topology of the configuration space and the phase transitions of the system

## Morse theory vs Floer homology

- Floer homology is a generalization of Morse theory to infinite-dimensional settings, such as the study of symplectic manifolds and gauge theories
- While Morse theory deals with functions on finite-dimensional manifolds, Floer homology is concerned with functionals on infinite-dimensional spaces, such as the space of loops or connections
- Floer homology provides a powerful tool for studying the topology of these infinite-dimensional spaces and has important applications in symplectic geometry and low-dimensional topology
- The relationship between Morse theory and Floer homology is an active area of research, with many deep connections and analogies between the two theories

# Unit 10 – Jacobi Fields and Conjugate Points

## 10.1 Jacobi fields and geodesic variations

Jacobi fields are vector fields along geodesics that describe nearby geodesics' behavior. They're crucial for understanding manifold geometry and geodesic stability, satisfying the Jacobi equation which involves the Riemann curvature tensor.

Geodesic variations are one-parameter families of geodesics that study nearby geodesics' behavior. They're closely linked to Jacobi fields, providing a way to examine geodesic stability and optimality through variation vector fields and the first variation formula.

### Jacobi fields

- Jacobi fields are vector fields along a geodesic that describe the behavior of nearby geodesics
- They play a crucial role in understanding the geometry of a manifold and the stability of geodesics

### Definition of Jacobi fields

- A Jacobi field is a vector field  $J(t)$  along a geodesic  $\gamma(t)$  satisfying the Jacobi equation
- $J(t)$  measures the deviation of a one-parameter family of geodesics from  $\gamma(t)$
- The initial conditions for a Jacobi field are  $J(0)$  and  $J'(0)$ , which determine the field uniquely

### Jacobi equation

- The Jacobi equation is a second-order linear differential equation that characterizes Jacobi fields
- It is given by  $J''(t) + R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0$ , where  $R$  is the Riemann curvature tensor
- The Jacobi equation describes how the curvature of the manifold affects the behavior of nearby geodesics

### Solutions of Jacobi equation

- The solutions of the Jacobi equation form a vector space of dimension  $2n$ , where  $n$  is the dimension of the manifold
- The space of Jacobi fields along a geodesic is isomorphic to the space of initial conditions  $(J(0), J'(0))$
- In flat Euclidean space, the Jacobi fields are linear combinations of the parallel transport of  $J(0)$  and  $tJ'(0)$

### Jacobi fields vs parallel vector fields

- Parallel vector fields along a geodesic are Jacobi fields, but not all Jacobi fields are parallel
- Parallel vector fields have constant length and maintain a fixed angle with the tangent vector of the geodesic

- Jacobi fields can change length and direction, depending on the curvature of the manifold (Riemannian manifold)

## Geodesic variations

- Geodesic variations are one-parameter families of geodesics that describe the behavior of nearby geodesics
- They are closely related to Jacobi fields and provide a way to study the stability and optimality of geodesics

## One-parameter variations of geodesics

- A one-parameter variation of a geodesic  $\gamma(t)$  is a smooth map  $\alpha : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$  such that  $\alpha(0, t) = \gamma(t)$
- For each fixed  $s \in (-\varepsilon, \varepsilon)$ , the curve  $\alpha_s(t) = \alpha(s, t)$  is a geodesic
- The variation vector field  $V(t) = \frac{\partial \alpha}{\partial s}(0, t)$  measures the deviation of the varied geodesics from  $\gamma(t)$

## Variation vector fields

- The variation vector field  $V(t)$  of a geodesic variation  $\alpha(s, t)$  is a Jacobi field along  $\gamma(t)$
- $V(t)$  satisfies the Jacobi equation and is determined by its initial conditions  $V(a)$  and  $V'(a)$
- The variation vector field describes how the varied geodesics deviate from the original geodesic  $\gamma(t)$  (tangent vector)

## Geodesic variations vs Jacobi fields

- Every Jacobi field along a geodesic can be realized as the variation vector field of some geodesic variation
- However, not every geodesic variation gives rise to a Jacobi field, as the varied curves may not be geodesics
- Geodesic variations provide a geometric interpretation of Jacobi fields and their properties

## First variation formula

- The first variation formula relates the length of a varied geodesic to the initial velocity of the variation vector field
- It states that  $\frac{d}{ds}|_{s=0} L(\alpha_s) = \langle V(b), \dot{\gamma}(b) \rangle - \langle V(a), \dot{\gamma}(a) \rangle$
- The first variation formula is used to derive the geodesic equations and study the optimality of geodesics (critical point)

## Conjugate points

- Conjugate points are pairs of points along a geodesic where the exponential map fails to be a local diffeomorphism
- They play a crucial role in the study of the global geometry of a manifold and the optimality of geodesics

## Definition of conjugate points

- Two points  $P$  and  $Q$  along a geodesic  $\gamma(t)$  are conjugate if there exists a non-zero Jacobi field  $J(t)$  along  $\gamma(t)$  such that  $J(P) = J(Q) = 0$
- Equivalently,  $P$  and  $Q$  are conjugate if there exists a non-trivial geodesic variation  $\alpha(s, t)$  of  $\gamma(t)$  such that  $\alpha(s, P) = \gamma(P)$  and  $\alpha(s, Q) = \gamma(Q)$  for all  $s$
- Conjugate points indicate the presence of nearby geodesics that intersect  $\gamma(t)$  at both  $P$  and  $Q$

## Existence of conjugate points

- The existence of conjugate points depends on the curvature of the manifold
- In positively curved manifolds (sphere), conjugate points always exist along any geodesic
- In negatively curved manifolds (hyperbolic space), there are no conjugate points along any geodesic
- In flat manifolds (Euclidean space), the only conjugate points are the endpoints of a geodesic

## Conjugate points vs critical points

- Conjugate points are related to the critical points of the energy functional on the space of paths between two points
- A geodesic  $\gamma(t)$  is a critical point of the energy functional if and only if it has no conjugate points
- The Morse index of a geodesic (number of conjugate points) is equal to the index of the critical point it represents

## Morse index theorem

- The Morse index theorem relates the number of conjugate points along a geodesic to the index of the corresponding critical point of the energy functional
- It states that the index of a geodesic is equal to the number of conjugate points along the geodesic, counted with multiplicity
- The Morse index theorem provides a way to study the global topology of a manifold using the critical points of the energy functional (Morse theory)

## Applications of Jacobi fields

- Jacobi fields have numerous applications in various areas of mathematics and physics, where they are used to study the behavior of geodesics and the geometry of manifolds

## Jacobi fields in Riemannian geometry

- In Riemannian geometry, Jacobi fields are used to study the curvature and topology of manifolds
- They provide a way to compute the Riemann curvature tensor and its various contractions (sectional curvature, Ricci curvature)
- Jacobi fields also play a role in the Morse theory of geodesics and the study of conjugate points and cut points

## Jacobi fields in general relativity

- In general relativity, Jacobi fields are used to study the behavior of light rays and test particles in curved spacetime
- They describe the deviation of nearby geodesics (deviation vector) and provide a way to compute the tidal forces experienced by an object
- Jacobi fields also play a role in the study of gravitational lensing and the formation of caustics

## Jacobi fields in calculus of variations

- In the calculus of variations, Jacobi fields arise as the solutions of the second variation equation for a variational problem
- They describe the behavior of nearby extremals and provide a way to study the stability and optimality of solutions
- Jacobi fields are used in the Morse theory of variational problems and the study of conjugate points and focal points

## Jacobi fields in optimal control theory

- In optimal control theory, Jacobi fields are used to study the sensitivity of optimal trajectories to changes in the initial conditions or control parameters
- They describe the deviation of nearby optimal trajectories and provide a way to compute the first-order necessary conditions for optimality (Pontryagin's maximum principle)
- Jacobi fields also play a role in the study of the second-order sufficient conditions for optimality and the stability of optimal control systems

## 10.2 Conjugate points and focal points

Conjugate points and focal points are key concepts in understanding the behavior of geodesics on curved surfaces. They reveal how nearby geodesics intersect and diverge, shedding light on the underlying geometry of the manifold.

These concepts connect to the broader study of geodesic variation fields, which examine how geodesics change when their initial conditions are altered. Understanding conjugate and focal points helps us analyze the stability of geodesics and global properties of manifolds.

## Geodesic variation fields

- Geodesic variation fields study how geodesics change when their initial conditions are varied, providing insights into the geometry of the underlying manifold
- Variation fields along a geodesic satisfy the Jacobi equation, a second-order linear differential equation that encodes information about the curvature of the manifold

## Jacobi fields

- Jacobi fields are vector fields along a geodesic that arise from variations of the geodesic and satisfy the Jacobi equation



- The behavior of Jacobi fields provides information about the stability of the geodesic and the presence of conjugate points
- Example: Given a one-parameter family of geodesics  $\gamma_s(t)$ , the Jacobi field is defined as  $J(t) = \frac{\partial}{\partial s} \gamma_s(t)|_{s=0}$

## Vanishing Jacobi fields

- Vanishing Jacobi fields are Jacobi fields that are zero at some point along the geodesic
- The existence of a non-trivial vanishing Jacobi field indicates the presence of a conjugate point
- Example: If a Jacobi field  $J(t)$  satisfies  $J(0) = J(t_0) = 0$  for some  $t_0 > 0$ , then the points  $\gamma(0)$  and  $\gamma(t_0)$  are conjugate points along the geodesic  $\gamma$

## Conjugate points

- Conjugate points are pairs of points along a geodesic where a non-trivial Jacobi field vanishes, indicating a degeneracy in the geodesic flow
- The presence of conjugate points has important implications for the global geometry of the manifold and the behavior of geodesics

## Definition of conjugate points

- Two points  $P$  and  $Q$  along a geodesic  $\gamma$  are called conjugate if there exists a non-trivial Jacobi field along  $\gamma$  that vanishes at both  $P$  and  $Q$
- Conjugate points can be thought of as points where nearby geodesics starting from  $P$  intersect at  $Q$
- Example: On the sphere, antipodal points are conjugate points along any geodesic (great circle) connecting them

## Multiplicity of conjugate points

- The multiplicity of a conjugate point is the dimension of the space of Jacobi fields that vanish at that point
- Higher multiplicity conjugate points indicate a greater degeneracy in the geodesic flow
- Example: On a surface of revolution, conjugate points along a meridian geodesic have multiplicity 1, while conjugate points at the poles have multiplicity 2

## Existence of conjugate points

- The existence of conjugate points depends on the curvature of the manifold
- In positively curved manifolds (e.g., spheres), conjugate points always exist along geodesics
- In negatively curved manifolds (e.g., hyperbolic space), there are no conjugate points along geodesics
- In flat manifolds (e.g., Euclidean space), conjugate points only occur at infinity

## Conjugate points vs cut points

- Cut points are points along a geodesic where the geodesic ceases to be minimizing
- Conjugate points are always cut points, but the converse is not true

- Example: On a flat torus, geodesics have no conjugate points but have cut points where they self-intersect

## Focal points

- Focal points are points where a family of geodesics emanating from a common point (or submanifold) intersect
- Focal points are related to conjugate points but are defined in terms of a family of geodesics rather than a single geodesic

## Definition of focal points

- A point  $q$  is a focal point along a geodesic  $Y$  emanating from a point  $p$  if there exists a variation of  $Y$  through geodesics starting at  $p$  such that the variation field vanishes at  $q$
- Focal points can also be defined for geodesics orthogonal to a submanifold, where the variation is through geodesics orthogonal to the submanifold
- Example: On a sphere, the poles are focal points for any family of meridian geodesics

## Properties of focal points

- The set of focal points along a geodesic forms a discrete set
- The multiplicity of a focal point is the dimension of the space of variation fields that vanish at that point
- Focal points depend on the initial submanifold and the geodesic, unlike conjugate points which are intrinsic to the geodesic itself

## Focal points vs conjugate points

- Focal points and conjugate points are closely related but distinct concepts
- Every conjugate point is a focal point, but not every focal point is a conjugate point
- Example: On a paraboloid, the vertex is a focal point for any family of geodesics emanating from a circle, but it is not a conjugate point along any individual geodesic

## Morse index theorem

- The Morse index theorem relates the number of conjugate points along a geodesic to the index of the critical point of the energy functional corresponding to the geodesic
- It provides a powerful tool for studying the global geometry of manifolds and the behavior of geodesics

## Statement of Morse index theorem

- Let  $Y$  be a geodesic segment between two non-conjugate points  $p$  and  $q$ . The index of the critical point of the energy functional corresponding to  $Y$  equals the number of conjugate points along  $Y$  (counted with multiplicity)
- The index of a critical point is the dimension of the largest subspace on which the Hessian of the energy functional is negative definite

## Applications of Morse index theorem

- The Morse index theorem can be used to prove the existence of conjugate points along certain geodesics
- It provides a lower bound for the number of geodesics connecting two points on a manifold
- The theorem has applications in the study of geodesic flows, Hamiltonian systems, and the stability of minimal surfaces

## Rauch comparison theorem

- The Rauch comparison theorem relates the distance between conjugate points (or focal points) to the curvature of the manifold
- It allows for the comparison of conjugate and focal points on different manifolds based on their curvature

## Statement of Rauch comparison theorem

- Let  $M$  and  $N$  be Riemannian manifolds with sectional curvatures satisfying  $K_M \leq K_N$ . If  $\gamma_M$  and  $\gamma_N$  are geodesics in  $M$  and  $N$ , respectively, with the same initial conditions, then the first conjugate point along  $\gamma_M$  occurs no earlier than the first conjugate point along  $\gamma_N$
- A similar statement holds for focal points when comparing geodesics orthogonal to submanifolds

## Implications for conjugate points

- In positively curved manifolds, conjugate points occur closer together than in flat or negatively curved manifolds
- The Rauch comparison theorem provides upper bounds for the distance between conjugate points based on the curvature of the manifold

## Implications for focal points

- The Rauch comparison theorem also applies to focal points, providing upper bounds for the distance between a submanifold and its focal points based on the curvature of the manifold
- In positively curved manifolds, focal points occur closer to the initial submanifold than in flat or negatively curved manifolds

## Geodesic bifurcation

- Geodesic bifurcation occurs when a family of geodesics splits into two or more branches at a conjugate point
- It is a phenomenon that arises in the study of geodesic flows and has connections to Morse theory and the stability of geodesics

## Definition of geodesic bifurcation

- A geodesic  $\gamma$  is said to bifurcate at a conjugate point  $q$  if there exist two or more distinct geodesics that agree with  $\gamma$  up to  $q$  but differ beyond it

- Bifurcation occurs when the multiplicity of the conjugate point is greater than one, indicating the presence of multiple independent Jacobi fields that vanish at  $q$

## Relationship to conjugate points

- Geodesic bifurcation can only occur at conjugate points along a geodesic
- The multiplicity of the conjugate point determines the number of branches that can bifurcate from the geodesic
- Example: On a surface of revolution, geodesic bifurcation occurs at the poles, where the multiplicity of the conjugate point is two

## Morse-Bott functions

- Geodesic bifurcation is closely related to the study of Morse-Bott functions, which are generalizations of Morse functions
- A Morse-Bott function is a smooth function whose critical points form submanifolds, and the Hessian of the function is non-degenerate in the normal directions to these submanifolds
- The energy functional for geodesics is an example of a Morse-Bott function, with critical points corresponding to geodesics and bifurcations occurring at conjugate points

## Geodesic loops

- Geodesic loops are closed geodesics, i.e., geodesics that start and end at the same point
- The study of geodesic loops is central to understanding the global geometry of manifolds and has connections to conjugate points and the fundamental group of the manifold

## Closed geodesics

- A closed geodesic is a geodesic  $\gamma : [a, b] \rightarrow M$  such that  $\gamma(a) = \gamma(b)$  and  $\dot{\gamma}(a) = \dot{\gamma}(b)$
- Closed geodesics can be viewed as periodic orbits of the geodesic flow on the unit tangent bundle of the manifold
- Example: On a sphere, the great circles are closed geodesics

## Multiplicity of geodesic loops

- The multiplicity of a geodesic loop is the dimension of the space of periodic Jacobi fields along the loop
- Higher multiplicity geodesic loops indicate the presence of bifurcating families of closed geodesics
- Example: On a flat torus, geodesic loops have multiplicity 2 if they are not self-intersecting and multiplicity 4 if they are self-intersecting

## Relationship to conjugate points

- Geodesic loops are closely related to conjugate points, as the endpoints of a loop are conjugate points along the geodesic
- The multiplicity of a geodesic loop is equal to the multiplicity of the conjugate point at the endpoint of the loop

- The existence of conjugate points along a geodesic can be used to prove the existence of closed geodesics on a manifold

## Variational properties

- Geodesics can be characterized as critical points of the energy functional on the space of paths between two points
- The variational properties of geodesics provide insights into their stability and the global geometry of the manifold

## Geodesics as critical points

- Geodesics are critical points of the energy functional  $E(\gamma) = \frac{1}{2} \int_a^b \|\dot{\gamma}(t)\|^2 dt$  on the space of paths  $\gamma : [a, b] \rightarrow M$  with fixed endpoints
- Critical points of the energy functional satisfy the geodesic equation, which is a second-order nonlinear differential equation
- Geodesics can also be characterized as critical points of the length functional  $L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt$ , subject to the constraint that the path is parametrized by arc length

## Second variation formula

- The second variation formula describes the behavior of the energy functional near a critical point (geodesic)
- It involves the Jacobi operator, which is a second-order linear differential operator acting on vector fields along the geodesic
- The sign of the second variation determines the stability of the geodesic: positive definite indicates stability, while indefinite or negative definite indicates instability

## Stability of geodesics

- A geodesic is stable if it locally minimizes the energy functional, i.e., if the second variation is positive definite for all variations
- Stable geodesics are surrounded by a neighborhood of non-intersecting geodesics
- Unstable geodesics are saddle points or local maxima of the energy functional and are sensitive to small perturbations in their initial conditions
- The Morse index of a geodesic is the number of independent unstable directions (negative eigenvalues of the second variation)

## 10.3 Morse index theorem

The Morse index theorem connects a manifold's topology to critical points of the energy functional on paths. It counts conjugate points along geodesics, providing insight into the global structure of Riemannian manifolds and the stability of their geodesics.

This theorem links the index of a geodesic to its minimizing properties and the spectrum of the second variation operator. It's crucial for understanding geodesic stability, manifold topology, and has applications in various areas of differential geometry.

## Definition of Morse index theorem

- Fundamental result in Riemannian geometry connects the topology of a manifold to the critical points of the energy functional on paths
- Provides a way to count the number of conjugate points along a geodesic
- Crucial tool for understanding the global structure of a Riemannian manifold and its geodesics

## Relationship to geodesics

- Geodesics are locally length-minimizing curves on a Riemannian manifold
- Morse index theorem studies the stability of geodesics under small perturbations
- Relates the index of a geodesic (number of conjugate points) to its minimizing properties

## Connection to second variation

- Second variation formula measures the change in length of a geodesic under small perturbations
- Morse index equals the number of negative eigenvalues of the second variation operator
- Links the stability of a geodesic to the spectrum of a self-adjoint operator

## Role of conjugate points

- Conjugate points are points along a geodesic where it fails to minimize length locally
- Presence of conjugate points indicates instability of the geodesic
- Morse index counts the number of conjugate points along a geodesic (with multiplicity)

## Computation of Morse index

- Involves analyzing the spectrum of the index form, a symmetric bilinear form associated with a geodesic
- Index form encodes information about the second variation of the energy functional
- Non-negativity of the index form implies stability of the geodesic

## Index form and its properties

- Defined using the Hessian of the energy functional on the space of paths
- Symmetric bilinear form on the space of vector fields along a geodesic
- Relates to the Jacobi equation, which governs the behavior of Jacobi fields

## Jacobi fields and conjugate points

- Jacobi fields are vector fields along a geodesic that satisfy the Jacobi equation
- Vanishing of a Jacobi field at two points indicates the presence of a conjugate point
- Dimension of the space of Jacobi fields determines the multiplicity of conjugate points

## Calculation via eigenvalues

- Morse index equals the number of negative eigenvalues of the index form (counted with multiplicity)
- Eigenvalue problem for the Jacobi equation provides a way to compute the index
- Spectral theory techniques (Sturm-Liouville theory) used to analyze the eigenvalues

## Geometric implications

- Morse index captures important geometric and topological information about a Riemannian manifold
- Relates the stability of geodesics to the global structure of the manifold
- Provides obstructions to the existence of certain types of metrics or geodesics

## Relationship between index and topology

- Closed geodesics with Morse index zero are stable and minimize length in their homotopy class
- Existence of closed geodesics with high Morse index implies non-trivial topology (non-contractible loops)
- Morse inequalities relate the Morse indices of closed geodesics to the Betti numbers of the manifold

## Closed geodesics and Morse index

- Closed geodesics are periodic solutions to the geodesic equation
- Morse index of a closed geodesic measures its stability under perturbations
- Lusternik-Fet theorem guarantees existence of closed geodesics in each non-trivial homotopy class

## Applications to sphere theorem

- Sphere theorem states that a Riemannian manifold with positive sectional curvature is homeomorphic to a sphere
- Proof relies on studying the Morse indices of closed geodesics
- Absence of conjugate points along closed geodesics implies positive curvature

## Generalizations and extensions

- Morse index theorem has been generalized to various settings beyond Riemannian geometry
- Extensions to infinite-dimensional manifolds, Lorentzian geometry, and sub-Riemannian geometry
- Connections to other areas of mathematics, such as calculus of variations and symplectic geometry

## Higher dimensional Morse index theorem

- Generalization to geodesic submanifolds of arbitrary dimension
- Relates the Morse index to the topology of the space of geodesic submanifolds
- Involves studying the spectrum of the Jacobi operator on sections of the normal bundle

## Focal points vs conjugate points

- Focal points are conjugate points along a geodesic submanifold
- Arise when studying the stability of minimal submanifolds
- Morse index theorem for focal points relates to the topology of the space of minimal submanifolds

## Infinite dimensional Morse index theorem

- Extension to infinite-dimensional Riemannian manifolds, such as loop spaces and path spaces
- Involves studying the spectrum of the Hessian operator on infinite-dimensional function spaces
- Applications to the calculus of variations and the study of geodesics on infinite-dimensional manifolds

## Proofs and techniques

- Various approaches to proving the Morse index theorem and its generalizations
- Rely on techniques from functional analysis, partial differential equations, and variational methods
- Comparison theorems and Riccati equations play a key role in many proofs

## Variational methods

- Morse index theorem can be proved using variational techniques, such as the min-max principle
- Involves studying the critical points of the energy functional on the space of paths
- Minimax characterization of eigenvalues used to relate the index to the spectrum of the Hessian

## Comparison theorems

- Rauch comparison theorem relates the Jacobi fields along geodesics to the sectional curvature
- Allows for the comparison of Morse indices between different Riemannian manifolds
- Used to prove the Morse index theorem by comparing with spaces of constant curvature

## Riccati equations and Jacobi fields

- Jacobi equation can be transformed into a Riccati equation for the shape operators
- Analysis of the Riccati equation provides information about conjugate points and the Morse index
- Sturm comparison theorem for Riccati equations used to estimate the location of conjugate points

## 10.4 Rauch comparison theorem

The Rauch comparison theorem is a key result in Riemannian geometry. It compares the geometry of a manifold with bounded sectional curvature to that of space forms. This allows us to control geodesics, Jacobi fields, and other geometric quantities on Riemannian manifolds.

The theorem links local properties like sectional curvature bounds to global properties of manifolds. It has important implications for edge lengths, angles, and volumes of geodesic balls. The theorem's proof involves techniques like the second variation formula and Sturm comparison theorem.



## Definition of Rauch comparison theorem

- Fundamental result in Riemannian geometry that compares the geometry of a manifold with bounds on its sectional curvature to the geometry of a space form (a complete simply connected Riemannian manifold of constant sectional curvature)
- Provides a way to control the behavior of geodesics, Jacobi fields, and other geometric quantities on a Riemannian manifold by comparing them to their counterparts in a space form
- Establishes a link between local properties (sectional curvature bounds) and global properties (topology and geometry of the manifold)

## Assumptions and setup

### Sectional curvature bounds

- Consider a complete Riemannian manifold  $M$  with sectional curvature  $K$  satisfying  $\delta \leq K \leq \Delta$  for some constants  $\delta$  and  $\Delta$
- The sectional curvature measures the Gaussian curvature of 2-dimensional subspaces (planes) in the tangent space at each point
- The bounds on sectional curvature provide control over the local geometry of the manifold

### Geodesic triangles

- A geodesic triangle in  $M$  consists of three points (vertices) connected by minimizing geodesics (edges)
- The comparison theorem relates the geometry of geodesic triangles in  $M$  to those in the corresponding space forms  $M_\delta$  and  $M_\Delta$  with constant sectional curvatures  $\delta$  and  $\Delta$ , respectively
- The space forms serve as models for understanding the behavior of geodesic triangles in  $M$

### Jacobi fields

- Jacobi fields are vector fields along a geodesic that describe the infinitesimal variation of the geodesic under small perturbations of its endpoints
- They satisfy the Jacobi equation, a second-order linear differential equation involving the Riemann curvature tensor
- The behavior of Jacobi fields is closely related to the sectional curvature and plays a crucial role in the comparison theorem

## Main results and implications

### Comparison of edge lengths

- The Rauch comparison theorem states that if  $M$  has sectional curvature bounds  $\delta \leq K \leq \Delta$ , then the edge lengths of a geodesic triangle in  $M$  are bounded above by the corresponding edge lengths in  $M_\Delta$  and below by those in  $M_\delta$
- This result allows us to estimate the distances between points in  $M$  using the geometry of the space forms

- The comparison of edge lengths has important consequences for the global geometry of  $M$ , such as the Bonnet-Myers theorem on diameter bounds

## Angles in geodesic triangles

- The comparison theorem also provides bounds on the angles of geodesic triangles in  $M$  in terms of the angles in the corresponding triangles in  $M_\delta$  and  $M_\Delta$
- If  $M$  has sectional curvature bounds  $\delta \leq K \leq \Delta$ , then the angles of a geodesic triangle in  $M$  are bounded below by the angles in  $M_\delta$  and above by the angles in  $M_\Delta$
- These angle comparisons are crucial for understanding the behavior of geodesics and the topology of the manifold

## Volume of geodesic balls

- The Rauch comparison theorem implies a comparison of the volumes of geodesic balls in  $M$  with those in the space forms
- If  $M$  has sectional curvature bounds  $\delta \leq K \leq \Delta$ , then the volume of a geodesic ball in  $M$  is bounded above by the volume of the corresponding ball in  $M_\Delta$  and below by the volume in  $M_\delta$
- This volume comparison has important applications in geometric analysis and the study of the spectrum of the Laplace-Beltrami operator

## Proof techniques and key lemmas

### Second variation formula

- The second variation formula describes the second derivative of the length functional along a family of geodesics
- It involves the Jacobi equation and provides a connection between the behavior of Jacobi fields and the sectional curvature
- The second variation formula is a key tool in the proof of the Rauch comparison theorem

### Index form and conjugate points

- The index form is a symmetric bilinear form associated with a geodesic that measures the positivity of the second variation
- Conjugate points along a geodesic are points where the index form degenerates, indicating the existence of nontrivial Jacobi fields
- The distribution of conjugate points is related to the sectional curvature bounds and plays a role in the proof of the comparison theorem

### Sturm comparison theorem

- The Sturm comparison theorem is a result in the theory of ordinary differential equations that compares the solutions of two second-order linear equations
- It is used to compare the behavior of Jacobi fields along geodesics in  $M$  with those in the space forms  $M_\delta$  and  $M_\Delta$

- The Sturm comparison theorem is a crucial ingredient in the proof of the Rauch comparison theorem

## Generalizations and extensions

### Sphere and hyperbolic space

- The Rauch comparison theorem can be formulated specifically for the cases where the space forms are the sphere (positive constant curvature) and the hyperbolic space (negative constant curvature)
- In these cases, the theorem provides more explicit comparisons and has additional geometric consequences
- The sphere and hyperbolic space serve as important models for understanding the geometry of manifolds with positive or negative curvature bounds

### Triangles vs hinges

- The comparison theorem can be extended to more general configurations called hinges, which consist of two geodesic segments emanating from a common point
- Hinges allow for the comparison of angles and lengths in a wider range of situations beyond just geodesic triangles
- The hinge version of the Rauch comparison theorem has applications in the study of Alexandrov spaces and other singular spaces

### Singular spaces and Alexandrov geometry

- The ideas behind the Rauch comparison theorem have been generalized to singular spaces, such as Alexandrov spaces, which are metric spaces with curvature bounds in a synthetic sense
- In Alexandrov geometry, the comparison theorems play a fundamental role in understanding the structure and properties of these spaces
- The generalization of the Rauch theorem to singular spaces has led to important developments in geometric topology and the study of metric spaces with curvature bounds

## Applications and related results

### Bonnet-Myers theorem

- The Bonnet-Myers theorem states that a complete Riemannian manifold with a positive lower bound on its Ricci curvature (a weaker condition than a positive lower bound on sectional curvature) must be compact and have a diameter bounded above by a constant depending on the curvature bound
- The proof of the Bonnet-Myers theorem relies on the Rauch comparison theorem to control the behavior of geodesics and estimate distances
- This theorem highlights the connection between curvature bounds and the global topology of the manifold

### Synge's theorem

- Synge's theorem is a result in Riemannian geometry that relates the existence of closed geodesics to the curvature and topology of the manifold

- It states that a compact even-dimensional Riemannian manifold with positive sectional curvature must have a closed geodesic
- The proof of Synge's theorem uses the Rauch comparison theorem to analyze the behavior of Jacobi fields and the index form along geodesics

## Sphere theorem

- The sphere theorem is a fundamental result in Riemannian geometry that characterizes manifolds with positive curvature
- It states that a complete Riemannian manifold with sectional curvature bounded below by a positive constant must be homeomorphic to a sphere
- The proof of the sphere theorem involves the Rauch comparison theorem and the analysis of geodesic triangles and their angles

## Gromov's Betti number theorem

- Gromov's Betti number theorem provides bounds on the Betti numbers (topological invariants) of a Riemannian manifold in terms of its curvature and diameter
- The theorem states that the Betti numbers of a compact Riemannian manifold with sectional curvature bounded above and diameter bounded below are bounded above by constants depending only on the dimension and the bounds
- The proof of Gromov's theorem uses the Rauch comparison theorem to estimate the volume of geodesic balls and control the growth of the manifold

## Historical context and development

### Rauch's original formulation

- The Rauch comparison theorem was first formulated by Harry Rauch in the 1950s
- Rauch's original version of the theorem focused on the comparison of edge lengths in geodesic triangles and the behavior of Jacobi fields
- This groundbreaking work laid the foundation for the development of comparison geometry and the study of manifolds with curvature bounds

### Berger's contribution

- Marcel Berger made significant contributions to the theory of comparison geometry and the Rauch comparison theorem
- Berger generalized and refined the theorem, extending it to more general settings and providing new applications
- His work helped to establish comparison geometry as a fundamental tool in Riemannian geometry and geometric analysis

## Modern perspectives and open problems

- The Rauch comparison theorem continues to be an active area of research, with ongoing developments and generalizations

- Recent work has focused on extending the theorem to new settings, such as singular spaces, metric measure spaces, and spaces with variable curvature bounds
- Open problems include the study of optimal constants in the comparison inequalities, the relationship between different notions of curvature, and the implications of the theorem for the topology and geometry of manifolds

## 10.5 Sphere theorems

Sphere theorems are a crucial part of Riemannian geometry, connecting curvature to topological and geometric properties of spheres. These powerful results provide insights into the structure and behavior of spheres across various dimensions, using concepts like Gaussian curvature, sectional curvature, and geodesics.

These theorems have wide-ranging applications in geometry and related fields. They form the foundation for comparison geometry, rigidity theory, and the study of manifolds with specific curvature properties. Sphere theorems continue to evolve, with generalizations and extensions pushing the boundaries of our understanding of curved spaces.

### Definition of sphere theorems

- Sphere theorems are a collection of results in Riemannian geometry that relate the curvature of a sphere to its topological and geometric properties
- These theorems provide powerful tools for understanding the structure and behavior of spheres in various dimensions
- Key concepts include Gaussian curvature, sectional curvature, geodesics, and the Euler characteristic

### Gauss-Bonnet theorem on spheres

- States that the integral of the Gaussian curvature over a compact, orientable surface is equal to  $2\pi$  times the Euler characteristic of the surface
- For a sphere, the Euler characteristic is always 2, implying that the integral of the Gaussian curvature is  $4\pi$
- Provides a deep connection between the curvature and topology of a surface

### Synge's theorem for even-dimensional spheres

- Asserts that a compact, orientable, even-dimensional Riemannian manifold with positive sectional curvature is simply connected
- Implies that even-dimensional spheres are the only compact, orientable, even-dimensional manifolds with positive sectional curvature
- Highlights the unique topological properties of spheres in even dimensions

### Diameter sphere theorem

- States that if a compact, connected Riemannian manifold has sectional curvature bounded below by a positive constant, then its diameter is bounded above by  $\pi/\sqrt{k}$ , where  $k$  is the lower curvature bound
- Establishes a relationship between the curvature and the size of a manifold

- Applies to spheres of any dimension, providing an upper bound on their diameter based on the lower curvature bound

## Topological implications of sphere theorems

- Sphere theorems reveal deep connections between the curvature of a sphere and its topological properties
- These results provide insight into the fundamental structure of spheres and their role in the classification of manifolds
- Key concepts include the Euler characteristic, fundamental group, and homotopy groups

## Relationship between curvature and topology

- Positive curvature tends to impose strong topological restrictions on a manifold
- Sphere theorems demonstrate that spheres are often the only manifolds that can support certain curvature conditions
- Examples include the Gauss-Bonnet theorem and Synge's theorem, which relate curvature to the Euler characteristic and simple connectivity, respectively

## Closed geodesics on spheres

- Geodesics are curves that minimize distance between points on a manifold
- On a sphere, all geodesics are great circles, which are closed curves
- The existence and properties of closed geodesics on spheres are closely related to the curvature and topology of the sphere

## Morse theory on spheres

- Morse theory studies the relationship between the critical points of a smooth function on a manifold and the topology of the manifold
- On a sphere, Morse functions can be used to analyze the topology and to construct cell decompositions
- The number and type of critical points of a Morse function on a sphere are determined by the Euler characteristic, which is related to the curvature via the Gauss-Bonnet theorem

## Geometric consequences of sphere theorems

- Sphere theorems provide powerful tools for understanding the geometric structure of spheres
- These results establish bounds and estimates for various geometric quantities, such as curvature, focal points, and injectivity radius
- Key concepts include sectional curvature, conjugate points, and the exponential map

## Sectional curvature bounds

- Many sphere theorems assume or imply bounds on the sectional curvature of a sphere

- For example, the diameter sphere theorem assumes a lower bound on sectional curvature, while Synge's theorem implies positive sectional curvature
- These bounds can be used to derive other geometric properties, such as estimates on the volume and diameter of the sphere

## Focal points and conjugate points

- Focal points and conjugate points are points along a geodesic where nearby geodesics converge or intersect
- The existence and distribution of these points are closely related to the curvature of the sphere
- Sphere theorems can be used to estimate the location of focal points and conjugate points based on curvature bounds

## Injectivity radius estimates

- The injectivity radius of a point on a manifold is the largest radius for which the exponential map at that point is a diffeomorphism
- On a sphere, the injectivity radius is related to the curvature and the diameter
- Sphere theorems, such as the diameter sphere theorem, can be used to estimate the injectivity radius based on curvature bounds

## Applications of sphere theorems

- Sphere theorems have numerous applications in Riemannian geometry and related fields
- These results provide a foundation for comparison geometry, rigidity theory, and the study of manifolds with specific curvature properties
- Key concepts include model spaces, isometry groups, and Einstein metrics

## Comparison geometry on spheres

- Comparison geometry uses spheres as model spaces to study the properties of more general Riemannian manifolds
- By comparing the curvature and geometry of a manifold to that of a sphere, one can derive bounds and estimates for various geometric quantities
- Sphere theorems provide the necessary tools for establishing these comparisons and deriving the resulting bounds

## Rigidity results for spherical metrics

- Rigidity theorems in Riemannian geometry characterize manifolds that are uniquely determined by their curvature or other geometric properties
- Sphere theorems often play a crucial role in establishing rigidity results for manifolds with spherical metrics or curvature bounds
- For example, Synge's theorem implies that even-dimensional spheres are the only compact, orientable manifolds with positive sectional curvature

## Sphere theorems in Riemannian geometry

- Sphere theorems are a fundamental part of the toolkit of Riemannian geometry
- These results are used to study a wide range of problems, from the classification of manifolds to the analysis of Einstein metrics and the Ricci flow
- The techniques and ideas developed in the context of sphere theorems have found applications in many other areas of geometry and topology

## Generalizations and extensions

- The classical sphere theorems have been generalized and extended in various directions
- These generalizations include analogues for manifolds with boundary, results in Lorentzian geometry, and higher-dimensional versions of the theorems
- Key concepts include geodesic boundary conditions, Lorentzian metrics, and curvature operators

## Sphere theorems for manifolds with boundary

- Some sphere theorems have been extended to the setting of manifolds with boundary
- These results typically involve additional assumptions on the geodesic boundary conditions, such as convexity or mean curvature bounds
- Examples include generalizations of the Gauss-Bonnet theorem and diameter bounds for manifolds with boundary

## Analogues in Lorentzian geometry

- Lorentzian geometry is the study of manifolds with a metric of signature  $(-, +, \dots, +)$ , which includes important examples such as spacetimes in general relativity
- Some sphere theorems have Lorentzian analogues, which relate the curvature of a Lorentzian manifold to its causal structure and topology
- These results often involve concepts specific to Lorentzian geometry, such as null geodesics and the chronology condition

## Higher-dimensional sphere theorems

- Many sphere theorems have been generalized to higher dimensions, both for Riemannian and Lorentzian manifolds
- These generalizations often involve more sophisticated curvature conditions, such as bounds on the Ricci curvature or the curvature operator
- Higher-dimensional sphere theorems have found applications in the study of Einstein manifolds, the Ricci flow, and the classification of manifolds with special holonomy

## Historical context and development

- The study of sphere theorems has a rich history, with contributions from many prominent geometers
- The development of these results has been closely intertwined with the evolution of Riemannian geometry and the study of manifolds with specific curvature properties



- Key figures include Rauch, Berger, Klingenberg, Toponogov, and many others

## Early work by Rauch and Berger

- The early development of sphere theorems can be traced back to the work of Harry Rauch and Marcel Berger in the 1950s and 1960s
- Rauch proved a comparison theorem for the Jacobi fields along geodesics, which laid the foundation for later work on focal points and conjugate points
- Berger established several important sphere theorems, including a diameter bound for manifolds with positive sectional curvature and a rigidity result for spherical metrics

## Contributions of Klingenberg and Toponogov

- Wilhelm Klingenberg and Victor Toponogov made significant contributions to the theory of sphere theorems in the 1960s and 1970s
- Klingenberg proved a generalization of Synge's theorem for odd-dimensional manifolds and established several injectivity radius estimates
- Toponogov developed a powerful comparison theorem for triangles in Riemannian manifolds, which has become a fundamental tool in comparison geometry

## Recent advances and open problems

- The study of sphere theorems remains an active area of research, with many recent advances and open problems
- Some notable developments include the proof of the differentiable sphere theorem by Brendle and Schoen, and the resolution of the Willmore conjecture by Marques and Neves
- Open problems include the generalization of sphere theorems to manifolds with non-positive curvature, and the study of sphere theorems in the context of Alexandrov spaces and metric measure spaces

## 10.6 Comparison geometry and Toponogov's theorem

Comparison geometry examines how a manifold's curvature affects its properties. It compares manifolds to model spaces like spheres or hyperbolic planes. This approach helps us understand global features of manifolds by studying local curvature.

Toponogov's theorem is a key result in comparison geometry. It relates triangles in curved spaces to those in constant-curvature models. This theorem gives us insights into how curvature shapes the overall structure of a manifold.

## Comparison geometry overview

- Comparison geometry is a subfield of differential geometry that studies the relationships between the curvature of a Riemannian manifold and its topological and geometric properties
- It involves comparing the geometry of a given manifold with that of a model space, such as a space of constant curvature (Euclidean space, sphere, or hyperbolic space)
- The main tools used in comparison geometry include sectional curvature, triangle comparison theorems, and convergence techniques like Gromov-Hausdorff convergence

## Sectional curvature in Riemannian manifolds

- Sectional curvature is a measure of the curvature of a Riemannian manifold along two-dimensional subspaces (sections) of the tangent space at each point
- It generalizes the notion of Gaussian curvature for surfaces to higher-dimensional manifolds
- The sign of the sectional curvature (positive, zero, or negative) determines the local geometry of the manifold and has implications for its global properties
- Manifolds with positive sectional curvature (e.g., spheres) have a tendency to focus geodesics, while those with negative sectional curvature (e.g., hyperbolic spaces) have a tendency to disperse geodesics

## Triangle comparison theorems

- Triangle comparison theorems relate the geometry of triangles in a Riemannian manifold to those in a model space of constant curvature
- The most well-known triangle comparison theorems are the Toponogov theorem for manifolds with lower curvature bounds and the Alexandrov-Toponogov theorem for manifolds with upper curvature bounds
- These theorems compare the angles and side lengths of triangles in the manifold with those in the model space, providing insights into the global geometry of the manifold
- For example, in a manifold with non-negative sectional curvature, the sum of the angles of a geodesic triangle is always greater than or equal to  $\pi$ , as in Euclidean space

## Alexandrov spaces

- Alexandrov spaces are a generalization of Riemannian manifolds that allow for certain types of singularities while maintaining curvature bounds in a synthetic sense
- They are defined using triangle comparison properties rather than smooth structures and are named after the Russian mathematician A.D. Alexandrov
- Alexandrov spaces with curvature bounded below (CBB) or curvature bounded above (CBA) play a crucial role in comparison geometry and the study of metric spaces with curvature bounds
- Many results from Riemannian geometry, such as the Toponogov theorem and the splitting theorem, have analogues in the context of Alexandrov spaces

## Gromov-Hausdorff convergence

- Gromov-Hausdorff convergence is a notion of convergence for metric spaces, introduced by Mikhail Gromov, that allows for the study of limit spaces of sequences of Riemannian manifolds or more general metric spaces
- It is based on the Hausdorff distance between subsets of a common metric space and provides a way to compare spaces that may not have a smooth structure or may have different dimensions
- Gromov-Hausdorff convergence has been used to prove many important results in comparison geometry, such as the compactness theorem for Riemannian manifolds with curvature and diameter bounds

- It has also been instrumental in the study of the structure of limit spaces, such as the Gromov-Hausdorff limits of sequences of Riemannian manifolds with Ricci curvature bounds

## Toponogov's theorem

### Statement of theorem

- Toponogov's theorem is a fundamental result in comparison geometry that relates the geometry of triangles in a Riemannian manifold with a lower sectional curvature bound to those in a model space of constant curvature
- It states that if  $M$  is a complete Riemannian manifold with sectional curvature  $K \geq k$ , where  $k$  is a constant, then for any geodesic triangle  $\Delta pqr$  in  $M$  and a corresponding comparison triangle  $\overline{\Delta pqr}$  in a model space of constant curvature  $k$ , the following inequality holds:  $d(p, q) \leq \overline{d}(p, q)$ , where  $d$  and  $\overline{d}$  denote the distances in  $M$  and the model space, respectively
- Intuitively, this means that triangles in a manifold with a lower curvature bound are "thinner" than those in the corresponding model space
- The theorem also provides a comparison for the angles of the triangles: the angles of  $\Delta pqr$  are no smaller than the angles of  $\overline{\Delta pqr}$

### Assumptions and conditions

- Toponogov's theorem assumes that the Riemannian manifold  $M$  is complete, which means that every Cauchy sequence in  $M$  converges to a point in  $M$
- The lower curvature bound  $K \geq k$  is assumed to hold globally on the manifold
- The theorem is stated for geodesic triangles, which are triangles whose sides are geodesics (shortest paths) in the manifold
- The comparison triangle  $\overline{\Delta pqr}$  is a triangle in the model space of constant curvature  $k$  with the same side lengths as  $\Delta pqr$

### Proof outline

- The proof of Toponogov's theorem typically proceeds by contradiction, assuming that the distance inequality fails for some geodesic triangle
- The idea is to construct a special geodesic hinge (a pair of geodesics joined at a point) in the manifold and compare it with a corresponding hinge in the model space
- By studying the behavior of these hinges and using the lower curvature bound, one can derive a contradiction to the assumed failure of the distance inequality
- The proof relies on various technical lemmas and the properties of geodesics in Riemannian manifolds, such as the first and second variation formulas

### Geometric consequences

- Toponogov's theorem has numerous geometric consequences and applications in Riemannian geometry and beyond
- It implies that manifolds with a lower curvature bound have a unique geodesic between any two points that are sufficiently close

- The theorem also leads to the Hadamard-Cartan theorem, which states that complete, simply connected Riemannian manifolds with non-positive sectional curvature are diffeomorphic to Euclidean space
- Toponogov's theorem and its variants play a crucial role in the study of the geometry and topology of manifolds with curvature bounds, as well as in the development of comparison geometry for more general metric spaces

## Applications of comparison geometry

### Sphere theorem

- The sphere theorem is a classic result in Riemannian geometry that characterizes Riemannian manifolds with positive sectional curvature that are close to being a sphere
- It states that if a complete, connected Riemannian manifold has sectional curvature satisfying  $1/4 < K \leq 1$ , then it is homeomorphic to a sphere
- The proof of the sphere theorem relies heavily on comparison geometry techniques, such as the Toponogov theorem and the Rauch comparison theorem
- The sphere theorem has been generalized in various ways, including the differentiable sphere theorem, which gives conditions for a manifold to be diffeomorphic to a sphere

### Finiteness theorems

- Finiteness theorems in Riemannian geometry aim to classify or bound the number of Riemannian manifolds satisfying certain curvature and topological conditions
- A notable example is the Cheeger finiteness theorem, which states that for given constants  $C, D, V > 0$  and an integer  $n$ , there are only finitely many diffeomorphism classes of compact  $n$ -dimensional Riemannian manifolds with sectional curvature  $|K| \leq C$ , diameter  $\leq D$ , and volume  $\geq V$
- The proof of the Cheeger finiteness theorem uses comparison geometry arguments, such as the Gromov compactness theorem and the Toponogov theorem
- Finiteness theorems have applications in the study of the moduli spaces of Riemannian metrics on a given manifold and in understanding the structure of Riemannian manifolds with curvature bounds

### Pinching theorems

- Pinching theorems in Riemannian geometry relate the topology of a manifold to its curvature by showing that if the curvature is sufficiently pinched (i.e., close to a constant value), then the manifold has a specific topological type
- A classic example is the Rauch-Berger-Klingenberg sphere theorem, which states that if a complete, simply connected Riemannian manifold has sectional curvature satisfying  $1/4 < K \leq 1$ , then it is homeomorphic to a sphere
- Pinching theorems often rely on comparison geometry techniques, such as the Toponogov theorem and the Rauch comparison theorem
- More general pinching theorems have been developed for various curvature conditions, such as positive Ricci curvature or almost non-negative sectional curvature, and for different topological conclusions, such as being homotopy equivalent to a compact symmetric space

## Fundamental groups of manifolds

- Comparison geometry techniques can be used to study the fundamental groups of Riemannian manifolds with curvature bounds
- The fundamental group is a topological invariant that captures information about the loops in a manifold and their homotopy classes
- The Synge theorem, which relies on the Toponogov theorem, states that a compact, orientable, even-dimensional Riemannian manifold with positive sectional curvature has a finite fundamental group
- The Gromov-Lawson-Rosenberg conjecture, now proven in many cases, relates the fundamental group of a manifold with positive scalar curvature to the fundamental group of a corresponding space of positive scalar curvature metrics
- Comparison geometry arguments, such as the Gromov-Lawson surgery theorem and the Cheeger-Gromoll soul theorem, have been crucial in the study of fundamental groups of manifolds with curvature bounds

## Generalizations and extensions

### Singular spaces

- Comparison geometry techniques have been extended to study singular spaces, such as metric spaces with curvature bounds in a generalized sense
- Alexandrov spaces, which are metric spaces with curvature bounds in the sense of triangle comparison inequalities, have been extensively studied using comparison geometry methods
- The Gromov-Hausdorff convergence has been used to study the structure and properties of limit spaces of sequences of singular spaces with curvature bounds
- Singular spaces arise naturally in various contexts, such as the Gromov-Hausdorff limits of Riemannian manifolds with curvature bounds or as quotients of Riemannian manifolds by isometric group actions

### Ricci curvature bounds

- While much of classical comparison geometry focuses on sectional curvature bounds, there has been significant interest in extending the theory to Ricci curvature bounds
- Ricci curvature is a trace of the sectional curvature tensor and provides information about the average curvature in each direction
- The Bishop-Gromov volume comparison theorem relates the growth of volumes of balls in a Riemannian manifold to a lower Ricci curvature bound
- The Cheeger-Gromoll splitting theorem states that a complete Riemannian manifold with non-negative Ricci curvature and a line (a geodesic that minimizes distance between any two of its points) splits isometrically as a product of the line and a lower-dimensional manifold
- Comparison geometry techniques have been developed for metric measure spaces with generalized Ricci curvature bounds, such as RCD spaces (metric measure spaces with Riemannian curvature dimension conditions)

## Alexandrov spaces with curvature bounds

- Alexandrov spaces with curvature bounded below (CBB) or above (CBA) are important generalizations of Riemannian manifolds that allow for certain types of singularities while maintaining curvature bounds in a synthetic sense
- Many classic results from Riemannian comparison geometry, such as the Toponogov theorem and the Gromov-Hausdorff compactness theorem, have analogues for Alexandrov spaces
- The Perelman stability theorem states that the class of compact Alexandrov spaces with curvature bounded below is closed under Gromov-Hausdorff convergence
- The Kapovitch stability theorem provides conditions under which a sequence of Alexandrov spaces with curvature bounded above converges to a limit space with the same lower curvature bound
- The study of Alexandrov spaces has led to important developments in the understanding of the structure and properties of metric spaces with curvature bounds

## Optimal transport and Ricci curvature

- Optimal transport theory, which studies the problem of finding the most efficient way to transport mass from one probability distribution to another, has deep connections to Ricci curvature and comparison geometry
- The Lott-Villani-Sturm theory of curvature-dimension conditions for metric measure spaces is based on the idea of convexity of certain functionals along geodesics in the space of probability measures (Wasserstein space)
- The curvature-dimension condition  $CD(K, N)$  for a metric measure space is a generalization of the notion of a Riemannian manifold with Ricci curvature bounded below by  $K$  and dimension bounded above by  $N$
- Optimal transport techniques have been used to prove functional inequalities, such as the Poincaré and log-Sobolev inequalities, for metric measure spaces with curvature-dimension conditions
- The development of optimal transport methods in the context of curvature bounds has led to a fruitful interplay between geometry, analysis, and probability theory, with applications in various fields, such as machine learning and data science

# Unit 11 – Comparison Theorems & Geometric Inequalities

## 11.1 Bonnet-Myers theorem

The Bonnet-Myers theorem connects a Riemannian manifold's curvature to its diameter. It states that a complete manifold with positive Ricci curvature must be compact, providing a powerful link between geometry and topology.

This result has far-reaching implications, from classifying manifolds to studying their fundamental groups. It's a cornerstone of Riemannian geometry, inspiring generalizations and applications in various mathematical fields beyond differential geometry.

### Bonnet-Myers theorem

- The Bonnet-Myers theorem is a fundamental result in Riemannian geometry that relates the curvature of a complete Riemannian manifold to its diameter
- It provides an upper bound on the diameter of a manifold based on a lower bound on its Ricci curvature
- The theorem has significant implications for the topology and geometry of Riemannian manifolds

### Statement of theorem

- If  $(M, g)$  is a complete Riemannian manifold of dimension  $n$  with Ricci curvature satisfying  $\text{Ric} \geq (n - 1)K > 0$  for some constant  $K$ , then the diameter of  $M$  is at most  $\frac{\pi}{\sqrt{K}}$
- In other words, if the Ricci curvature is bounded below by a positive constant, then the manifold must be compact and have a finite diameter
- The theorem can be stated in terms of the mean curvature vector field  $H$  as well: if  $|H| \geq c > 0$  for some constant  $C$ , then the diameter of  $M$  is at most  $\frac{n-1}{c}$

### Assumptions and conditions

- The manifold  $(M, g)$  must be complete, meaning that every Cauchy sequence of points in  $M$  converges to a point in  $M$  with respect to the Riemannian distance function
- The Ricci curvature, which measures the average sectional curvature of planes containing a given direction, must be bounded below by a positive constant  $(n - 1)K$
- The dimension  $n$  of the manifold is assumed to be finite

### Implications of theorem

- The Bonnet-Myers theorem implies that a complete Riemannian manifold with strictly positive Ricci curvature must be compact (closed and bounded)
- It provides a link between the curvature of a manifold and its topological properties, such as compactness and finite fundamental group

- The theorem is a key tool in the study of the geometry and topology of Riemannian manifolds, with applications in areas such as the sphere theorems and the classification of manifolds

## Riemannian manifolds

- A Riemannian manifold is a smooth manifold equipped with a Riemannian metric, which is a smoothly varying inner product on each tangent space
- Riemannian manifolds are the main objects of study in Riemannian geometry, a branch of differential geometry that generalizes Euclidean geometry to curved spaces
- The Riemannian metric allows for the definition of geometric concepts such as length, angle, and curvature on the manifold

## Definition of Riemannian manifold

- A Riemannian manifold is a pair  $(M, g)$ , where  $M$  is a smooth manifold and  $g$  is a Riemannian metric on  $M$
- The Riemannian metric  $g$  assigns to each point  $p \in M$  an inner product  $g_p$  on the tangent space  $T_p M$ , which varies smoothly with respect to  $p$
- The inner product  $g_p$  allows for the computation of lengths of tangent vectors and angles between them

## Geodesics in Riemannian manifolds

- Geodesics are the generalization of straight lines in Euclidean space to Riemannian manifolds
- A geodesic is a curve  $\gamma: I \rightarrow M$  that locally minimizes the length functional  $L(\gamma) = \int_I \sqrt{g(\gamma'(t), \gamma'(t))} dt$
- Geodesics are characterized by the geodesic equation  $\nabla_{\gamma'} \gamma' = 0$ , where  $\nabla$  is the Levi-Civita connection associated with the Riemannian metric  $g$

## Sectional curvature

- Sectional curvature is a measure of the curvature of a Riemannian manifold that assigns a real number to each plane (2-dimensional subspace) in the tangent space at each point
- Given a plane  $\sigma = \text{span}\{X, Y\} \subset T_p M$ , the sectional curvature  $K(\sigma)$  is defined as  $K(\sigma) = \frac{R(X, Y, Y, X)}{|X \wedge Y|^2}$ , where  $R$  is the Riemann curvature tensor and  $|X \wedge Y|$  is the area of the parallelogram spanned by  $X$  and  $Y$
- Sectional curvature controls the behavior of geodesics and the growth of the volume of balls in the manifold

## Diameter and completeness

- The diameter and completeness are two important concepts in the study of Riemannian manifolds that are closely related to the Bonnet-Myers theorem
- The diameter of a Riemannian manifold measures the maximum distance between any two points, while completeness ensures that the manifold has no "holes" or "missing points"



- The Bonnet-Myers theorem provides a connection between these two notions and the curvature of the manifold

## Definition of diameter

- The diameter of a Riemannian manifold  $(M, g)$  is defined as  $\text{diam}(M) = \sup \{d(p, q) : p, q \in M\}$ , where  $d$  is the Riemannian distance function induced by the metric  $g$
- In other words, the diameter is the maximum distance between any two points in the manifold
- A manifold with finite diameter is said to be bounded

## Completeness of Riemannian manifolds

- A Riemannian manifold  $(M, g)$  is said to be complete if every Cauchy sequence of points in  $M$  converges to a point in  $M$  with respect to the Riemannian distance function  $d$
- Completeness ensures that the manifold has no "missing points" and that geodesics can be extended indefinitely
- The Hopf-Rinow theorem states that completeness is equivalent to the condition that every bounded closed subset of  $M$  is compact

## Relationship between diameter and completeness

- The Bonnet-Myers theorem establishes a connection between the diameter and completeness of a Riemannian manifold and its Ricci curvature
- If a complete Riemannian manifold has Ricci curvature bounded below by a positive constant, then it must have finite diameter and be compact
- Conversely, if a Riemannian manifold has finite diameter, then it must be complete (by the Hopf-Rinow theorem) and have Ricci curvature bounded below (by the Bonnet-Myers theorem)

## Applications of Bonnet-Myers theorem

- The Bonnet-Myers theorem has numerous applications in differential geometry and related fields
- It is a powerful tool for studying the topology and geometry of Riemannian manifolds, particularly in the context of compact manifolds and the fundamental group
- The theorem also plays a key role in the development of sphere theorems, which classify manifolds based on their curvature properties

## Compact Riemannian manifolds

- The Bonnet-Myers theorem implies that a complete Riemannian manifold with strictly positive Ricci curvature must be compact
- Compact Riemannian manifolds have many special properties, such as finite volume, finite diameter, and finite fundamental group
- The theorem provides a way to prove the compactness of a manifold based on its curvature, without explicitly constructing a compact subset

## Fundamental group of manifolds

- The fundamental group of a topological space is a group that encodes information about the space's loops and their homotopy classes
- For a compact Riemannian manifold, the Bonnet-Myers theorem implies that the fundamental group must be finite
- This is because the universal cover of the manifold must also be compact (by the theorem), and the fundamental group acts freely and properly discontinuously on the universal cover

## Sphere theorems

- Sphere theorems are a class of results in Riemannian geometry that provide conditions under which a manifold must be homeomorphic to a sphere
- The Bonnet-Myers theorem is a key ingredient in the proofs of many sphere theorems, such as the quarter-pinched sphere theorem and the diameter sphere theorem
- These theorems classify manifolds based on their curvature properties and have important implications for the topology of Riemannian manifolds

## Proofs of Bonnet-Myers theorem

- The Bonnet-Myers theorem can be proved using a variety of techniques from Riemannian geometry and analysis
- The original proof by Bonnet and Myers relied on the study of Jacobi fields and the behavior of geodesics under curvature bounds
- Subsequent proofs have used different approaches, such as the Bochner technique and the comparison theory for Riccati equations

## Outline of proof

1. Consider a unit-speed geodesic  $\gamma : [0, \infty) \rightarrow M$  in a complete Riemannian manifold  $(M, g)$  with  $\text{Ric} \geq (n-1)K > 0$
2. Study the behavior of Jacobi fields along  $\gamma$ , which measure the deviation of nearby geodesics from  $\gamma$
3. Use the Riccati equation for the shape operator of the distance function from a point to show that the Jacobi fields must vanish at some finite time
4. Conclude that  $\gamma$  must stop minimizing distance at some finite time, implying that the manifold has finite diameter

## Key steps and techniques

- The proof relies on the analysis of Jacobi fields, which are vector fields along a geodesic that satisfy the Jacobi equation  $J'' + R(J, \gamma')\gamma' = 0$ , where  $R$  is the Riemann curvature tensor
- The Riccati equation for the shape operator  $S(t)$  of the distance function from a point is given by  $S' + S^2 + R_{\gamma'} = 0$ , where  $R_{\gamma'}(X) = R(X, \gamma')\gamma'$
- Under the Ricci curvature bound  $\text{Ric} \geq (n-1)K$ , the Riccati equation implies that  $S(t)$  must blow up in finite time, which in turn implies that the Jacobi fields must vanish

## Alternative proofs and approaches

- The Bochner technique provides an alternative proof of the Bonnet-Myers theorem using the Bochner-Weitzenböck formula for the Laplacian of the distance function
- The comparison theory for Riccati equations allows for the proof of the theorem by comparing the shape operator of the manifold with that of a model space of constant curvature
- Other proofs have used techniques from optimal transport theory and the study of the heat kernel on Riemannian manifolds

## Generalizations and extensions

- The Bonnet-Myers theorem has been generalized and extended in various directions, both by weakening the curvature condition and by considering different geometric settings
- These generalizations have led to a deeper understanding of the relationship between curvature, diameter, and topology in Riemannian and Finsler geometry
- The theorem has also inspired analogous results in other geometric contexts, such as sub-Riemannian geometry and metric measure spaces

## Weakening the curvature condition

- The Ricci curvature lower bound in the Bonnet-Myers theorem can be weakened to an integral condition, such as  $\int_M Ric_-^{n/2} d\mu < \infty$ , where  $Ric_-$  is the negative part of the Ricci curvature and  $\mu$  is the Riemannian volume measure
- The theorem has also been generalized to manifolds with Ricci curvature bounded below by a function  $f(r)$  that depends on the distance  $r$  from a fixed point
- These generalizations allow for the application of the theorem to a wider class of manifolds, including some non-compact ones

## Bonnet-Myers theorem in Finsler geometry

- Finsler geometry is a generalization of Riemannian geometry in which the Riemannian metric is replaced by a Finsler metric, which may depend on both the point and the direction in the tangent space
- The Bonnet-Myers theorem has been extended to the Finsler setting, where the Ricci curvature is replaced by the weighted Ricci curvature or the flag curvature
- The generalized theorem provides diameter bounds and compactness results for Finsler manifolds satisfying suitable curvature conditions

## Analogues in other geometric settings

- The Bonnet-Myers theorem has inspired analogous results in other geometric contexts, such as sub-Riemannian geometry and metric measure spaces
- In sub-Riemannian geometry, where the metric structure is determined by a distribution of horizontal subspaces in the tangent bundle, a version of the theorem has been proved using the generalized Ricci curvature
- In metric measure spaces, which are metric spaces equipped with a measure satisfying certain compatibility conditions, a synthetic notion of Ricci curvature has been developed that allows for the

proof of diameter bounds and compactness results analogous to the Bonnet-Myers theorem

## Historical context and significance

- The Bonnet-Myers theorem is named after the French mathematicians Pierre Ossian Bonnet and Sumner Byron Myers, who independently proved the result in the late 1930s and early 1940s
- The theorem represents a major milestone in the development of Riemannian geometry and has had a profound impact on the field and its applications
- The Bonnet-Myers theorem continues to inspire new research and remains an active area of study in contemporary differential geometry

## Origins of theorem

- The Bonnet-Myers theorem has its roots in the work of Bernhard Riemann, who introduced the concept of Riemann curvature and laid the foundations for the study of Riemannian manifolds in the mid-19th century
- In the early 20th century, mathematicians such as Hermann Weyl and Élie Cartan developed the theory of Riemannian geometry and explored the relationship between curvature and topology
- Bonnet and Myers independently proved the theorem in the late 1930s and early 1940s, building on the work of their predecessors and introducing new techniques from analysis and differential equations

## Impact on differential geometry

- The Bonnet-Myers theorem has had a profound impact on the development of differential geometry, particularly in the study of Riemannian manifolds and their topology
- The theorem has served as a key tool in the proof of many important results, such as the sphere theorems and the classification of compact manifolds with positive curvature
- The theorem has also inspired the development of new techniques and approaches in Riemannian geometry, such as the Bochner technique and the comparison theory for Riccati equations

## Current research and open problems

- The Bonnet-Myers theorem remains an active area of research in contemporary differential geometry, with ongoing work on generalizations, extensions, and applications of the theorem
- One current area of interest is the study of Ricci curvature bounds in metric measure spaces and their relation to diameter bounds and compactness results
- Another open problem is the extension of the Bonnet-Myers theorem to infinite-dimensional manifolds, such as the space of Riemannian metrics on a given manifold
- The theorem also has potential applications in other areas of mathematics, such as geometric group theory and the study of partial differential equations on manifolds

## 11.2 Synge's theorem

Synge's theorem connects a manifold's topology to its curvature. It shows how closed geodesics on Riemannian manifolds relate to sectional curvature, shedding light on their global structure. This result is key in Metric Differential Geometry.

The theorem focuses on even-dimensional Riemannian manifolds with specific sectional curvature conditions. It proves the existence of closed geodesics, distinguishing between orientable and non-orientable manifolds. The proof uses Jacobi fields and the index form.

## Synge's theorem overview

- Synge's theorem is a fundamental result in Riemannian geometry that relates the topology of a manifold to its curvature
- Establishes a connection between the existence of closed geodesics on a Riemannian manifold and its sectional curvature
- Plays a crucial role in understanding the global structure of Riemannian manifolds and has significant implications in the study of Metric Differential Geometry

## Assumptions of Synge's theorem

### Riemannian manifolds

- Synge's theorem applies to Riemannian manifolds, which are smooth manifolds equipped with a Riemannian metric
- Riemannian metric allows for the measurement of lengths, angles, and curvature on the manifold
- Riemannian manifolds provide a natural setting for studying geometric properties and geodesics

### Even-dimensional manifolds

- Synge's theorem specifically considers even-dimensional Riemannian manifolds
- Even-dimensional manifolds have dimensions that are multiples of 2 (2, 4, 6, etc.)
- The theorem exploits the properties and symmetries inherent in even-dimensional spaces

### Sectional curvature conditions

- Synge's theorem assumes certain conditions on the sectional curvature of the Riemannian manifold
- Sectional curvature measures the curvature of two-dimensional subspaces (planes) tangent to the manifold at a point
- The theorem typically requires the sectional curvature to be either strictly positive or non-negative

## Statement of Synge's theorem

### Closed geodesics

- Synge's theorem asserts the existence of closed geodesics on certain Riemannian manifolds
- Closed geodesics are paths on the manifold that start and end at the same point and are locally length-minimizing
- The theorem provides conditions under which a manifold must contain at least one closed geodesic

## Orientable vs non-orientable manifolds

- Synge's theorem distinguishes between orientable and non-orientable even-dimensional Riemannian manifolds
- Orientable manifolds have a consistent choice of orientation (e.g., the 2-sphere)
- Non-orientable manifolds do not admit a consistent orientation (e.g., the real projective plane)
- The theorem states different conclusions for orientable and non-orientable manifolds

## Proof of Synge's theorem

### Contradiction approach

- The proof of Synge's theorem often proceeds by contradiction
- Assumes the non-existence of closed geodesics and derives a contradiction under the given curvature assumptions
- This contradiction implies that the initial assumption must be false, establishing the existence of closed geodesics

## Jacobi fields

- Jacobi fields play a crucial role in the proof of Synge's theorem
- Jacobi fields describe the infinitesimal variations of geodesics on a Riemannian manifold
- They capture the behavior of nearby geodesics and provide information about conjugate points and the index form

## Index form and conjugate points

- The index form is a quadratic form associated with a geodesic that measures the second variation of its length
- Conjugate points are points along a geodesic where the index form vanishes
- The proof of Synge's theorem analyzes the properties of the index form and the existence of conjugate points
- The presence or absence of conjugate points is related to the sectional curvature assumptions

## Applications of Synge's theorem

### Topology of even-dimensional manifolds

- Synge's theorem has implications for the topology of even-dimensional Riemannian manifolds
- It provides obstructions to the existence of certain topological structures based on curvature conditions
- For example, it can be used to show that certain manifolds cannot be simply connected or have specific homotopy groups

## Sphere theorem

- Synge's theorem is closely related to the sphere theorem in Riemannian geometry
- The sphere theorem states that a compact, simply connected Riemannian manifold with sectional curvature bounded below by a positive constant is homeomorphic to a sphere
- Synge's theorem can be used as a key ingredient in proving the sphere theorem

## Generalizations of Synge's theorem

### Odd-dimensional manifolds

- While Synge's theorem originally focuses on even-dimensional manifolds, generalizations to odd-dimensional manifolds have been explored
- These generalizations often require additional assumptions or modifications to the curvature conditions
- Odd-dimensional versions of Synge's theorem provide insights into the geometry and topology of manifolds with odd dimensions

## Weaker curvature assumptions

- Researchers have investigated variations of Synge's theorem under weaker curvature assumptions
- Instead of strictly positive or non-negative sectional curvature, more general curvature conditions have been considered
- These generalizations aim to extend the applicability of Synge's theorem to a broader class of Riemannian manifolds

## Relationship to other theorems

### Comparison with Myers' theorem

- Synge's theorem is often compared and contrasted with Myers' theorem in Riemannian geometry
- Myers' theorem states that a complete Riemannian manifold with Ricci curvature bounded below by a positive constant is compact and has a finite fundamental group
- While both theorems relate curvature to topological properties, they focus on different curvature conditions and yield distinct conclusions

## Connection to Bonnet-Myers theorem

- Synge's theorem is closely connected to the Bonnet-Myers theorem
- The Bonnet-Myers theorem provides an upper bound on the diameter of a complete Riemannian manifold with Ricci curvature bounded below by a positive constant
- Synge's theorem and the Bonnet-Myers theorem both highlight the interplay between curvature and the global structure of Riemannian manifolds

## Historical context of Synge's theorem

### Synge's contributions to geometry

- Synge's theorem is named after the Irish mathematician John Lighton Synge
- Synge made significant contributions to differential geometry and relativity theory
- His work laid the foundation for many important results in Riemannian geometry, including the theorem that bears his name

### Influence on later developments

- Synge's theorem has had a profound impact on the development of Riemannian geometry and related fields
- It has inspired numerous generalizations, extensions, and applications
- The theorem continues to be a valuable tool in the study of the geometry and topology of Riemannian manifolds
- Synge's ideas have influenced research in areas such as geodesic flows, Morse theory, and the geometry of submanifolds

## 11.3 Gromov-Hausdorff convergence

Gromov-Hausdorff convergence is a powerful tool for comparing metric spaces that may not be directly comparable. It allows us to measure how "close" two spaces are, even if they're not subsets of the same space or have different dimensions.

This concept is crucial in metric differential geometry, as it helps us understand how sequences of metric spaces behave. We can study limits of spaces, analyze their properties, and gain insights into the structure of complex geometric objects.

### Definition of Gromov-Hausdorff distance

- Gromov-Hausdorff distance is a way to measure the dissimilarity between two metric spaces
- It allows for comparing spaces that may not be isometric but are "close" in some sense
- The concept is fundamental in understanding the convergence of sequences of metric spaces in the context of Metric Differential Geometry

### Hausdorff distance between metric spaces

- Hausdorff distance is a metric on the space of compact subsets of a metric space
- For two compact subsets  $A$  and  $B$  of a metric space  $(X, d)$ , the Hausdorff distance  $d_H(A, B)$  is defined as the infimum of all  $\varepsilon > 0$  such that  $A$  is contained in the  $\varepsilon$ -neighborhood of  $B$  and vice versa
- Intuitively, it measures how far two sets are from each other

### Infimum over all isometric embeddings

- Gromov-Hausdorff distance extends the notion of Hausdorff distance to compare metric spaces



- For two compact metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , the Gromov-Hausdorff distance  $d_{GH}(X, Y)$  is defined as the infimum of the Hausdorff distance between  $f(X)$  and  $g(Y)$  over all isometric embeddings  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  into a common metric space  $(Z, d_Z)$
- This allows for comparing spaces that may not be subsets of the same ambient space

## Gromov-Hausdorff distance as a pseudometric

- Gromov-Hausdorff distance satisfies the properties of a pseudometric on the space of compact metric spaces
- Non-negativity:  $d_{GH}(X, Y) \geq 0$  for all  $X, Y$
- Symmetry:  $d_{GH}(X, Y) = d_{GH}(Y, X)$  for all  $X, Y$
- Triangle inequality:  $d_{GH}(X, Z) \leq d_{GH}(X, Y) + d_{GH}(Y, Z)$  for all  $X, Y, Z$
- However,  $d_{GH}(X, Y) = 0$  does not necessarily imply  $X$  and  $Y$  are isometric, making it a pseudometric rather than a metric

## Properties of Gromov-Hausdorff convergence

- Gromov-Hausdorff convergence is a notion of convergence for sequences of compact metric spaces
- It captures the idea of spaces becoming more and more similar in terms of their metric structure
- Understanding the properties of this convergence is crucial in studying the limiting behavior of sequences of metric spaces

## Completeness of the space of compact metric spaces

- The space of compact metric spaces, equipped with the Gromov-Hausdorff distance, is complete
- This means that every Cauchy sequence of compact metric spaces converges to a compact metric space with respect to the Gromov-Hausdorff distance
- Completeness is essential for the existence of limits of sequences of metric spaces

## Precompactness and total boundedness

- A family of compact metric spaces is said to be precompact if every sequence in the family has a convergent subsequence with respect to the Gromov-Hausdorff distance
- Precompactness is equivalent to total boundedness in the Gromov-Hausdorff metric
- Total boundedness means that the space can be covered by finitely many balls of arbitrarily small radius
- These properties are useful in establishing the existence of convergent subsequences

## Relationship to uniform convergence

- Gromov-Hausdorff convergence is related to the notion of uniform convergence of metric spaces
- If a sequence of compact metric spaces converges uniformly to a metric space, then it also converges in the Gromov-Hausdorff sense

- However, the converse is not true in general, as Gromov-Hausdorff convergence allows for more flexibility in the limiting space

## Gromov-Hausdorff convergence vs Lipschitz convergence

- Lipschitz convergence is another notion of convergence for metric spaces, based on the existence of almost isometric maps between the spaces
- Gromov-Hausdorff convergence is weaker than Lipschitz convergence
- If a sequence of compact metric spaces converges in the Lipschitz sense, then it also converges in the Gromov-Hausdorff sense
- However, Gromov-Hausdorff convergence is more widely applicable and allows for studying a broader range of limiting spaces

## Examples and applications

- Gromov-Hausdorff convergence has found numerous applications in various areas of mathematics
- It provides a framework for studying the limiting behavior of sequences of metric spaces and understanding the geometry of limit spaces
- The following examples illustrate the versatility and importance of this concept

## Convergence of Riemannian manifolds

- Gromov-Hausdorff convergence can be used to study the limit of a sequence of Riemannian manifolds
- For instance, a sequence of Riemannian manifolds with curvature bounds may converge to a metric space that is not necessarily a smooth manifold (Alexandrov space)
- This allows for understanding the asymptotic geometry of manifolds and the formation of singularities

## Convergence of metric graphs

- Metric graphs, such as finite graphs with edge lengths, can be studied using Gromov-Hausdorff convergence
- Sequences of metric graphs may converge to limit spaces that exhibit interesting geometric and topological properties (fractals, real trees)
- This has applications in the study of random graphs, network analysis, and combinatorics

## Gromov-Hausdorff convergence in geometric group theory

- In geometric group theory, Gromov-Hausdorff convergence is used to study the asymptotic geometry of finitely generated groups
- The Cayley graphs of a sequence of groups, equipped with suitable metrics, may converge in the Gromov-Hausdorff sense to a limit space (asymptotic cone)
- This provides insights into the large-scale structure and properties of groups

## Applications to the study of limit spaces

- Gromov-Hausdorff convergence is a powerful tool for studying the geometry and topology of limit spaces
- Limit spaces arising from Gromov-Hausdorff convergence often exhibit interesting properties (dimension, curvature bounds, symmetries)
- Understanding the structure of limit spaces has applications in various areas, such as metric geometry, topology, and analysis

## Techniques for proving Gromov-Hausdorff convergence

- Proving Gromov-Hausdorff convergence of a sequence of metric spaces often involves a combination of various techniques
- These techniques aim to establish the existence of suitable approximations and embeddings that capture the convergence
- The following methods are commonly used in the study of Gromov-Hausdorff convergence

## Establishing existence of $\varepsilon$ -approximations

- An  $\varepsilon$ -approximation between two metric spaces is a map that distorts distances by at most  $\varepsilon$
- Proving the existence of  $\varepsilon$ -approximations with  $\varepsilon$  tending to zero is a key step in establishing Gromov-Hausdorff convergence
- This often involves constructing suitable maps between the spaces and estimating the distortion of distances

## Constructing suitable isometric embeddings

- Gromov-Hausdorff convergence can be established by constructing isometric embeddings of the spaces into a common metric space
- The goal is to find embeddings such that the Hausdorff distance between the embedded spaces tends to zero
- Techniques such as the isometric embedding theorem and the use of ultraproducts are often employed

## Utilizing the Gromov compactness theorem

- The Gromov compactness theorem states that a family of compact metric spaces with uniformly bounded diameter is precompact in the Gromov-Hausdorff metric
- This theorem is a powerful tool for establishing the existence of convergent subsequences
- It reduces the problem of proving convergence to showing that the spaces have uniformly bounded diameter

## Relationship to measured Gromov-Hausdorff convergence

- Measured Gromov-Hausdorff convergence is an extension of Gromov-Hausdorff convergence that incorporates measure-theoretic information

- It considers the convergence of metric measure spaces, which are metric spaces equipped with probability measures
- Techniques from measured Gromov-Hausdorff convergence, such as the use of optimal transport and the Prokhorov metric, can be adapted to the Gromov-Hausdorff setting

## Limits of sequences of metric spaces

- Gromov-Hausdorff convergence provides a framework for studying the limits of sequences of metric spaces
- The limit spaces arising from Gromov-Hausdorff convergence often exhibit interesting geometric and topological properties
- Understanding the structure and behavior of these limit spaces is a central theme in the study of Gromov-Hausdorff convergence

## Tangent cones and blow-ups

- Tangent cones are limit spaces that describe the infinitesimal geometry of a metric space at a point
- They are obtained by rescaling the space around the point and taking the Gromov-Hausdorff limit of the rescaled spaces
- Tangent cones provide local information about the space and can be used to study singularities and curvature properties

## Alexandrov spaces as limits of Riemannian manifolds

- Alexandrov spaces are metric spaces with curvature bounds in the sense of comparison geometry
- They naturally arise as Gromov-Hausdorff limits of sequences of Riemannian manifolds with curvature bounds
- Studying the geometry of Alexandrov spaces provides insights into the asymptotic behavior of Riemannian manifolds and the formation of singularities

## Limits of Cayley graphs and asymptotic cones

- Cayley graphs are geometric objects associated with finitely generated groups, encoding their algebraic structure
- The Gromov-Hausdorff limits of sequences of Cayley graphs, suitably rescaled, are called asymptotic cones
- Asymptotic cones capture the large-scale geometry of groups and provide information about their asymptotic properties (growth, amenability)

## Stability of geometric properties under Gromov-Hausdorff convergence

- Gromov-Hausdorff convergence preserves certain geometric properties of metric spaces
- Properties such as compactness, connectedness, and doubling dimension are stable under Gromov-Hausdorff convergence
- Studying the stability of geometric properties is important for understanding the behavior of limit spaces and the preservation of structure under convergence

## 11.4 Bishop-Gromov volume comparison

Nonnegative Ricci curvature in Riemannian manifolds leads to favorable geometric properties and important consequences. It constrains volume growth, impacts topology, and provides insights into the relationship between curvature and global structure.

The Bishop-Gromov inequality is a key result, comparing volume growth in manifolds with nonnegative Ricci curvature to spaces of constant sectional curvature. This powerful tool helps study the geometry and topology of these manifolds.

### Riemannian manifolds of nonnegative Ricci curvature

- Fundamental objects in Riemannian geometry that exhibit favorable geometric properties due to the nonnegativity of their Ricci curvature tensor
- Play a crucial role in understanding the relationship between curvature and the global structure of manifolds
- Nonnegative Ricci curvature imposes restrictions on the geometry and topology of the manifold, leading to important consequences such as volume growth estimates and topological implications

### Ricci curvature bound

- Condition on the Ricci curvature tensor, requiring it to be greater than or equal to zero at every point of the manifold
- Ricci curvature measures the average sectional curvature in each tangent direction and provides information about the local geometry of the manifold
- Nonnegative Ricci curvature implies nonnegative sectional curvature in all two-dimensional subspaces of the tangent space at each point

### Volume growth of geodesic balls

- Nonnegative Ricci curvature constrains the growth rate of the volume of geodesic balls as their radius increases
- Volume of geodesic balls in manifolds with nonnegative Ricci curvature grows at most polynomially with respect to the radius (Euclidean space)
- Contrasts with the exponential volume growth observed in manifolds with negative Ricci curvature (hyperbolic space)

### Asymptotic volume ratio

- Quantity that measures the limiting behavior of the ratio between the volume of a geodesic ball and the volume of a Euclidean ball of the same radius as the radius tends to infinity
- In manifolds with nonnegative Ricci curvature, the asymptotic volume ratio is always less than or equal to 1
- Equality holds if and only if the manifold is isometric to Euclidean space
- Provides a way to compare the large-scale geometry of a manifold with that of Euclidean space

## Bishop-Gromov inequality

- Fundamental result in Riemannian geometry that relates the volume growth of geodesic balls in a manifold with nonnegative Ricci curvature to the volume growth in a space of constant sectional curvature
- Establishes an upper bound on the volume ratio of geodesic balls in terms of the volume ratio in a comparison space
- Serves as a powerful tool for studying the geometry and topology of manifolds with nonnegative Ricci curvature

## Statement of inequality

- Let  $M$  be a complete Riemannian manifold with nonnegative Ricci curvature and let  $B(p, r)$  denote the geodesic ball of radius  $r$  centered at a point  $p$  in  $M$
- The Bishop-Gromov inequality states that for any  $0 < r \leq R$ :
- $$\frac{\text{Vol}(B(p, R))}{\text{Vol}(B(p, r))} \leq \frac{\text{Vol}(B_K(R))}{\text{Vol}(B_K(r))}$$
- $B_K(r)$  and  $B_K(R)$  are geodesic balls in the space of constant sectional curvature  $K$

## Proof using Ricci curvature bound

- The proof of the Bishop-Gromov inequality relies on the nonnegativity of the Ricci curvature
- Key steps involve comparing the Jacobi fields along geodesics in the manifold with those in the comparison space
- Nonnegative Ricci curvature implies that the Jacobi fields in the manifold grow at most as fast as the corresponding Jacobi fields in the comparison space
- Volume comparison is obtained by integrating the Jacobi fields and applying the Ricci curvature bound

## Rigidity of equality case

- The Bishop-Gromov inequality is sharp, meaning that equality holds if and only if the manifold is isometric to the comparison space up to the radius  $R$
- If equality holds for all  $r \leq R$ , then the geodesic ball  $B(p, R)$  in the manifold is isometric to the corresponding geodesic ball in the comparison space
- Rigidity result highlights the strong influence of nonnegative Ricci curvature on the local geometry of the manifold

## Geometric consequences

- The assumption of nonnegative Ricci curvature has far-reaching implications for the geometry and topology of Riemannian manifolds
- Leads to various geometric properties and constraints on the global structure of the manifold
- Enables the derivation of important results in geometric analysis and provides insights into the interplay between curvature and topology

## Bounding topology of manifolds

- Nonnegative Ricci curvature imposes restrictions on the topology of the manifold
- Fundamental group of a compact manifold with nonnegative Ricci curvature is finite (Myers' theorem)
- Higher homotopy groups of such manifolds are also constrained, leading to topological implications
- Manifolds with nonnegative Ricci curvature have a well-behaved topology and exhibit a certain level of rigidity

## Maximal volume growth

- Manifolds with nonnegative Ricci curvature have the maximal volume growth among all manifolds with the same dimension
- Volume of geodesic balls grows at most as fast as the volume of balls in Euclidean space (Bishop-Gromov inequality)
- Maximal volume growth property characterizes the large-scale geometry of the manifold and provides information about its asymptotic behavior

## Optimal packing of geodesic balls

- Nonnegative Ricci curvature allows for efficient packing of geodesic balls in the manifold
- Packing density of geodesic balls in manifolds with nonnegative Ricci curvature is always greater than or equal to the packing density in Euclidean space
- Optimal packing results have applications in coding theory and the study of sphere packings

## Spaces of directions

- Important objects associated with Riemannian manifolds that capture the infinitesimal geometry at each point
- Spaces of directions provide a way to study the local structure of the manifold and its tangent spaces
- Play a crucial role in understanding the behavior of geodesics and the convergence of sequences of manifolds

## Tangent cones

- Tangent cones are metric spaces that arise as the limit of rescaled neighborhoods of a point in the manifold
- Obtained by zooming in on the manifold around a point and considering the limit of the rescaled metrics
- Tangent cones encode the infinitesimal geometry of the manifold at the point and provide a first-order approximation of the local structure

## Gromov-Hausdorff convergence

- Notion of convergence for metric spaces, including Riemannian manifolds, that captures the idea of two spaces becoming increasingly similar in terms of their metric structure

- Sequence of Riemannian manifolds converges in the Gromov-Hausdorff sense if there exist distance-preserving maps between the manifolds that become increasingly isometric
- Gromov-Hausdorff convergence allows for the study of limits of sequences of manifolds and the analysis of their geometric properties

## Cone structure of limit spaces

- Limit spaces obtained as Gromov-Hausdorff limits of sequences of manifolds with nonnegative Ricci curvature often exhibit a cone-like structure
- Tangent cones at points of the limit space are metric cones over a cross-sectional space called the space of directions
- Cone structure provides insights into the local geometry of the limit space and allows for the study of its singularities and regular points

## Generalizations

- The concepts and results related to manifolds with nonnegative Ricci curvature have been extended and generalized to various settings
- Generalizations allow for the study of spaces with weaker curvature conditions or different notions of curvature
- Provide a broader framework for understanding the interplay between curvature and geometry in metric spaces

## Ricci curvature bounded below

- Extension of the notion of nonnegative Ricci curvature to allow for a lower bound on the Ricci curvature tensor
- Manifolds with Ricci curvature bounded below by a constant  $K$  satisfy similar geometric properties as manifolds with nonnegative Ricci curvature
- Lower bound on Ricci curvature provides control over the geometry of the manifold and leads to volume comparison and topological implications

## Alexandrov spaces

- Metric spaces that satisfy a notion of curvature bound in terms of comparison triangles
- Generalize the concept of sectional curvature bounds to non-smooth settings
- Alexandrov spaces with curvature bounded below exhibit geometric properties similar to manifolds with lower bounds on sectional curvature
- Provide a framework for studying singular spaces and their geometry

## Optimal transport approach

- An approach to defining generalized notions of Ricci curvature using optimal transport theory
- Ricci curvature is characterized in terms of the behavior of the optimal transport map between probability measures on the space



- Optimal transport approach allows for the extension of Ricci curvature bounds to metric measure spaces and provides a unifying framework for various notions of generalized Ricci curvature
- Enables the study of geometric and functional inequalities in non-smooth settings

## 11.5 Sobolev inequalities on manifolds

Sobolev inequalities on manifolds provide a powerful framework for analyzing functions and their derivatives in geometric settings. They extend classical results to curved spaces, offering crucial tools for studying partial differential equations and variational problems on manifolds.

These inequalities connect function spaces, embedding theorems, and geometric properties of manifolds. They play a key role in proving existence and regularity of solutions to PDEs, as well as in geometric analysis and the study of minimal surfaces and harmonic maps.

### Sobolev spaces on manifolds

- Sobolev spaces extend the notion of differentiability to functions on manifolds, allowing for a more general framework to study PDEs and geometric analysis
- Sobolev spaces provide a natural setting for studying variational problems and energy functionals on manifolds

### Definition of Sobolev spaces

- Let  $M$  be a smooth, compact Riemannian manifold and  $1 \leq p \leq \infty$
- For  $k \in \mathbb{N}$ , the Sobolev space  $W^{k,p}(M)$  consists of functions  $u \in L^p(M)$  such that  $\nabla^j u \in L^p(M)$  for all  $0 \leq j \leq k$ , where  $\nabla$  is the covariant derivative on  $M$
- The norm on  $W^{k,p}(M)$  is defined as

$$\|u\|_{W^{k,p}(M)} = \left( \sum_{j=0}^k \|\nabla^j u\|_{L^p(M)}^p \right)^{1/p}$$

- For  $p = 2$ , the Sobolev space  $H^k(M) := W^{k,2}(M)$  is a Hilbert space with inner product

$$\langle u, v \rangle_{H^k(M)} = \sum_{j=0}^k \langle \nabla^j u, \nabla^j v \rangle_{L^2(M)}$$

### Properties of Sobolev spaces

- Sobolev spaces  $W^{k,p}(M)$  are Banach spaces for  $1 \leq p \leq \infty$  and  $k \in \mathbb{N}$
- For  $p = 2$ ,  $H^k(M)$  is a Hilbert space
- Sobolev spaces are reflexive for  $1 < p < \infty$
- The space  $C^\infty(M)$  of smooth functions is dense in  $W^{k,p}(M)$  for  $1 \leq p < \infty$
- Sobolev spaces satisfy interpolation inequalities, allowing for control of intermediate derivatives

### Sobolev embedding theorems

- Sobolev embedding theorems relate the integrability of functions and their derivatives on manifolds

- For  $M$  a compact Riemannian manifold of dimension  $n$ , the Sobolev embedding theorem states that if  $kp > n$ , then  $W^{k,p}(M) \hookrightarrow C^{k-[n/p]}(M)$ , where  $[n/p]$  denotes the integer part of  $n/p$
- In particular, for  $p > n/k$ , functions in  $W^{k,p}(M)$  are continuous (Morrey's inequality)
- The Rellich-Kondrachov compactness theorem asserts that for  $kp < n$ , the embedding  $W^{k,p}(M) \hookrightarrow L^q(M)$  is compact for  $1 \leq q < p^* = np/(n - kp)$

## Sobolev inequalities on manifolds

- Sobolev inequalities on manifolds provide a quantitative description of the Sobolev embedding theorems, giving explicit control on the norms of functions in different spaces
- These inequalities play a crucial role in the study of PDEs, geometric analysis, and the calculus of variations on manifolds

## Gagliardo-Nirenberg-Sobolev inequality

- The Gagliardo-Nirenberg-Sobolev inequality on a compact Riemannian manifold  $M$  of dimension  $n$  states that for  $1 \leq p < n$  and  $1 \leq q \leq p^* = np/(n - p)$ , there exists a constant  $C > 0$  such that  $\|u\|_{L^q(M)} \leq C \|\nabla u\|_{L^p(M)}^\theta \|u\|_{L^r(M)}^{1-\theta}$  for all  $u \in W^{1,p}(M)$ , where  $\theta = (n/p - n/q)/(1 - n/p + n/r)$  and  $1 \leq r \leq \infty$
- This inequality interpolates between the Sobolev and Hölder inequalities, providing a refined control on the integrability of functions
- The Gagliardo-Nirenberg-Sobolev inequality is a key tool in proving existence and regularity of solutions to PDEs on manifolds

## Trudinger-Moser inequality

- The Trudinger-Moser inequality is a limiting case of the Sobolev embedding theorem for  $p = n$ , providing an exponential integrability result
- For  $M$  a compact Riemannian manifold of dimension  $n$  and  $\alpha > 0$ , there exists a constant  $C > 0$  such that  $\int_M e^{\alpha|u|^{n/(n-1)}} dV_g \leq C$  for all  $u \in W^{1,n}(M)$  with  $\|\nabla u\|_{L^n(M)} \leq 1$
- This inequality is sharp in the sense that the exponential integrability fails for exponents larger than  $n/(n - 1)$
- The Trudinger-Moser inequality has applications in the study of conformal geometry and the prescribed curvature problem

## Concentration-compactness principle

- The concentration-compactness principle, introduced by P.L. Lions, is a powerful tool for analyzing the behavior of sequences of functions with bounded Sobolev norms
- It states that for a sequence  $(u_k)$  bounded in  $W^{1,p}(M)$ , there are three possible scenarios: 1. Compactness: a subsequence of  $(u_k)$  converges strongly in  $L^p(M)$  2. Concentration: a subsequence of  $(u_k)$  concentrates at a point, i.e., a significant portion of the  $L^p$  norm is captured in a small neighborhood 3. Dichotomy:  $(u_k)$  can be split into a sum of two sequences, one exhibiting compactness and the other concentration
- The concentration-compactness principle is crucial in proving existence of solutions to PDEs and geometric variational problems, especially in the presence of critical exponents

## Applications of Sobolev inequalities

- Sobolev inequalities on manifolds have numerous applications in various areas of mathematics, including partial differential equations, geometric analysis, and the calculus of variations
- They provide a fundamental tool for controlling the behavior of functions and their derivatives, leading to existence, regularity, and qualitative properties of solutions

## Existence of solutions to PDEs

- Sobolev inequalities are essential in proving the existence of weak solutions to PDEs on manifolds using variational methods
- For example, consider the semilinear elliptic equation  $-\Delta_g u + u = f(u)$  on a compact Riemannian manifold  $(M, g)$ , where  $f$  satisfies certain growth conditions
- The Sobolev embedding theorems and the Trudinger-Moser inequality can be used to show that the associated energy functional satisfies the Palais-Smale condition, leading to the existence of a critical point, which corresponds to a weak solution of the PDE
- Similar techniques apply to a wide range of PDEs, including the Yamabe problem, the prescribed scalar curvature problem, and the Einstein-scalar field equations

## Regularity of solutions

- Sobolev inequalities play a crucial role in establishing the regularity of weak solutions to PDEs on manifolds
- The Gagliardo-Nirenberg-Sobolev inequality and the Sobolev embedding theorems can be used in bootstrap arguments to improve the integrability and differentiability of solutions
- For instance, consider a weak solution  $u \in H^1(M)$  to a semilinear elliptic equation
- By applying the Gagliardo-Nirenberg-Sobolev inequality and the Sobolev embedding theorems iteratively, one can show that  $u \in W^{k,p}(M)$  for all  $k \in \mathbb{N}$  and  $p < \infty$ , implying that  $u$  is smooth by the Sobolev embedding theorem
- These techniques are fundamental in the regularity theory of elliptic and parabolic PDEs on manifolds

## Geometric applications

- Sobolev inequalities have significant applications in geometric analysis and the study of the relationships between the geometry and topology of manifolds
- In the Yamabe problem, which asks whether every Riemannian metric on a compact manifold is conformal to a metric of constant scalar curvature, the Sobolev inequalities are used to control the conformal factor and prove the existence of a solution
- Sobolev inequalities also play a key role in the study of minimal surfaces and harmonic maps between Riemannian manifolds, providing estimates on the energy and regularity of these objects
- In the context of Ricci flow, Sobolev inequalities are used to control the evolution of geometric quantities and establish convergence results

## Relationship with other inequalities

- Sobolev inequalities on manifolds are closely related to and can be derived from other fundamental inequalities in functional analysis and geometry
- These relationships provide additional insight into the properties of Sobolev spaces and the role of geometry in functional inequalities

## Poincaré inequality on manifolds

- The Poincaré inequality on a compact Riemannian manifold  $(M, g)$  states that there exists a constant  $C > 0$  such that  $\|u - \bar{u}\|_{L^2(M)} \leq C \|\nabla u\|_{L^2(M)}$  for all  $u \in H^1(M)$ , where  $\bar{u} = \frac{1}{\text{Vol}(M)} \int_M u dV_g$  is the average of  $u$  over  $M$
- This inequality implies that the  $H^1(M)$  norm is equivalent to the norm  $(\|\nabla u\|_{L^2(M)}^2 + |\bar{u}|^2)^{1/2}$
- The Poincaré inequality can be used to prove the Rellich-Kondrachov compactness theorem for Sobolev spaces on manifolds
- Generalizations of the Poincaré inequality, such as the Poincaré-Sobolev inequality, can be used to derive Sobolev inequalities on manifolds

## Nash inequality on manifolds

- The Nash inequality on a compact Riemannian manifold  $(M, g)$  of dimension  $n$  states that there exists a constant  $C > 0$  such that  $\|u\|_{L^2(M)}^{1+2/n} \leq C \|\nabla u\|_{L^2(M)} \|u\|_{L^1(M)}^{2/n}$  for all  $u \in H^1(M)$
- This inequality is a key tool in the study of heat kernel estimates and the behavior of the heat equation on manifolds
- The Nash inequality can be used to derive the Sobolev inequality for  $p = 2$  via interpolation with the Poincaré inequality

## Log-Sobolev inequality on manifolds

- The log-Sobolev inequality on a compact Riemannian manifold  $(M, g)$  states that there exists a constant  $C > 0$  such that  $\int_M u^2 \log \left( \frac{u^2}{\|u\|_{L^2(M)}^2} \right) dV_g \leq C \|\nabla u\|_{L^2(M)}^2$  for all  $u \in H^1(M)$  with  $\|u\|_{L^2(M)} = 1$
- This inequality provides a strengthening of the Sobolev inequality for  $p = 2$  and has applications in the study of the heat equation, Ricci curvature, and the Gaussian concentration of measure
- The log-Sobolev inequality is related to the notion of hypercontractivity of the heat semigroup and can be used to derive exponential decay estimates for solutions to the heat equation on manifolds

## Extensions and generalizations

- Sobolev inequalities on manifolds can be extended and generalized in various directions, adapting to different geometric settings and incorporating fractional or higher-order derivatives
- These extensions provide a broader framework for studying PDEs and geometric problems in more general contexts

## Sobolev inequalities on Riemannian manifolds

- Sobolev inequalities can be studied on complete, non-compact Riemannian manifolds with suitable geometric assumptions, such as bounds on the Ricci curvature or the injectivity radius
- In this setting, the inequalities often involve additional terms that depend on the geometry of the manifold, such as the isoperimetric profile or the heat kernel
- For example, on a complete Riemannian manifold with non-negative Ricci curvature, the Sobolev inequality for  $p = 2$  takes the form  $\|u\|_{L^{2n/(n-2)}(M)} \leq C \|\nabla u\|_{L^2(M)}$  for all  $u \in C_c^\infty(M)$ , where the constant  $C$  depends on the dimension  $n$  and the volume growth of the manifold
- Sobolev inequalities on non-compact manifolds have applications in the study of the Yamabe problem, the structure of Riemannian manifolds with non-negative Ricci curvature, and the behavior of harmonic functions

## Sobolev inequalities on sub-Riemannian manifolds

- Sub-Riemannian geometry is a generalization of Riemannian geometry, where the metric structure is defined only on a subbundle of the tangent bundle, leading to a constrained notion of length and geodesics
- Sobolev spaces on sub-Riemannian manifolds are defined using the horizontal gradient, which takes into account only the admissible directions of differentiation
- Sobolev inequalities in the sub-Riemannian setting involve the sub-Riemannian dimension, also known as the homogeneous dimension, which depends on the growth of the volume of sub-Riemannian balls
- For example, on the Heisenberg group  $\mathbb{H}^n$ , which is a fundamental example of a sub-Riemannian manifold, the following Sobolev inequality holds:  $\|u\|_{L^{Q/(Q-1)}(\mathbb{H}^n)} \leq C \|\nabla_H u\|_{L^1(\mathbb{H}^n)}$  for all  $u \in C_c^\infty(\mathbb{H}^n)$ , where  $Q = 2n + 2$  is the homogeneous dimension and  $\nabla_H$  denotes the horizontal gradient
- Sub-Riemannian Sobolev inequalities have applications in the study of hypoelliptic PDEs, control theory, and the geometry of sub-Riemannian manifolds

## Fractional Sobolev inequalities on manifolds

- Fractional Sobolev spaces on manifolds extend the notion of differentiability to non-integer orders, allowing for a more refined analysis of the regularity of functions
- For  $s \in (0, 1)$  and  $1 \leq p < \infty$ , the fractional Sobolev space  $W^{s,p}(M)$  on a compact Riemannian manifold  $(M, g)$  is defined as the space of functions  $u \in L^p(M)$  such that  $\iint_{M \times M} \frac{|u(x) - u(y)|^p}{d_g(x, y)^{n+sp}} dV_g(x) dV_g(y) < \infty$ , where  $d_g$  denotes the Riemannian distance on  $M$
- Fractional Sobolev inequalities on manifolds provide a bridge between the classical Sobolev inequalities and the isoperimetric inequality, which corresponds to the limiting case  $s = 1$
- For example, on a compact Riemannian manifold of dimension  $n$ , the following fractional Sobolev inequality holds for  $1 \leq p < n/s$ :  $\|u\|_{L^{p^*}(M)} \leq C \left( \iint_{M \times M} \frac{|u(x) - u(y)|^p}{d_g(x, y)^{n+sp}} dV_g(x) dV_g(y) \right)^{1/p}$ , where  $p^* = np/(n - sp)$

## 11.6 Eigenvalues of the Laplacian

Eigenvalues of the Laplacian are key to understanding geometric properties of Riemannian manifolds. They provide insights into the manifold's shape, size, and curvature. By studying these eigenvalues, we can uncover hidden relationships between a manifold's spectrum and its intrinsic geometry.

This topic connects spectral theory with differential geometry, revealing how abstract mathematical concepts relate to physical phenomena. From heat diffusion to quantum mechanics, eigenvalues of the Laplacian have wide-ranging applications in physics and engineering, making them a crucial area of study in modern mathematics.

### Definition of eigenvalues

- Eigenvalues are scalar values associated with linear operators or matrices that characterize their behavior under transformation
- In the context of Metric Differential Geometry, eigenvalues play a crucial role in understanding the properties of the Laplacian operator on Riemannian manifolds
- Studying the eigenvalues of the Laplacian provides insights into the geometric and topological properties of the underlying manifold

### Eigenvalues of linear operators

- Given a linear operator  $A$  on a vector space  $V$ , a scalar  $\lambda$  is an eigenvalue of  $A$  if there exists a nonzero vector  $v \in V$  such that  $Av = \lambda v$
- The vector  $v$  is called an eigenvector corresponding to the eigenvalue  $\lambda$
- Eigenvalues and eigenvectors capture the stretching or scaling behavior of the linear operator in certain directions

### Laplacian as linear operator

- The Laplacian  $\Delta$  is a second-order differential operator that acts on functions defined on a Riemannian manifold  $M$
- In local coordinates, the Laplacian is given by  $\Delta = -\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j)$ , where  $g$  is the metric tensor and  $g^{ij}$  are the components of its inverse
- The Laplacian is a self-adjoint linear operator on the space of square-integrable functions  $L^2(M)$

### Eigenfunctions of Laplacian

- An eigenfunction of the Laplacian is a non-zero function  $f$  on the manifold  $M$  that satisfies the equation  $\Delta f = \lambda f$  for some scalar  $\lambda$
- The scalar  $\lambda$  is the corresponding eigenvalue of the Laplacian
- Eigenfunctions of the Laplacian form a basis for the function space  $L^2(M)$  and play a central role in the spectral analysis of the manifold

## Spectral geometry

- Spectral geometry is the study of the relationship between the geometric properties of a Riemannian manifold and the spectrum of its Laplacian operator
- It aims to extract geometric information about the manifold from the eigenvalues and eigenfunctions of the Laplacian
- Spectral geometry provides a powerful tool for understanding the intrinsic geometry of manifolds and has applications in various areas of mathematics and physics

## Spectrum of Laplacian

- The spectrum of the Laplacian is the set of all eigenvalues  $\{\lambda_i\}$  of the Laplacian operator
- On a compact Riemannian manifold, the spectrum is discrete and consists of an increasing sequence of non-negative real numbers  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$
- The multiplicity of each eigenvalue represents the dimension of the corresponding eigenspace

## Geometric information from spectrum

- The spectrum of the Laplacian encodes various geometric properties of the manifold
- For example, the first non-zero eigenvalue  $\lambda_1$  is related to the Cheeger constant, which measures the isoperimetric ratio of the manifold
- The asymptotic behavior of the eigenvalues provides information about the dimension, volume, and curvature of the manifold

## Isospectrality vs isometry

- Two Riemannian manifolds are said to be isospectral if they have the same spectrum of the Laplacian
- Isospectral manifolds share many geometric properties, but they may not be isometric (i.e., have the same metric)
- The question of whether isospectrality implies isometry is a fundamental problem in spectral geometry, and counterexamples have been constructed (Sunada's method)

## Computation of eigenvalues

- Computing the eigenvalues of the Laplacian is a central problem in spectral geometry and has important applications in various fields
- Several techniques and principles are used to estimate and approximate the eigenvalues, both theoretically and numerically
- Understanding the computational aspects of eigenvalues is crucial for practical applications and gaining insights into the geometric properties of manifolds

## Variational characterization

- The eigenvalues of the Laplacian can be characterized variationally using the Rayleigh quotient
- For a function  $f \in H^1(M)$ , the Rayleigh quotient is defined as  $R(f) = \frac{\int_M |\nabla f|^2 dV}{\int_M f^2 dV}$

- The eigenvalues can be obtained by minimizing the Rayleigh quotient over suitable function spaces, leading to the min-max principle

## Min-max principle

- The min-max principle provides a variational characterization of the eigenvalues
- The  $k$ -th eigenvalue  $\lambda_k$  can be obtained as  $\lambda_k = \min_{V_k} \max_{f \in V_k \setminus \{0\}} R(f)$ , where  $V_k$  ranges over all  $k$ -dimensional subspaces of  $H^1(M)$
- The min-max principle allows for the estimation and approximation of eigenvalues using finite-dimensional subspaces and numerical methods

## Courant-Fischer theorem

- The Courant-Fischer theorem is a generalization of the min-max principle for self-adjoint operators
- It provides a variational characterization of the eigenvalues in terms of the Rayleigh quotient and orthogonal subspaces
- The theorem states that  $\lambda_k = \min_{V_{k+1}} \max_{f \in V_{k+1} \setminus \{0\}} R(f)$ , where  $V_{k+1}$  ranges over all  $(k+1)$ -dimensional subspaces of  $H^1(M)$

## Weyl's law

- Weyl's law describes the asymptotic behavior of the eigenvalues of the Laplacian on a compact Riemannian manifold as the index tends to infinity
- It provides a powerful tool for understanding the relationship between the eigenvalues and the geometric properties of the manifold
- Weyl's law has important implications in spectral geometry, as it connects the spectrum of the Laplacian to the dimension, volume, and curvature of the manifold

## Asymptotic behavior of eigenvalues

- Weyl's law states that the eigenvalues  $\lambda_k$  of the Laplacian on a compact Riemannian manifold  $M$  of dimension  $n$  satisfy the asymptotic relation  $\lambda_k \sim C_n \left(\frac{k}{\text{Vol}(M)}\right)^{2/n}$  as  $k \rightarrow \infty$
- The constant  $C_n$  depends only on the dimension  $n$  and is given by  $C_n = 4\pi^2 \left(\frac{\Gamma(\frac{n}{2} + 1)}{|S^{n-1}|}\right)^{2/n}$ , where  $|S^{n-1}|$  denotes the volume of the  $(n-1)$ -dimensional unit sphere
- Weyl's law provides a precise estimate of the growth rate of the eigenvalues and their distribution for large indices

## Weyl's asymptotic formula

- Weyl's asymptotic formula gives a more refined estimate of the counting function  $N(\lambda) = \#\{k : \lambda_k \leq \lambda\}$ , which counts the number of eigenvalues less than or equal to a given value  $\lambda$
- The formula states that  $N(\lambda) \sim \frac{\text{Vol}(M)}{(4\pi)^{n/2} \Gamma(\frac{n}{2} + 1)} \lambda^{n/2}$  as  $\lambda \rightarrow \infty$
- Weyl's asymptotic formula captures the leading order term in the asymptotic expansion of the counting function and provides a more precise description of the distribution of eigenvalues



## Geometric interpretation

- Weyl's law and asymptotic formula have important geometric interpretations
- The leading order term in Weyl's asymptotic formula depends on the volume of the manifold, indicating that the asymptotic behavior of the eigenvalues is determined by the global size of the manifold
- The second-order term in the asymptotic expansion involves the scalar curvature of the manifold, revealing a connection between the eigenvalues and the local geometry
- Weyl's law demonstrates the interplay between the spectrum of the Laplacian and the geometric properties of the underlying manifold

## Heat kernel

- The heat kernel is a fundamental object in spectral geometry that describes the diffusion of heat on a Riemannian manifold
- It is closely related to the eigenvalues and eigenfunctions of the Laplacian and provides a powerful tool for studying the geometry of the manifold
- The heat kernel has important applications in various areas, including heat equation, spectral analysis, and geometric analysis

## Heat equation on manifolds

- The heat equation on a Riemannian manifold  $M$  is a partial differential equation that describes the evolution of a temperature distribution  $u(x, t)$  over time
- It is given by  $\frac{\partial u}{\partial t} = \Delta u$ , where  $\Delta$  is the Laplacian operator on the manifold
- The heat equation models the diffusion of heat on the manifold and is a fundamental equation in geometric analysis

## Heat kernel as eigenfunction expansion

- The heat kernel  $K(x, y, t)$  is the fundamental solution of the heat equation on the manifold
- It can be expressed as an infinite series expansion in terms of the eigenfunctions  $\phi_k(x)$  and eigenvalues  $\lambda_k$  of the Laplacian:

$$K(x, y, t) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y)$$

- The heat kernel encodes information about the geometry of the manifold through its dependence on the eigenvalues and eigenfunctions

## Short-time asymptotic expansion

- The short-time asymptotic expansion of the heat kernel provides valuable information about the local geometry of the manifold
- As  $t \rightarrow 0$ , the heat kernel admits an asymptotic expansion of the form
$$K(x, y, t) \sim \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(x,y)^2}{4t}} (a_0(x, y) + a_1(x, y)t + a_2(x, y)t^2 + \dots)$$

- The coefficients  $a_i(X, Y)$  in the expansion are determined by the local geometric invariants of the manifold, such as the curvature and its derivatives
- The short-time asymptotic expansion reveals the interplay between the heat kernel and the geometric properties of the manifold

## Nodal sets

- Nodal sets are the zero sets of eigenfunctions of the Laplacian on a Riemannian manifold
- They provide a geometric and topological perspective on the behavior of eigenfunctions and have important connections to the geometry of the manifold
- The study of nodal sets is a central topic in spectral geometry and has applications in various areas, including quantum mechanics and pattern formation

## Definition of nodal sets

- Given an eigenfunction  $\phi$  of the Laplacian on a Riemannian manifold  $M$ , the nodal set of  $\phi$  is defined as  $N(\phi) = \{x \in M : \phi(x) = 0\}$
- The nodal set is the set of points where the eigenfunction vanishes and separates the manifold into regions where the eigenfunction has opposite signs
- Nodal sets are  $(n - 1)$ -dimensional submanifolds of  $M$  and have a rich geometric and topological structure

## Nodal domains of eigenfunctions

- A nodal domain of an eigenfunction  $\phi$  is a connected component of the complement of its nodal set, i.e., a region where  $\phi$  has a constant sign
- The nodal domains partition the manifold into regions where the eigenfunction is positive or negative
- The study of nodal domains provides insights into the oscillatory behavior of eigenfunctions and their relation to the geometry of the manifold

## Courant's nodal domain theorem

- Courant's nodal domain theorem is a fundamental result in spectral geometry that bounds the number of nodal domains of an eigenfunction
- It states that the number of nodal domains of the  $k$ -th eigenfunction  $\phi_k$  is at most  $k + 1$
- The theorem provides a topological constraint on the behavior of eigenfunctions and has important implications for the study of nodal sets and their properties

## Applications

- The study of eigenvalues of the Laplacian on Riemannian manifolds has numerous applications in various fields of mathematics, physics, and engineering
- Spectral geometry techniques and results find applications in shape optimization, inverse problems, quantum mechanics, and the study of chaotic systems
- Understanding the eigenvalues and their properties opens up new avenues for solving practical problems and gaining insights into the behavior of physical systems

## Shape optimization problems

- Shape optimization problems involve finding the optimal shape of a domain or manifold that minimizes or maximizes a certain functional while satisfying given constraints
- Eigenvalues of the Laplacian play a crucial role in shape optimization, as they are often used as objective functionals or constraints
- Spectral geometry techniques, such as the min-max principle and Weyl's law, are employed to characterize optimal shapes and develop efficient optimization algorithms

## Inverse spectral problems

- Inverse spectral problems aim to reconstruct the geometry of a manifold from its spectrum or partial spectral data
- The goal is to determine the metric or other geometric properties of the manifold based on the knowledge of its eigenvalues or eigenfunctions
- Spectral geometry provides the theoretical foundation and computational tools for tackling inverse spectral problems and has applications in areas such as seismology and medical imaging

## Quantum chaos on manifolds

- Quantum chaos studies the quantum mechanical behavior of systems whose classical counterparts exhibit chaotic dynamics
- Riemannian manifolds serve as natural models for quantum chaotic systems, and the eigenvalues and eigenfunctions of the Laplacian play a central role in understanding their properties
- Spectral geometry techniques, such as Weyl's law and the heat kernel, are used to analyze the statistical properties of eigenvalues and the distribution of eigenfunctions in the semiclassical limit
- The study of quantum chaos on manifolds has important implications for understanding the correspondence between classical and quantum mechanics and the behavior of complex quantum systems

# Unit 12 – Physics Applications in Differential Geometry

## 12.1 General relativity and spacetime geometry

General relativity revolutionized our understanding of gravity, describing it as the curvature of spacetime caused by mass and energy. This theory explains phenomena like black holes, gravitational waves, and the expansion of the universe.

Spacetime geometry is central to general relativity. It's described using a four-dimensional manifold with a Lorentzian metric, allowing us to mathematically represent the warping of space and time by massive objects.

### Spacetime as a pseudo-Riemannian manifold

- Spacetime is a 4-dimensional manifold that combines the 3 spatial dimensions and 1 time dimension into a single geometric structure
- In general relativity, spacetime is described as a pseudo-Riemannian manifold, which is a generalization of a Riemannian manifold that allows for indefinite metrics
- The metric tensor of spacetime determines the geometry of the manifold and how distances and angles are measured between events

### Metrics of spacetime

#### Minkowski metric for special relativity

- The Minkowski metric is a flat spacetime metric used in special relativity to describe the geometry of spacetime in the absence of gravity
- It is given by the line element  $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$ , where  $c$  is the speed of light and  $dt$ ,  $dx$ ,  $dy$ , and  $dz$  are the infinitesimal changes in time and spatial coordinates
- The Minkowski metric has a signature of  $(-, +, +, +)$ , indicating one timelike dimension and three spacelike dimensions

#### Lorentzian metrics for general relativity

- Lorentzian metrics are used in general relativity to describe the geometry of spacetime in the presence of gravity
- They are pseudo-Riemannian metrics with a signature of  $(-, +, +, +)$  or  $(+, -, -, -)$ , depending on the convention used
- The metric tensor in general relativity is determined by the distribution of matter and energy in spacetime through Einstein's field equations

## Curvature of spacetime

### Riemann curvature tensor

- The Riemann curvature tensor is a mathematical object that describes the intrinsic curvature of a manifold
- It measures how parallel transport of vectors around closed loops fails to return them to their original orientation
- In general relativity, the Riemann tensor is determined by the metric tensor and its derivatives

### Ricci curvature tensor

- The Ricci curvature tensor is obtained by contracting the Riemann tensor over two of its indices
- It provides a measure of the average curvature in different directions at a given point in spacetime
- The Ricci tensor appears in Einstein's field equations and relates the curvature of spacetime to the distribution of matter and energy

### Scalar curvature

- The scalar curvature, also known as the Ricci scalar, is a scalar quantity obtained by further contracting the Ricci tensor
- It gives a single number that characterizes the overall curvature of spacetime at a given point
- The scalar curvature appears in the Einstein-Hilbert action, which is used to derive Einstein's field equations from a variational principle

## Geodesics in spacetime

### Timelike vs spacelike geodesics

- Geodesics are the straightest possible paths between two points in a curved spacetime
- Timelike geodesics represent the worldlines of massive particles and always have a tangent vector with a negative norm ( $ds^2 < 0$ )
- Spacelike geodesics have a tangent vector with a positive norm ( $ds^2 > 0$ ) and cannot be physically traversed by massive particles

### Null geodesics for light paths

- Null geodesics have a tangent vector with a zero norm ( $ds^2 = 0$ ) and represent the paths followed by light rays in spacetime
- They are the boundaries between timelike and spacelike regions in spacetime
- The paths of light are bent by the curvature of spacetime, leading to phenomena such as gravitational lensing

## Geodesic deviation

- Geodesic deviation describes how initially parallel geodesics can diverge or converge due to the curvature of spacetime
- The relative acceleration between nearby geodesics is governed by the Riemann curvature tensor
- Geodesic deviation is responsible for the tidal forces experienced by extended objects in a gravitational field

## Einstein's field equations

### Energy-momentum tensor

- The energy-momentum tensor is a symmetric rank-2 tensor that describes the density and flux of energy and momentum in spacetime
- It includes contributions from matter, radiation, and other fields present in the universe
- The energy-momentum tensor acts as the source term in Einstein's field equations, determining how matter and energy curve spacetime

### Coupling of matter and geometry

- Einstein's field equations relate the curvature of spacetime, described by the Einstein tensor (constructed from the Ricci tensor and scalar curvature), to the energy-momentum tensor
- The equations take the form  $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ , where  $G_{\mu\nu}$  is the Einstein tensor,  $T_{\mu\nu}$  is the energy-momentum tensor,  $G$  is Newton's gravitational constant, and  $c$  is the speed of light
- This coupling between matter and geometry is a fundamental principle of general relativity, showing that matter and energy determine the curvature of spacetime, which in turn affects the motion of matter and energy

### Vacuum solutions

- Vacuum solutions to Einstein's field equations are spacetimes in which the energy-momentum tensor vanishes ( $T_{\mu\nu} = 0$ )
- These solutions describe regions of spacetime that are devoid of matter and energy, but can still have non-trivial curvature
- Examples of vacuum solutions include the Schwarzschild metric (describing the spacetime around a non-rotating, uncharged black hole) and the Kerr metric (describing the spacetime around a rotating black hole)

### Schwarzschild solution

#### Spherical symmetry

- The Schwarzschild solution is a spherically symmetric, static solution to Einstein's field equations in vacuum
- It describes the spacetime geometry around a non-rotating, uncharged, spherically symmetric massive object, such as a black hole or a star

- The metric is given by  $ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$ , where  $M$  is the mass of the central object and  $d\Omega^2$  is the line element on a unit 2-sphere

## Event horizon and singularity

- The Schwarzschild solution has an event horizon at the Schwarzschild radius,  $r_s = \frac{2GM}{c^2}$ , which is the boundary of the black hole
- Inside the event horizon, the roles of time and radial coordinate are switched, making it impossible for anything, including light, to escape from the black hole
- At the center of the black hole ( $r = 0$ ), there is a gravitational singularity where the curvature of spacetime becomes infinite and general relativity breaks down

## Orbits in Schwarzschild spacetime

- The orbits of particles and light in Schwarzschild spacetime are determined by the geodesic equations
- Stable circular orbits exist only for radii greater than  $3r_s$ , while unstable circular orbits exist between  $1.5r_s$  and  $3r_s$
- At the photon sphere, located at  $1.5r_s$ , light can orbit the black hole in unstable circular orbits
- Particles and light can also follow bound elliptical orbits or unbound hyperbolic trajectories, depending on their initial conditions

## Kerr solution for rotating black holes

### Stationary axisymmetric spacetimes

- The Kerr solution describes the spacetime geometry around a rotating, uncharged black hole
- It is a stationary and axisymmetric solution to Einstein's field equations in vacuum, meaning that the metric is independent of time and azimuthal angle
- The Kerr metric is given by  $ds^2 = -\left(1 - \frac{2GMr}{c^2 \rho^2}\right)c^2 dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2GMr a^2 \sin^2 \theta}{c^2 \rho^2}\right) \sin^2 \theta d\phi^2 - \frac{4GMr a \sin^2 \theta}{c \rho^2} c dt d\phi$ , where  $M$  is the mass of the black hole,  $a$  is the angular momentum per unit mass,  $\rho^2 = r^2 + a^2 \cos^2 \theta$ , and  $\Delta = r^2 - \frac{2GMr}{c^2} + a^2$

### Ergosphere and extracting energy

- The Kerr solution features an ergosphere, a region outside the event horizon where spacetime is dragged along with the rotation of the black hole
- Within the ergosphere, it is impossible for particles to remain stationary with respect to a distant observer, as they must rotate in the same direction as the black hole
- The existence of the ergosphere allows for the extraction of energy from a rotating black hole through the Penrose process, in which a particle enters the ergosphere, splits into two, and one of the resulting particles escapes with more energy than the original particle

## Orbits in Kerr spacetime

- The orbits of particles and light in Kerr spacetime are more complex than in Schwarzschild spacetime due to the presence of the black hole's rotation
- The innermost stable circular orbit (ISCO) depends on the black hole's angular momentum and the particle's orbital direction (prograde or retrograde)
- For a maximally rotating black hole, the ISCO is located at the event horizon for prograde orbits and at  $9GM/c^2$  for retrograde orbits
- The Kerr metric also exhibits frame-dragging, or the Lense-Thirring effect, where a rotating black hole drags nearby spacetime and influences the orbits of particles and light

## Cosmological solutions

### Friedmann-Lemaître-Robertson-Walker metric

- The Friedmann-Lemaître-Robertson-Walker (FLRW) metric is a solution to Einstein's field equations that describes a homogeneous and isotropic universe
- It is the foundation of modern cosmology and is given by  $ds^2 = -c^2 dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]$ , where  $a(t)$  is the scale factor that describes the expansion or contraction of the universe, and  $k$  is the curvature parameter ( $k = 0$  for a flat universe,  $k > 0$  for a closed universe, and  $k < 0$  for an open universe)
- The FLRW metric is used to study the evolution of the universe on large scales and to derive the Friedmann equations that relate the expansion rate to the matter and energy content of the universe

### Cosmological constant and dark energy

- The cosmological constant, denoted by  $\Lambda$ , is a term that can be added to Einstein's field equations to represent a constant energy density of the vacuum
- It was originally introduced by Einstein to achieve a static universe, but was later abandoned after the discovery of the expansion of the universe
- In the context of modern cosmology, the cosmological constant is often associated with dark energy, a hypothetical form of energy that permeates all of space and drives the accelerated expansion of the universe
- The presence of a positive cosmological constant leads to a repulsive gravitational effect that becomes dominant at large scales, causing the universe to expand at an increasing rate

### Expansion history of the universe

- The expansion history of the universe describes how the scale factor  $a(t)$  evolves with time
- It is determined by the Friedmann equations, which relate the expansion rate to the matter and energy content of the universe (including ordinary matter, radiation, and dark energy)
- The universe began with a Big Bang singularity at  $t = 0$ , followed by a period of rapid inflation, during which the universe expanded exponentially
- After inflation, the universe underwent a period of radiation domination, followed by a period of matter domination



- Recent observations indicate that the universe is currently in a phase of accelerated expansion, driven by dark energy (which may be represented by a cosmological constant or a more general form of energy with negative pressure)

## Gravitational waves

### Linearized gravity

- Linearized gravity is an approximation to general relativity that is valid when the spacetime curvature is weak and the gravitational fields are small
- In this approximation, the metric tensor can be written as the sum of the Minkowski metric and a small perturbation:  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ , where  $\eta_{\mu\nu}$  is the Minkowski metric and  $h_{\mu\nu}$  is the perturbation (with  $|h_{\mu\nu}| \ll 1$ )
- Linearized gravity is used to study the propagation of gravitational waves, which are small ripples in the fabric of spacetime that travel at the speed of light

### Sources of gravitational waves

- Gravitational waves are generated by accelerating masses, similar to how electromagnetic waves are generated by accelerating charges
- The most significant sources of gravitational waves are compact binary systems, such as binary black holes, binary neutron stars, or a black hole-neutron star binary
- These systems emit gravitational waves as they orbit each other, losing energy and angular momentum in the process, which causes their orbits to decay and eventually leads to a merger
- Other sources of gravitational waves include supernovae, rapidly rotating neutron stars (pulsars), and the early universe (e.g., primordial gravitational waves from inflation)

### Detection of gravitational waves

- Gravitational waves were first directly detected in 2015 by the Laser Interferometer Gravitational-Wave Observatory (LIGO), a century after their prediction by Einstein's general theory of relativity
- LIGO and other ground-based interferometers, such as Virgo and KAGRA, detect gravitational waves by measuring the tiny changes in the relative distances between suspended mirrors caused by the passing gravitational wave
- These interferometers are sensitive to gravitational waves in the frequency range of about 10 Hz to 10 kHz, which corresponds to the mergers of stellar-mass compact objects
- Future space-based interferometers, such as the Laser Interferometer Space Antenna (LISA), will be sensitive to gravitational waves in the millihertz range, which corresponds to the mergers of supermassive black holes and other extreme-mass-ratio inspirals

## 12.2 Hamiltonian mechanics on manifolds

Hamiltonian mechanics on manifolds provides a geometric framework for studying classical mechanics. It uses symplectic manifolds to represent phase spaces, with Hamiltonian vector fields generating dynamics. This approach emphasizes conserved quantities and symmetries in physical systems.

Key concepts include symplectic forms, Poisson brackets, and canonical transformations. Hamilton's equations describe system evolution on symplectic manifolds. Understanding this geometric perspective illuminates the deep structure underlying classical mechanics and its connection to modern physics.

## Symplectic manifolds

- Symplectic manifolds provide the geometric framework for studying Hamiltonian mechanics, a formulation of classical mechanics that emphasizes the role of conserved quantities and symmetries
- A symplectic manifold is an even-dimensional smooth manifold equipped with a closed, non-degenerate 2-form called the symplectic form, which encodes the geometric structure necessary for Hamiltonian dynamics

## Definition of symplectic manifold

- A symplectic manifold is a pair  $(M, \omega)$  where  $M$  is a smooth manifold and  $\omega$  is a symplectic form on  $M$
- A symplectic form  $\omega$  is a closed ( $d\omega = 0$ ), non-degenerate 2-form, meaning that for every non-zero tangent vector  $v \in T_p M$ , there exists a tangent vector  $w \in T_p M$  such that  $\omega(v, w) \neq 0$
- The non-degeneracy condition implies that symplectic manifolds are always even-dimensional

## Examples of symplectic manifolds

- The cotangent bundle  $T^*M$  of any smooth manifold  $M$  is a symplectic manifold with the canonical symplectic form

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i$$

, where  $(q_i, p_i)$  are local coordinates on  $T^*M$

- The product of two symplectic manifolds  $(M_1, \omega_1)$  and  $(M_2, \omega_2)$  is a symplectic manifold with the symplectic form  $\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$ , where  $\pi_1$  and  $\pi_2$  are the projection maps
- Kähler manifolds, which are complex manifolds with a compatible Riemannian metric and symplectic form, are examples of symplectic manifolds (complex projective spaces)

## Darboux's theorem

- Darboux's theorem states that locally, all symplectic manifolds of the same dimension are indistinguishable
- More precisely, for any point  $P$  in a symplectic manifold  $(M, \omega)$ , there exists a neighborhood  $U$  of  $P$  and local coordinates  $(q_1, \dots, q_n, p_1, \dots, p_n)$  on  $U$  such that

$$\omega|_U = \sum_{i=1}^n dq_i \wedge dp_i$$

- This theorem implies that symplectic manifolds have no local invariants other than the dimension, and the study of symplectic manifolds focuses on their global properties

## Hamiltonian vector fields

- Hamiltonian vector fields are the fundamental objects that generate the dynamics on a symplectic manifold

- They are defined in terms of a smooth function called the Hamiltonian, which represents the total energy of the system

## Definition of Hamiltonian vector field

- Given a smooth function  $H : M \rightarrow \mathbb{R}$  on a symplectic manifold  $(M, \omega)$ , the Hamiltonian vector field  $X_H$  associated with  $H$  is the unique vector field satisfying  $dH = \omega(X_H, \cdot)$
- In local coordinates  $(q_i, p_i)$ , the Hamiltonian vector field is given by

$$X_H = \sum_{i=1}^n \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right)$$

## Properties of Hamiltonian vector fields

- Hamiltonian vector fields preserve the symplectic form:  $\mathcal{L}_{X_H} \omega = 0$ , where  $\mathcal{L}_{X_H}$  denotes the Lie derivative along  $X_H$
- The flow generated by a Hamiltonian vector field consists of symplectomorphisms, which are diffeomorphisms that preserve the symplectic form
- The Poisson bracket of two functions  $f$  and  $g$  on a symplectic manifold is related to the Lie bracket of their Hamiltonian vector fields:  $\{f, g\} = \omega(X_f, X_g)$

## Poisson brackets

- The Poisson bracket is a bilinear operation on smooth functions that encodes the geometry of the symplectic manifold and the algebra of observables in Hamiltonian mechanics
- Given two smooth functions  $f, g : M \rightarrow \mathbb{R}$ , their Poisson bracket is defined as  $\{f, g\} = \omega(X_f, X_g)$
- In local coordinates  $(q_i, p_i)$ , the Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

- The Poisson bracket satisfies the Jacobi identity:  $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ , making the space of smooth functions a Lie algebra under the Poisson bracket operation

## Hamiltonian dynamics

- Hamiltonian dynamics describes the time evolution of a physical system in terms of the symplectic geometry of its phase space, which is typically the cotangent bundle of the configuration manifold
- The dynamics are governed by Hamilton's equations, which are first-order differential equations involving the Hamiltonian function

## Hamilton's equations

- Given a Hamiltonian function  $H : M \rightarrow \mathbb{R}$  on a symplectic manifold  $(M, \omega)$ , Hamilton's equations are a system of first-order differential equations that describe the time evolution of the system:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

- The solutions to Hamilton's equations are integral curves of the Hamiltonian vector field  $X_H$ , which generate the Hamiltonian flow on the symplectic manifold

## Conservation of energy

- A key feature of Hamiltonian systems is the conservation of energy along the Hamiltonian flow
- If the Hamiltonian  $H$  does not explicitly depend on time, then  $\frac{dH}{dt} = 0$  along the solutions of Hamilton's equations
- This means that the Hamiltonian function, which typically represents the total energy of the system, remains constant along the trajectories of the system

## Liouville's theorem

- Liouville's theorem states that the Hamiltonian flow preserves the symplectic volume form  $\omega^n = \omega \wedge \cdots \wedge \omega$  ( $n$  times) on the symplectic manifold
- As a consequence, the volume of any region in the phase space remains constant under the Hamiltonian flow
- Liouville's theorem is a manifestation of the incompressibility of the Hamiltonian flow and has important implications for statistical mechanics and the study of ensembles of Hamiltonian systems

## Canonical transformations

- Canonical transformations are coordinate transformations on a symplectic manifold that preserve the symplectic structure
- They play a crucial role in simplifying the analysis of Hamiltonian systems and in the study of symmetries and conserved quantities

## Definition of canonical transformation

- A canonical transformation is a diffeomorphism  $\phi : M \rightarrow M$  on a symplectic manifold  $(M, \omega)$  that preserves the symplectic form:  $\phi^* \omega = \omega$
- Equivalently, a canonical transformation is a coordinate transformation  $(q_i, p_i) \mapsto (Q_i, P_i)$  that preserves the form of Hamilton's equations:

$$\frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i}, \quad \frac{dP_i}{dt} = -\frac{\partial K}{\partial Q_i}$$

where  $K(Q_i, P_i, t)$  is the transformed Hamiltonian

## Generating functions

- Canonical transformations can be described using generating functions, which are scalar functions that implicitly define the transformation
- There are four common types of generating functions, each depending on a different combination of old and new coordinates:

$$1. F_1(q_i, Q_i, t): p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$$

$$2. F_2(q_i, P_i, t): p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$3. F_3(p_i, Q_i, t): q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$4. F_4(p_i, P_i, t): q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i}$$

- The transformed Hamiltonian  $K$  is related to the original Hamiltonian  $H$  and the generating function  $F$  by the equation:

$$K(Q_I, P_I, t) = H(q_I, p_I, t) + \frac{\partial F}{\partial t}$$

## Examples of canonical transformations

- Point transformations, which are coordinate transformations that depend only on the coordinates  $(q_I, Q_I)$ , are canonical transformations with generating functions of type  $F_1$
- The Legendre transformation, which relates the Lagrangian and Hamiltonian formulations of mechanics, is a canonical transformation with a generating function of type  $F_2$
- The action-angle transformation, which is used in the study of integrable systems, is a canonical transformation that separates the dynamics into periodic motions and conserved quantities

## Symplectic group

- The symplectic group is the group of linear canonical transformations on a symplectic vector space
- It plays a fundamental role in the study of the symmetries and conservation laws of Hamiltonian systems

## Definition of symplectic group

- The symplectic group  $Sp(2n, \mathbb{R})$  is the group of linear transformations  $A$  on a symplectic vector space  $(V, \omega)$  of dimension  $2n$  that preserve the symplectic form:  $\omega(Av, Aw) = \omega(v, w)$  for all  $v, w \in V$
- In matrix form, a symplectic matrix  $A$  satisfies the condition  $A^T J A = J$ , where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

is the standard symplectic matrix and  $I_n$  is the  $n \times n$  identity matrix

## Lie algebra of symplectic group

- The Lie algebra  $\mathfrak{sp}(2n, \mathbb{R})$  of the symplectic group consists of the matrices  $X$  satisfying  $X^T J + J X = 0$
- The elements of the Lie algebra generate infinitesimal symplectic transformations and are closely related to the conserved quantities of Hamiltonian systems
- The Lie algebra  $\mathfrak{sp}(2n, \mathbb{R})$  is isomorphic to the space of quadratic polynomials in  $q_I$  and  $p_I$  under the Poisson bracket operation

## Symplectic matrices

- Symplectic matrices have several important properties:
- The inverse of a symplectic matrix is also symplectic:  $(A^T J A = J) \Rightarrow (A^{-1})^T = -J A^T J$
- The product of two symplectic matrices is symplectic:  $(A^T J A = J, B^T J B = J) \Rightarrow ((AB)^T J (AB) = J)$
- The determinant of a symplectic matrix is equal to 1:  $\det(A) = 1$
- Examples of symplectic matrices include rotation matrices in the plane, which have the form

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

, and the Pauli matrices  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$  in quantum mechanics

## Hamiltonian systems on cotangent bundles

- Many physical systems, such as particles moving in a potential or rigid body dynamics, are naturally described as Hamiltonian systems on the cotangent bundle of a configuration manifold
- The cotangent bundle has a canonical symplectic structure that makes it a natural setting for Hamiltonian mechanics

## Cotangent bundles as symplectic manifolds

- Given a smooth manifold  $Q$ , its cotangent bundle  $T^*Q$  is the space of covectors (linear functionals) on the tangent spaces of  $Q$
- The cotangent bundle  $T^*Q$  has a canonical symplectic form  $\omega = dq_i \wedge dp_i$ , where  $(q_i, p_i)$  are local coordinates on  $T^*Q$  induced by coordinates  $q_i$  on  $Q$
- The canonical symplectic form makes  $T^*Q$  a symplectic manifold, and the projection  $\pi : T^*Q \rightarrow Q$  is called the cotangent bundle projection

## Legendre transform

- The Legendre transform is a canonical transformation that relates the Lagrangian and Hamiltonian formulations of mechanics
- Given a Lagrangian function  $L(q_i, \dot{q}_i, t)$ , the Legendre transform defines the conjugate momenta  $p_i$  and the Hamiltonian function  $H(q_i, p_i, t)$  as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad H(q_i, p_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$$

- The Legendre transform is a canonical transformation with a generating function of type  $F_2(q_i, P_i, t) = P_i \dot{q}_i - L(q_i, \dot{q}_i, t)$ , where  $P_i$  are the new momenta

## Hamiltonian formulation of Lagrangian mechanics

- The Hamiltonian formulation of mechanics is obtained by applying the Legendre transform to the Lagrangian formulation
- The Euler-Lagrange equations of motion in Lagrangian mechanics,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0,$$

are transformed into Hamilton's equations in the Hamiltonian formulation:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

- The Hamiltonian formulation provides a more symmetric treatment of coordinates and momenta and is better suited for studying symmetries, conservation laws, and canonical transformations

## Moment maps

- Moment maps are a generalization of conserved quantities in the presence of symmetries, and they play a crucial role in the study of Hamiltonian systems with symmetries

- They provide a link between the symmetries of a Hamiltonian system and the geometry of the symplectic manifold

## Definition of moment map

- Let  $(M, \omega)$

## 12.3 Geometric mechanics and symplectic geometry

Geometric mechanics and symplectic geometry provide a powerful framework for understanding classical mechanical systems. These fields use manifolds to describe configuration and phase spaces, allowing for coordinate-free analysis of system dynamics.

Lagrangian and Hamiltonian mechanics are two key formulations within this framework. They use different approaches to derive equations of motion, but both rely on the underlying geometric structure of the system's phase space.

### Manifolds in classical mechanics

- Manifolds provide a mathematical framework for describing the configuration and phase spaces of classical mechanical systems
- They allow for a coordinate-free and intrinsic description of the geometry and dynamics of these systems

### Configuration space manifolds

- Configuration space manifolds represent the possible positions or configurations of a mechanical system
- Each point on the manifold corresponds to a specific configuration of the system (pendulum, double pendulum)
- The dimension of the configuration space manifold equals the number of degrees of freedom of the system
- Tangent vectors to the configuration space manifold represent velocities or changes in configuration

### Phase space manifolds

- Phase space manifolds describe the state of a mechanical system, including both its position and momentum
- Each point in the phase space represents a complete state of the system at a given instant
- The dimension of the phase space manifold is twice the number of degrees of freedom
- Cotangent vectors to the phase space manifold represent momentum or conjugate variables

### Cotangent bundles

- Cotangent bundles are a special type of phase space manifold constructed from a configuration space manifold
- They consist of the collection of all cotangent vectors at each point of the configuration space
- The cotangent bundle has a natural symplectic structure, making it suitable for Hamiltonian mechanics

- The canonical coordinates on the cotangent bundle are positions and their conjugate momenta  $(q, p)$  coordinates)

## Lagrangian mechanics

- Lagrangian mechanics is a formulation of classical mechanics that uses the principle of least action to derive equations of motion
- It is based on the Lagrangian function, which depends on the system's configuration and velocity

## Lagrangian function

- The Lagrangian function  $L(q, \dot{q}, t)$  is defined as the difference between the kinetic energy  $T$  and the potential energy  $V$  of the system
- It is a function of the generalized coordinates  $q$ , generalized velocities  $\dot{q}$ , and time  $t$
- The Lagrangian encodes the dynamics of the system and is used to derive the equations of motion
- The action functional  $S[q] = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$  is the integral of the Lagrangian over a path in configuration space

## Euler-Lagrange equations

- The Euler-Lagrange equations are the fundamental equations of motion in Lagrangian mechanics
- They are derived by applying the principle of least action to the action functional
- For each generalized coordinate  $q_i$ , the Euler-Lagrange equation is given by  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$
- These equations determine the time evolution of the system given initial conditions

## Variational principles

- Variational principles, such as the principle of least action, form the foundation of Lagrangian mechanics
- The principle of least action states that the path taken by a system between two points in configuration space is the one that minimizes the action functional
- Variational principles provide a powerful tool for deriving equations of motion and studying the behavior of mechanical systems
- They also establish a connection between classical mechanics and other areas of physics, such as quantum mechanics and field theory

## Noether's theorem and symmetries

- Noether's theorem relates symmetries of the Lagrangian to conserved quantities in the system
- If the Lagrangian is invariant under a continuous symmetry transformation, there exists a corresponding conserved quantity
- Examples of symmetries include time translation (energy conservation), spatial translation (linear momentum conservation), and rotation (angular momentum conservation)



- Noether's theorem provides a systematic way to identify conserved quantities and understand their physical significance

## Hamiltonian mechanics

- Hamiltonian mechanics is a formulation of classical mechanics that uses generalized coordinates and momenta to describe the state and evolution of a system
- It is based on the Hamiltonian function, which is obtained from the Lagrangian through a Legendre transformation

## Hamiltonian function

- The Hamiltonian function  $H(q, p, t)$  represents the total energy of the system, expressed in terms of generalized coordinates  $q$ , generalized momenta  $p$ , and time  $t$
- It is defined as the Legendre transform of the Lagrangian:  $H(q, p, t) = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t)$
- The Hamiltonian is a function on the phase space manifold and generates the time evolution of the system
- The level sets of the Hamiltonian correspond to constant energy surfaces in phase space

## Hamilton's equations

- Hamilton's equations are the fundamental equations of motion in Hamiltonian mechanics
- They are a system of first-order differential equations for the generalized coordinates and momenta
- For each pair of generalized coordinate  $q_i$  and momentum  $p_i$ , Hamilton's equations are given by  $\dot{q}_i = \frac{\partial H}{\partial p_i}$  and  $\dot{p}_i = -\frac{\partial H}{\partial q_i}$
- These equations describe the time evolution of the system in phase space and are equivalent to the Euler-Lagrange equations

## Legendre transform

- The Legendre transform is a mathematical operation that converts the Lagrangian formulation to the Hamiltonian formulation
- It introduces the generalized momenta  $p_i$  as new variables, defined by  $p_i = \frac{\partial L}{\partial \dot{q}_i}$
- The Legendre transform expresses the Hamiltonian as a function of the generalized coordinates and momenta,  $H(q, p, t)$
- The Legendre transform is a coordinate-free construction and preserves the geometric structure of the system

## Poisson brackets

- Poisson brackets are a fundamental operation in Hamiltonian mechanics that encode the geometric structure of phase space
- The Poisson bracket of two functions  $f(q, p)$  and  $g(q, p)$  is defined as  $\{f, g\} = \sum_i \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$
- Poisson brackets satisfy certain properties, such as antisymmetry, bilinearity, and the Jacobi identity

- The time evolution of any function  $f(q, p)$  is given by its Poisson bracket with the Hamiltonian:  
 $\dot{f} = \{f, H\}$

## Symplectic manifolds

- Symplectic manifolds are a class of smooth manifolds equipped with a closed, nondegenerate 2-form called the symplectic form
- They provide a natural geometric framework for Hamiltonian mechanics and other areas of mathematical physics

## Symplectic forms

- A symplectic form  $\omega$  is a closed, nondegenerate 2-form on a smooth manifold  $M$
- In local coordinates  $(q_i, p_i)$ , the symplectic form can be written as  $\omega = \sum_i dq_i \wedge dp_i$
- The nondegeneracy of the symplectic form implies that the manifold has even dimension and that there exists a symplectic vector bundle isomorphism between tangent and cotangent spaces
- The closedness of the symplectic form,  $d\omega = 0$ , is a crucial property that distinguishes symplectic manifolds from other types of manifolds

## Symplectomorphisms

- A symplectomorphism is a diffeomorphism between two symplectic manifolds that preserves the symplectic form
- More precisely, if  $(M_1, \omega_1)$  and  $(M_2, \omega_2)$  are symplectic manifolds, a symplectomorphism is a smooth map  $\phi : M_1 \rightarrow M_2$  such that  $\phi^* \omega_2 = \omega_1$
- Symplectomorphisms form a group under composition and are the natural symmetries of symplectic manifolds
- Hamiltonian mechanics is invariant under symplectomorphisms, which leads to the conservation of phase space volume (Liouville's theorem)

## Darboux's theorem

- Darboux's theorem states that locally, all symplectic manifolds of the same dimension are isomorphic
- More precisely, for any point  $P$  in a symplectic manifold  $(M, \omega)$ , there exists a neighborhood  $U$  of  $P$  and local coordinates  $(q_i, p_i)$  on  $U$  such that  $\omega|_U = \sum_i dq_i \wedge dp_i$
- These coordinates are called Darboux coordinates or canonical coordinates
- Darboux's theorem implies that symplectic manifolds have no local invariants other than the dimension

## Symplectic vector fields

- A symplectic vector field  $X$  on a symplectic manifold  $(M, \omega)$  is a vector field that preserves the symplectic form under the Lie derivative:  $\mathcal{L}_X \omega = 0$
- Equivalently, the flow generated by a symplectic vector field consists of symplectomorphisms

- Hamiltonian vector fields, defined by  $\omega(X_H, \cdot) = dH$  for a smooth function  $H$ , are a special class of symplectic vector fields
- The Poisson bracket of two functions can be expressed in terms of the symplectic form and their Hamiltonian vector fields:  $\{f, g\} = \omega(X_f, X_g)$

## Symplectic geometry in mechanics

- Symplectic geometry provides a natural and powerful framework for studying classical mechanical systems
- The phase space of a mechanical system is a symplectic manifold, with the symplectic form encoding the system's dynamics

## Symplectic structure of phase space

- The phase space of a mechanical system is a symplectic manifold, typically a cotangent bundle  $T^*Q$  of the configuration space manifold  $Q$
- The canonical symplectic form on  $T^*Q$  is given by  $\omega = \sum_i dq_i \wedge dp_i$ , where  $(q_i, p_i)$  are canonical coordinates
- The symplectic structure of phase space provides a geometric interpretation of Hamilton's equations and Poisson brackets
- Symplectic geometry unifies the Lagrangian and Hamiltonian formalisms, as the variational principle in the Lagrangian formalism can be seen as a consequence of the symplectic structure

## Hamiltonian vector fields

- In the context of symplectic geometry, the time evolution of a mechanical system is generated by the Hamiltonian vector field  $X_H$
- The Hamiltonian vector field is defined by the equation  $\omega(X_H, \cdot) = dH$ , where  $H$  is the Hamiltonian function
- Integral curves of the Hamiltonian vector field are solutions to Hamilton's equations and describe the motion of the system in phase space
- The flow generated by the Hamiltonian vector field consists of symplectomorphisms, which leads to the conservation of phase space volume (Liouville's theorem)

## Symplectic integrators

- Symplectic integrators are numerical methods for solving Hamilton's equations that preserve the symplectic structure of phase space
- They are designed to approximate the flow of the Hamiltonian vector field while maintaining the geometric properties of the system
- Examples of symplectic integrators include the symplectic Euler method, the Störmer-Verlet method, and the leapfrog method
- Symplectic integrators have excellent long-time stability properties and are widely used in computational physics and chemistry

## Geometric quantization

- Geometric quantization is a mathematical procedure that associates a quantum mechanical system to a classical mechanical system described by a symplectic manifold
- It aims to construct a Hilbert space of quantum states and a correspondence between classical observables and quantum operators
- The key ingredients of geometric quantization are a prequantum line bundle over the symplectic manifold and a polarization, which selects a subspace of the prequantum Hilbert space
- Geometric quantization provides a rigorous framework for understanding the relationship between classical and quantum mechanics and has applications in various areas of physics

## Poisson manifolds

- Poisson manifolds are a generalization of symplectic manifolds that allow for a more general notion of Poisson brackets
- They provide a unified framework for studying both symplectic and non-symplectic systems in classical mechanics

## Poisson brackets on manifolds

- A Poisson manifold is a smooth manifold  $M$  equipped with a bilinear operation  $\{\cdot, \cdot\}$  on the space of smooth functions  $C^\infty(M)$ , called the Poisson bracket
- The Poisson bracket satisfies the properties of antisymmetry, bilinearity, the Leibniz rule, and the Jacobi identity
- In local coordinates  $(x_i)$ , the Poisson bracket is determined by a bivector field  $\Pi = \sum_{i,j} \Pi^{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ , called the Poisson tensor
- The Poisson bracket of two functions  $f, g \in C^\infty(M)$  is given by  $\{f, g\} = \Pi(df, dg) = \sum_{i,j} \Pi^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$

## Poisson maps

- A Poisson map between two Poisson manifolds  $(M_1, \{\cdot, \cdot\}_1)$  and  $(M_2, \{\cdot, \cdot\}_2)$  is a smooth map  $\phi: M_1 \rightarrow M_2$  that preserves the Poisson brackets
- More precisely, for any functions  $f, g \in C^\infty(M_2)$ , the pullback of their Poisson bracket is equal to the Poisson bracket of their pullbacks:  $\phi^* \{f, g\}_2 = \{\phi^* f, \phi^* g\}_1$
- Poisson maps are the natural morphisms in the category of Poisson manifolds
- Symplectomorphisms between symplectic manifolds are examples of Poisson maps

## Symplectic leaves

- A symplectic leaf in a Poisson manifold  $(M, \{\cdot, \cdot\})$  is a maximal connected submanifold  $S \subset M$  such that the restriction of the Poisson bracket to  $S$  is non-degenerate
- Equivalently, a symplectic leaf is an integral manifold of the distribution generated by the Hamiltonian vector fields

- Every Poisson manifold is foliated by its symplectic leaves, which are themselves symplectic manifolds
- The symplectic leaves of a Poisson manifold provide a decomposition of the manifold into symplectic submanifolds

## Poisson vs symplectic manifolds

- Every symplectic manifold is a Poisson manifold, with the Poisson bracket defined by the symplectic form:  $\{f, g\} = \omega(X_f, X_g)$
- However, not every Poisson manifold is symplectic, as the Poisson tensor may be degenerate
- Symplectic manifolds are a special case of Poisson manifolds, where the Poisson tensor is non-degenerate and the symplectic leaves coincide with the entire manifold
- Poisson manifolds provide a more general framework for studying systems with degenerate Poisson structures, such as constrained mechanical systems and integrable systems

## Symmetries and conservation laws

- Symmetries play a crucial role in classical mechanics, as they lead to conservation laws and simplify the analysis of mechanical systems
- Symplectic geometry provides a natural framework for studying symmetries and their associated conservation laws

## Momentum maps

- A momentum map is a generalization of the concept of conserved quantities in the presence of symmetries
- Given a symplectic manifold  $(M, \omega)$  and a Lie group  $G$  acting on  $M$  by symplectomorphisms, a momentum map is a smooth map  $J : M \rightarrow \mathfrak{g}^*$ , where  $\mathfrak{g}^*$  is the dual of the Lie algebra of  $G$
- The momentum map satisfies the condition  $\omega(X_\xi, \cdot) = d(J^\xi)$  for every  $\xi \in \mathfrak{g}$ , where  $X_\xi$  is the vector field generated by the infinitesimal action of  $\xi$  and  $J^\xi = \langle J, \xi \rangle$  is the component of  $J$  along  $\xi$
- The components of the momentum map corresponding to the generators of the Lie algebra are the conserved quantities associated with the symmetry

## Noether's theorem revisited

- Noether's theorem in the context of sym

## 12.4 Gauge theory and fiber bundles

Gauge theory and fiber bundles are powerful tools in differential geometry, unifying various geometric structures through connections on principal bundles. They provide a framework for understanding Riemannian metrics, symplectic forms, and complex structures, with applications in mathematical physics.

Fiber bundles generalize product spaces, allowing fibers to vary smoothly over a base manifold. This concept is crucial for studying geometric structures like vector fields, differential forms, and metrics that change across a manifold's surface.

## Gauge theory fundamentals

- Gauge theory is a fundamental tool in modern differential geometry that describes the geometry of fiber bundles and their connections
- It provides a unified framework for understanding various geometric structures, such as Riemannian metrics, symplectic forms, and complex structures, in terms of the properties of connections on principal bundles
- Gauge theory has important applications in mathematical physics, particularly in the study of Yang-Mills fields and the geometry of four-manifolds

## Connections on principal bundles

- A connection on a principal bundle is a way to specify how to lift paths from the base manifold to the total space of the bundle in a consistent manner
- Connections allow for the definition of parallel transport, which moves elements along paths in the base manifold while respecting the bundle structure
- The space of connections on a principal bundle is an affine space, with the difference between two connections being a tensorial 1-form with values in the Lie algebra of the structure group

## Curvature of connections

- The curvature of a connection measures the extent to which parallel transport around infinitesimal loops fails to preserve the horizontal subspaces of the bundle
- Curvature is a tensorial 2-form with values in the Lie algebra of the structure group, satisfying the Bianchi identity
- Flat connections are those with vanishing curvature, and they correspond to locally trivial bundles with constant transition functions

## Chern-Weil theory

- Chern-Weil theory is a powerful tool that relates the topology of a principal bundle to the curvature of a connection on the bundle
- It provides a way to construct characteristic classes, which are cohomology classes that measure the non-triviality of the bundle
- The Chern-Weil homomorphism maps invariant polynomials on the Lie algebra of the structure group to cohomology classes of the base manifold

## Characteristic classes

- Characteristic classes are cohomological invariants associated to principal bundles, which provide obstructions to the existence of certain geometric structures on the bundle (flat connections, sections, etc.)
- Examples of characteristic classes include the Euler class for oriented vector bundles, Chern classes for complex vector bundles, and Pontryagin classes for real vector bundles
- Characteristic classes are natural under pull-backs and satisfy certain axioms, such as the Whitney product formula and the splitting principle

## Fiber bundles

- Fiber bundles are a fundamental concept in differential geometry that generalize the notion of a product space by allowing the fibers to vary smoothly over the base manifold
- A fiber bundle consists of a total space, a base manifold, and a projection map satisfying the local triviality condition
- Fiber bundles provide a natural framework for studying geometric structures that vary smoothly over a manifold, such as vector fields, differential forms, and metrics

## Definition and examples

- A fiber bundle is a tuple  $(E, B, \pi, F)$  where  $E$  is the total space,  $B$  is the base manifold,  $\pi : E \rightarrow B$  is a continuous surjection, and  $F$  is the typical fiber, such that every point in  $B$  has a neighborhood  $U$  for which  $\pi^{-1}(U)$  is homeomorphic to  $U \times F$
- Examples of fiber bundles include the Möbius strip (non-orientable bundle over  $S^1$  with fiber  $\mathbb{R}$ ), the tangent bundle of a manifold (bundle over  $M$  with fiber  $\mathbb{R}^n$ ), and the frame bundle of a vector bundle (principal  $GL(n, \mathbb{R})$ -bundle)

## Local trivializations

- A local trivialization of a fiber bundle is a collection of open sets  $\{U_i\}$  covering the base manifold, together with homeomorphisms  $\phi_i : \pi^{-1}(U_i) \rightarrow U_i \times F$  that commute with the projection onto  $U_i$
- Local trivializations provide a way to work with fiber bundles locally as product spaces, which is often easier than working with the global bundle structure
- The existence of local trivializations is equivalent to the local triviality condition in the definition of a fiber bundle

## Transition functions

- Transition functions are the data that encode how the local trivializations of a fiber bundle are related on the overlaps of their domains
- Given two local trivializations  $\phi_i$  and  $\phi_j$  over overlapping sets  $U_i$  and  $U_j$ , the transition function  $g_{ij} : U_i \cap U_j \rightarrow G$  is defined by  $\phi_i \circ \phi_j^{-1}(x, f) = (x, g_{ij}(x) \cdot f)$ , where  $G$  is the structure group of the bundle acting on the fibers
- Transition functions satisfy the cocycle condition  $g_{ij}g_{jk} = g_{ik}$  on triple overlaps, and they completely determine the bundle up to isomorphism

## Associated bundles

- Given a principal  $G$ -bundle  $P \rightarrow M$  and a left  $G$ -space  $F$ , the associated bundle is the fiber bundle  $P \times_G F \rightarrow M$  obtained by taking the quotient of  $P \times F$  by the diagonal  $G$ -action
- The fiber of the associated bundle over a point  $x \in M$  is the orbit space  $F_x = P_x \times_G F$ , where  $P_x$  is the fiber of  $P$  over  $x$  and  $G$  acts on  $F$  via the given left action
- Many important fiber bundles in differential geometry, such as vector bundles and principal bundles, can be constructed as associated bundles to a given principal bundle

## Principal bundles

- Principal bundles are a special class of fiber bundles where the fibers are equipped with a free and transitive action of a Lie group  $G$ , called the structure group
- The total space of a principal bundle is a manifold  $P$  on which  $G$  acts freely and properly from the right, with orbit space isomorphic to the base manifold  $M$
- Principal bundles provide a natural framework for studying geometric structures that are invariant under the action of a Lie group, such as Riemannian metrics, connections, and gauge fields

## Definition and properties

- A principal  $G$ -bundle is a fiber bundle  $\pi : P \rightarrow M$  together with a free right action of  $G$  on  $P$  that preserves the fibers and is compatible with the local trivializations
- The fibers of a principal bundle are diffeomorphic to the structure group  $G$ , and the action of  $G$  on each fiber is free and transitive
- Principal bundles satisfy the local triviality condition, meaning that every point in the base manifold has a neighborhood  $U$  such that  $\pi^{-1}(U)$  is  $G$ -equivariantly diffeomorphic to  $U \times G$

## Principal $G$ -bundles

- A principal  $G$ -bundle is completely determined by its transition functions, which are smooth maps  $g_{ij} : U_i \cap U_j \rightarrow G$  satisfying the cocycle condition
- The set of isomorphism classes of principal  $G$ -bundles over a manifold  $M$  is in bijection with the set of conjugacy classes of homomorphisms from the fundamental group of  $M$  to  $G$
- Principal  $G$ -bundles can be constructed by specifying a  $G$ -valued Čech 1-cocycle on a suitable open cover of the base manifold

## Gauge transformations

- A gauge transformation of a principal  $G$ -bundle  $P \rightarrow M$  is a  $G$ -equivariant diffeomorphism  $f : P \rightarrow P$  that preserves the fibers and covers the identity map on  $M$
- The group of gauge transformations, denoted  $\mathcal{G}(P)$ , is the group of smooth sections of the associated bundle  $P \times_G G \rightarrow M$ , where  $G$  acts on itself by conjugation
- Gauge transformations act on the space of connections on  $P$  by pulling back the connection 1-forms, and the curvature of a connection transforms in a covariant way under gauge transformations

## Reduction of structure group

- A reduction of the structure group of a principal  $G$ -bundle  $P \rightarrow M$  to a subgroup  $H \subset G$  is a principal  $H$ -subbundle  $Q \subset P$  such that  $P \cong Q \times_H G$
- Reductions of structure group correspond to additional geometric structures on the bundle that are compatible with the  $G$ -action, such as orientation, Riemannian metrics, or almost complex structures
- The existence of a reduction of structure group can be characterized by the vanishing of certain characteristic classes of the bundle, such as the first Stiefel-Whitney class for orientability or the second Stiefel-Whitney class for spin structures



## Vector bundles

- Vector bundles are fiber bundles whose fibers are vector spaces and whose transition functions are linear maps between the fibers
- They provide a natural framework for studying geometric objects that vary smoothly over a manifold, such as vector fields, differential forms, and metrics
- Many important constructions in differential geometry, such as the tangent bundle and the cotangent bundle, are examples of vector bundles

## Definition and examples

- A vector bundle of rank  $n$  over a manifold  $M$  is a fiber bundle  $\pi : E \rightarrow M$  such that each fiber  $E_x = \pi^{-1}(x)$  is an  $n$ -dimensional vector space, and the transition functions are linear isomorphisms between the fibers
- Examples of vector bundles include the trivial bundle  $M \times \mathbb{R}^n \rightarrow M$ , the tangent bundle  $TM \rightarrow M$ , the cotangent bundle  $T^*M \rightarrow M$ , and the bundle of  $k$ -forms  $\Lambda^k(T^*M) \rightarrow M$

## Frame bundles

- The frame bundle of a vector bundle  $E \rightarrow M$  is the principal  $GL(n, \mathbb{R})$ -bundle  $F(E) \rightarrow M$  whose fiber over a point  $x \in M$  consists of all ordered bases (frames) of the vector space  $E_x$
- A section of the frame bundle is called a frame field, and it provides a way to trivialize the vector bundle locally
- The frame bundle of a manifold  $M$  is the frame bundle of its tangent bundle, and it plays a fundamental role in the study of Riemannian geometry and general relativity

## Sections of vector bundles

- A section of a vector bundle  $E \rightarrow M$  is a smooth map  $s : M \rightarrow E$  such that  $\pi \circ s = id_M$ , where  $\pi : E \rightarrow M$  is the projection map
- The space of sections of a vector bundle, denoted  $\Gamma(E)$ , is an infinite-dimensional vector space over  $\mathbb{R}$  (or  $\mathbb{C}$  for complex vector bundles) and a module over the ring of smooth functions on  $M$
- Many important geometric objects, such as vector fields, differential forms, and metrics, can be viewed as sections of appropriate vector bundles

## Tangent and cotangent bundles

- The tangent bundle of a manifold  $M$  is the vector bundle  $TM \rightarrow M$  whose fiber over a point  $x \in M$  is the tangent space  $T_x M$
- The cotangent bundle of  $M$  is the dual vector bundle  $T^*M \rightarrow M$ , whose fiber over  $x$  is the dual space  $(T_x M)^*$  of linear functionals on  $T_x M$
- The tangent and cotangent bundles are fundamental objects in differential geometry, as they encode the infinitesimal structure of the manifold and allow for the definition of important geometric notions such as flows, Lie derivatives, and differential forms

## Connections and curvature

- Connections on vector bundles provide a way to differentiate sections of the bundle and to compare fibers at different points of the base manifold
- The curvature of a connection measures the extent to which parallel transport around infinitesimal loops fails to preserve the fibers, and it encodes important geometric and topological information about the bundle
- Connections and curvature play a central role in gauge theory and the study of geometric structures on manifolds, such as Riemannian metrics and symplectic forms

## Connections on vector bundles

- A connection on a vector bundle  $E \rightarrow M$  is a linear map  $\nabla : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E)$  satisfying the Leibniz rule  $\nabla(fs) = df \otimes s + f\nabla s$  for all smooth functions  $f$  and sections  $s$
- Connections allow for the definition of parallel transport, which is a way to move elements of the fibers along paths in the base manifold while preserving the linear structure of the fibers
- The space of connections on a vector bundle is an affine space, with the difference between two connections being a tensorial 1-form with values in the endomorphism bundle  $\text{End}(E)$

## Covariant derivatives

- The covariant derivative of a section  $s$  of a vector bundle  $E$  with respect to a connection  $\nabla$  is the section  $\nabla s$  of the tensor product bundle  $T^*M \otimes E$
- In local coordinates, the covariant derivative is given by  $\nabla_{\partial_i}s = \partial_i s + A_i s$ , where  $A_i$  are the connection coefficients (Christoffel symbols) and  $\partial_i$  are the local coordinate vector fields
- The covariant derivative satisfies the Leibniz rule and is compatible with the tensor product and contraction operations on vector bundles

## Parallel transport

- Parallel transport along a curve  $\gamma : [0, 1] \rightarrow M$  with respect to a connection  $\nabla$  on a vector bundle  $E \rightarrow M$  is a linear isomorphism  $P_\gamma : E_{\gamma(0)} \rightarrow E_{\gamma(1)}$  between the fibers over the endpoints of the curve
- Parallel transport is defined by solving the linear ODE  $\nabla_{\dot{\gamma}}s = 0$  for sections  $s$  of  $E$  along  $\gamma$ , with initial condition  $s(\gamma(0)) = v$  for some  $v \in E_{\gamma(0)}$
- Parallel transport is compatible with composition of curves and is invariant under reparametrization, making it a fundamental tool for comparing fibers at different points of the base manifold

## Curvature tensor

- The curvature of a connection  $\nabla$  on a vector bundle  $E \rightarrow M$  is the tensorial 2-form  $R \in \Omega^2(M, \text{End}(E))$  defined by  $R(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]}s$  for all vector fields  $X, Y$  and sections  $s$
- In local coordinates, the curvature tensor is given by  $R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ , where  $A_i$  are the connection coefficients and  $[\cdot, \cdot]$  is the Lie bracket of matrices
- The curvature tensor measures the non-commutativity of the covariant derivative and the failure of parallel transport around infinitesimal loops to preserve the fibers

## Bianchi identities

- The Bianchi identities are a set of relations satisfied by the curvature tensor of a connection on a vector bundle, expressing its algebraic and differential properties
- The first Bianchi identity states that the curvature tensor is antisymmetric in its last two arguments, i.e.,  $R(X, Y) = -R(Y, X)$  for all vector fields  $X, Y$
- The second Bianchi identity, also known as the differential Bianchi identity, relates the covariant derivative of the curvature tensor to its cyclic permutations, i.e.,  $(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0$  for all vector fields  $X, Y, Z$

## Yang-Mills theory

- Yang-Mills theory is a gauge theory that describes the dynamics of connections on principal bundles and their curvature, with applications in theoretical physics and geometry
- The Yang-Mills equations are a system of nonlinear partial differential equations for a connection on a principal bundle, generalizing Maxwell's equations and the equations of general relativity
- Solutions to the Yang-Mills equations, known as instantons and anti-instantons, have important topological and geometric properties and are used to define invariants of four-manifolds

## Yang-Mills equations

- The Yang-Mills equations for a connection  $A$  on a principal  $G$ -bundle  $P \rightarrow M$  are the Euler-Lagrange equations for the Yang-Mills functional  $S(A) = \int_M \langle F_A, F_A \rangle$ , where  $F_A$  is the curvature of  $A$  and  $\langle \cdot, \cdot \rangle$  is an invariant inner product on the Lie algebra of  $G$

## 12.5 Geometric flows and Ricci flow

Geometric flows are powerful tools in differential geometry that evolve geometric objects over time. They're governed by partial differential equations, typically based on curvature, aiming to deform objects into more desirable shapes while preserving certain properties.

The Ricci flow, introduced by Richard Hamilton in 1982, is a prime example. It evolves Riemannian metrics according to their Ricci curvature. This flow has been instrumental in proving major conjectures in topology and geometry, including the Poincaré conjecture.

### Definition of geometric flows

- Geometric flows are a powerful tool in differential geometry that evolve a geometric object over time according to a specific partial differential equation
- The evolution is typically governed by the curvature of the object, with the goal of deforming it into a more desirable or canonical shape while preserving certain properties (topology, symmetries, etc.)

### Smooth family of metrics

- In the context of Riemannian manifolds, a geometric flow is often defined as a smooth family of Riemannian metrics  $g(t)$  parameterized by time  $t$
- The metrics evolve according to a PDE that depends on the curvature tensor of  $g(t)$ , such as the Ricci tensor or the scalar curvature

- The initial metric  $g(0)$  is given, and the goal is to study the long-time behavior of the flow and its possible singularities

## Evolution equations for metrics

- The evolution equation for a geometric flow takes the general form  $\frac{\partial}{\partial t}g(t) = F(g(t))$ , where  $F$  is a smooth function of the metric and its curvature
- The choice of  $F$  determines the specific type of geometric flow (Ricci flow, mean curvature flow, Yamabe flow, etc.)
- The evolution equation is a nonlinear PDE that can be challenging to solve explicitly, but its qualitative behavior can often be analyzed using geometric and analytic techniques

## Examples of geometric flows

### Ricci flow

- The Ricci flow, introduced by Richard Hamilton in 1982, is defined by the evolution equation  $\frac{\partial}{\partial t}g(t) = -2\text{Ric}(g(t))$ , where  $\text{Ric}$  is the Ricci curvature tensor
- It has been used to prove the Poincaré conjecture in dimension 3 and the differentiable sphere theorem in dimension 4
- The Ricci flow tends to smooth out positive curvature and concentrate negative curvature, leading to the formation of singularities in finite time for most initial metrics

### Mean curvature flow

- The mean curvature flow deforms a submanifold  $M^n$  in a Riemannian manifold  $N^{n+k}$  by moving each point in the direction of its mean curvature vector  $\vec{H}$
- The evolution equation is  $\frac{\partial}{\partial t}F(p, t) = \vec{H}(p, t)$ , where  $F : M \times [0, T] \rightarrow N$  is a smooth family of immersions
- It has applications in geometric topology, image processing, and material science (modeling the motion of interfaces and phase boundaries)

### Yamabe flow

- The Yamabe flow is a conformal deformation of a Riemannian metric  $g$  that evolves according to the equation  $\frac{\partial}{\partial t}g(t) = -R(g(t))g(t)$ , where  $R$  is the scalar curvature
- It is designed to find a metric of constant scalar curvature within a given conformal class, thus solving the Yamabe problem
- The Yamabe flow always converges to a metric of constant scalar curvature on compact manifolds, but the limiting metric may have singularities

## Properties of Ricci flow

### Evolution of curvature

- Under the Ricci flow, the Riemann curvature tensor satisfies a nonlinear heat-type equation that involves the full curvature tensor

- The scalar curvature evolves according to  $\frac{\partial}{\partial t} R = \Delta R + 2|\text{Ric}|^2$ , which is similar to a reaction-diffusion equation
- The evolution equations for curvature can be used to derive important estimates and monotonicity formulas for the Ricci flow

## Existence and uniqueness

- For compact manifolds, the Ricci flow has a unique solution for a short time starting from any smooth initial metric
- The maximal time of existence depends on the initial metric and the dimension of the manifold
- In higher dimensions ( $n \geq 4$ ), the Ricci flow may develop singularities in finite time even for smooth initial data

## Maximum principles

- Various maximum principles hold for the Ricci flow, allowing for the comparison of curvature quantities at different points and times
- For example, the minimum of the scalar curvature is non-decreasing along the flow, and the maximum of the norm of the Riemann curvature tensor is non-increasing
- These principles are crucial for understanding the long-time behavior of the flow and the formation of singularities

## Convergence in low dimensions

- In dimension 2, the Ricci flow is equivalent to the heat equation for the Gaussian curvature and always converges to a metric of constant curvature (uniformization theorem)
- In dimension 3, the Ricci flow with surgery, developed by Perelman, can be used to prove the geometrization conjecture for all compact 3-manifolds
- In higher dimensions, the convergence of the Ricci flow is more complicated and may require additional assumptions (e.g., Kähler manifolds, positive curvature, etc.)

## Singularities of Ricci flow

### Classification of singularities

- Singularities of the Ricci flow can be classified into Type I, Type II, and Type III, depending on the rate of blow-up of the curvature as the singularity time is approached
- Type I singularities are modeled on shrinking self-similar solutions (Ricci solitons) and have a blow-up rate of the form  $|\text{Rm}| \leq \frac{C}{T-t}$ , where  $T$  is the singularity time
- Type II singularities are more complicated and may involve the formation of neck-like regions or the splitting of the manifold into multiple components

### Formation of singularities

- In dimension 3, the only way a singularity can form is through the pinching off of a neck-like region, as shown by Hamilton's neck theorem

- In higher dimensions, singularities can also form through the collapse of certain submanifolds or the formation of cusps
- The precise structure of singularities is still not fully understood, especially in dimensions  $n \geq 4$

## Blow-up analysis

- To study the local behavior of the Ricci flow near a singularity, one can perform a blow-up analysis by rescaling the metric and the time parameter
- The rescaled metrics converge (in a suitable sense) to a self-similar solution of the Ricci flow, called a singularity model
- The classification of singularity models is a crucial step in understanding the global behavior of the flow and in developing a surgery procedure to continue the flow past singularities

## Applications of Ricci flow

### Uniformization of surfaces

- In dimension 2, the Ricci flow provides a new proof of the uniformization theorem, which states that every compact Riemann surface admits a metric of constant curvature
- The Ricci flow starting from any initial metric converges to a metric of constant curvature, which is unique up to scaling and isometry
- This gives a canonical way to parameterize the moduli space of Riemann surfaces

### Geometrization of 3-manifolds

- The geometrization conjecture, formulated by Thurston, states that every compact 3-manifold can be decomposed into pieces that admit one of eight homogeneous geometries
- Perelman used the Ricci flow with surgery to prove the conjecture, by showing that the flow either converges to a metric of constant curvature or collapses the manifold along incompressible tori
- This work revolutionized the field of 3-dimensional topology and earned Perelman the Fields Medal (which he declined)

### Canonical metrics on Kähler manifolds

- On a Kähler manifold, the Ricci flow preserves the Kähler condition and is equivalent to a scalar PDE for the Kähler potential
- The Kähler-Ricci flow can be used to find canonical metrics, such as Kähler-Einstein metrics or constant scalar curvature metrics
- It has important applications in complex geometry, algebraic geometry, and mathematical physics (e.g., string theory, mirror symmetry)

## Connections to other areas

### Relationship with harmonic map flow

- The Ricci flow can be viewed as a generalization of the harmonic map flow between Riemannian manifolds

- If  $(M, g)$  and  $(N, h)$  are Riemannian manifolds and  $f: M \rightarrow N$  is a smooth map, the harmonic map flow is defined by  $\frac{\partial}{\partial t} f(x, t) = \tau_g f(x, t)$ , where  $\tau_g$  is the tension field of  $f$  with respect to the metric  $g$
- When  $M = N$  and  $f$  is the identity map, the harmonic map flow reduces to the Ricci flow (up to a constant factor)

## Analogy with heat equation

- The Ricci flow shares many similarities with the heat equation  $\frac{\partial}{\partial t} u = \Delta u$ , which describes the diffusion of heat in a medium
- Both equations have smoothing properties, maximum principles, and exhibit finite-time blow-up for certain initial data
- However, the Ricci flow is a nonlinear PDE that involves the curvature of the evolving metric, making its analysis more challenging

## Gradient flows vs reaction-diffusion equations

- Geometric flows can be broadly classified into two types: gradient flows and reaction-diffusion equations
- Gradient flows, such as the heat equation or the Yamabe flow, are driven by the gradient of a functional (e.g., the Dirichlet energy or the total scalar curvature) and tend to minimize this functional over time
- Reaction-diffusion equations, such as the Ricci flow or the mean curvature flow, involve terms that can be interpreted as "reaction" terms (e.g., quadratic curvature terms) and may exhibit more complex behavior, such as the formation of singularities or pattern formation

## 12.6 Geometric analysis and partial differential equations

Geometric analysis combines differential geometry and partial differential equations to study manifolds and their structures. It explores how a manifold's intrinsic geometry interacts with differential operators and equations defined on it, providing a framework for investigating geometric and analytic problems.

This powerful approach tackles classical theories like minimal surfaces and modern research in general relativity and string theory. By merging geometric intuition with analytical techniques, geometric analysis offers deep insights into the nature of space and mathematical structures.

### Foundations of geometric analysis

- Geometric analysis combines differential geometry and partial differential equations to study the geometry of manifolds and their associated structures
- Focuses on understanding the interplay between the intrinsic geometry of a manifold and the behavior of differential operators and equations defined on it
- Provides a powerful framework for studying a wide range of geometric and analytic problems, from the classical theory of minimal surfaces to modern research in general relativity and string theory

### Manifolds and Riemannian metrics

- Manifolds are topological spaces that locally resemble Euclidean space and provide a natural setting for studying geometric objects and structures

- Riemannian metrics are symmetric positive-definite tensor fields that define an inner product on each tangent space, allowing the measurement of lengths, angles, and volumes on the manifold
- The choice of Riemannian metric determines the geometric properties of the manifold, such as curvature and geodesics (shortest paths between points)

## Sobolev spaces on manifolds

- Sobolev spaces are function spaces that incorporate both the values of a function and its derivatives, providing a suitable framework for studying partial differential equations on manifolds
- The definition of Sobolev spaces on manifolds requires the use of coordinate charts and partition of unity to handle the local-to-global transition
- Sobolev spaces on manifolds inherit many of the properties of their Euclidean counterparts, such as density of smooth functions and Sobolev embedding theorems

## Embedding theorems and compactness

- Embedding theorems, such as the Sobolev embedding theorem and the Kondrakov embedding theorem, describe the relationships between different Sobolev spaces and their continuous embeddings into other function spaces
- These embeddings often provide crucial estimates and compactness properties that are essential for solving partial differential equations and variational problems on manifolds
- Compact embeddings, such as the Rellich-Kondrakov theorem, play a key role in the existence theory for elliptic partial differential equations and the calculus of variations

## Laplace-Beltrami operator

- The Laplace-Beltrami operator is a natural generalization of the Euclidean Laplacian to Riemannian manifolds, serving as a fundamental object in geometric analysis
- It is defined as the divergence of the gradient and can be expressed in local coordinates using the metric tensor and Christoffel symbols
- The Laplace-Beltrami operator is a second-order elliptic partial differential operator that encodes important geometric information about the manifold, such as curvature and topology

## Definition and basic properties

- In local coordinates, the Laplace-Beltrami operator acting on a smooth function  $u$  is given by  $\Delta_g u = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j u)$ , where  $g = (g_{ij})$  is the Riemannian metric tensor and  $|g| = \det(g_{ij})$
- The Laplace-Beltrami operator is self-adjoint with respect to the  $L^2$  inner product induced by the Riemannian volume form, which allows for the development of a rich spectral theory
- The Laplace-Beltrami operator satisfies important properties such as the maximum principle, which states that a subharmonic function (satisfying  $\Delta_g u \geq 0$ ) attains its maximum on the boundary of the domain

## Eigenvalues and eigenfunctions

- Eigenvalues and eigenfunctions of the Laplace-Beltrami operator play a crucial role in understanding the geometry and topology of the underlying manifold



- Eigenfunctions are smooth functions that satisfy the equation  $\Delta_g u = \lambda u$  for some constant eigenvalue  $\lambda$ , and they form an orthonormal basis for the  $L^2$  space on the manifold
- The spectrum of the Laplace-Beltrami operator, consisting of all eigenvalues, is discrete and non-negative for compact manifolds, and it encodes important geometric and topological information (Weyl's law, heat kernel asymptotics)

## Heat kernel and heat equation

- The heat equation  $\partial_t u = \Delta_g u$  describes the diffusion of heat or other quantities on a Riemannian manifold, with the Laplace-Beltrami operator governing the spatial evolution
- The heat kernel  $K(t, x, y)$  is the fundamental solution to the heat equation, representing the temperature at point  $y$  and time  $t$  due to an initial unit heat source at point  $x$
- The heat kernel has important asymptotic properties for small times, which are closely related to the geometry of the manifold (heat kernel expansion, Varadhan's formula)

## Green's functions and representation formulas

- Green's functions are fundamental solutions to the Poisson equation  $\Delta_g u = f$  on a Riemannian manifold, providing a way to represent solutions in terms of the source function  $f$
- On compact manifolds, the Green's function can be expressed as a series involving the eigenfunctions and eigenvalues of the Laplace-Beltrami operator
- Representation formulas, such as the Green's formula and the Poisson formula, allow for the expression of harmonic functions (satisfying  $\Delta_g u = 0$ ) in terms of their boundary values using the Green's function

## Elliptic partial differential equations

- Elliptic partial differential equations (PDEs) are a class of equations that generalize the Poisson equation and play a central role in geometric analysis
- These equations are characterized by their ellipticity condition, which ensures that the principal symbol of the differential operator is positive definite
- Elliptic PDEs on Riemannian manifolds arise in various contexts, such as the study of harmonic functions, minimal surfaces, and curvature equations

## Laplacian and Poisson equation

- The Poisson equation  $\Delta_g u = f$  is the prototypical elliptic PDE, where the Laplace-Beltrami operator is the leading term and  $f$  is a given source function
- Solutions to the Poisson equation can be interpreted as potentials or fields generated by the source function, with the Laplace-Beltrami operator describing the spatial dependence
- The Dirichlet problem for the Poisson equation involves finding a solution with prescribed boundary values, while the Neumann problem involves prescribed normal derivatives on the boundary

## Maximum principles and uniqueness

- Maximum principles are powerful tools for studying elliptic PDEs, providing a priori estimates and comparison results for solutions

- The weak maximum principle states that a subharmonic function (satisfying  $\Delta_g u \geq 0$ ) attains its maximum on the boundary of the domain, while the strong maximum principle asserts that the maximum is strict unless the function is constant
- Maximum principles can be used to establish uniqueness results for elliptic boundary value problems, showing that the solution is uniquely determined by its boundary data

## Regularity theory and Schauder estimates

- Regularity theory aims to study the smoothness properties of solutions to elliptic PDEs, showing that solutions inherit regularity from the coefficients and the boundary data
- Schauder estimates provide  $C^{k,\alpha}$  estimates for solutions in terms of the  $C^{k-2,\alpha}$  norms of the coefficients and the boundary data, where  $C^{k,\alpha}$  denotes the Hölder space of functions with  $k$  continuous derivatives and  $\alpha$ -Hölder continuous  $k$ -th derivatives
- These estimates are crucial for establishing the existence and regularity of solutions to nonlinear elliptic PDEs using fixed point arguments and the implicit function theorem

## Variational methods and existence theorems

- Variational methods provide a powerful framework for studying the existence and properties of solutions to elliptic PDEs by reformulating them as minimization problems for suitable energy functionals
- The direct method in the calculus of variations can be used to establish the existence of minimizers for coercive and lower semicontinuous functionals on reflexive Banach spaces, such as Sobolev spaces
- The Lax-Milgram theorem provides an existence and uniqueness result for variational problems associated with bounded and coercive bilinear forms, which arise from the weak formulation of elliptic PDEs

## Geometric applications of elliptic PDEs

- Elliptic partial differential equations play a crucial role in various areas of geometric analysis, providing a bridge between the analytic and geometric aspects of the theory
- These equations often arise as the Euler-Lagrange equations for geometric variational problems or as the governing equations for curvature flows and other geometric evolutions
- The study of elliptic PDEs in geometric contexts has led to significant advances in our understanding of the structure and properties of Riemannian manifolds

## Harmonic functions and minimal surfaces

- Harmonic functions are solutions to the Laplace equation  $\Delta_g u = 0$  and can be characterized as critical points of the Dirichlet energy functional  $\int_M |\nabla u|^2 dV_g$
- Minimal surfaces are hypersurfaces that locally minimize the area functional and can be characterized as solutions to the minimal surface equation, a nonlinear elliptic PDE involving the mean curvature
- The theory of harmonic functions and minimal surfaces has important applications in complex analysis, conformal geometry, and the study of soap films and bubbles

## Yamabe problem and scalar curvature

- The Yamabe problem asks whether every Riemannian manifold admits a conformal metric with constant scalar curvature, which can be reduced to solving a nonlinear elliptic PDE known as the Yamabe equation
- The scalar curvature of a Riemannian manifold is a function that describes the average Gaussian curvature of geodesic balls and plays a key role in the Yamabe problem and other geometric questions
- The positive mass theorem in general relativity, which states that the total mass of an asymptotically flat manifold with non-negative scalar curvature is always non-negative, can be proved using techniques from elliptic PDEs and geometric measure theory

## Einstein metrics and Ricci flow

- Einstein metrics are Riemannian metrics that satisfy the Einstein equation  $\text{Ric}_g = \lambda g$ , where  $\text{Ric}_g$  is the Ricci curvature tensor and  $\lambda$  is a constant, and they are critical points of the Einstein-Hilbert action in general relativity
- The study of Einstein metrics leads to important questions in geometric analysis, such as the Calabi conjecture on the existence of Kähler-Einstein metrics and the positive mass theorem for asymptotically flat manifolds
- The Ricci flow, introduced by Richard Hamilton, is a geometric evolution equation that deforms a Riemannian metric in the direction of its Ricci curvature and has been used to prove the Poincaré conjecture and classify 3-dimensional manifolds

## Monge-Ampère equation and Kähler geometry

- The Monge-Ampère equation is a fully nonlinear elliptic PDE that arises in various geometric contexts, such as the prescribed Gaussian curvature problem and the Calabi conjecture in Kähler geometry
- In complex geometry, the complex Monge-Ampère equation is closely related to the notion of Kähler-Einstein metrics and the existence of canonical metrics on complex manifolds
- The Yau-Tian-Donaldson conjecture, which characterizes the existence of Kähler-Einstein metrics in terms of algebraic stability conditions, has been a major driving force in the development of complex geometric analysis

## Parabolic and hyperbolic equations

- Parabolic and hyperbolic partial differential equations are two other important classes of equations in geometric analysis, describing time-dependent processes and wave propagation phenomena
- Parabolic equations, such as the heat equation, are characterized by a diffusive behavior and exhibit smoothing properties, while hyperbolic equations, such as the wave equation, exhibit finite propagation speed and preserve singularities
- The study of parabolic and hyperbolic equations on Riemannian manifolds leads to important questions in geometric analysis, such as the long-time behavior of curvature flows and the geometry of wave propagation

## Heat equation and diffusion on manifolds

- The heat equation  $\partial_t u = \Delta_g u$  describes the diffusion of heat or other quantities on a Riemannian manifold, with the Laplace-Beltrami operator governing the spatial evolution
- Solutions to the heat equation exhibit a smoothing effect, with initial singularities being instantly regularized and the solution becoming smooth for positive times
- The heat kernel, which is the fundamental solution to the heat equation, plays a crucial role in the study of diffusion processes on manifolds and has important asymptotic properties related to the geometry of the manifold

## Wave equation and finite propagation speed

- The wave equation  $\partial_t^2 u = \Delta_g u$  describes the propagation of waves on a Riemannian manifold, such as sound waves or electromagnetic waves
- Solutions to the wave equation exhibit finite propagation speed, meaning that disturbances propagate at a finite speed determined by the geometry of the manifold (speed of light in the case of electromagnetic waves)
- The study of wave propagation on manifolds leads to important questions in geometric analysis, such as the relationship between the geometry of the manifold and the behavior of geodesic flow and the formation of caustics

## Nonlinear equations and reaction-diffusion systems

- Nonlinear parabolic and hyperbolic equations, such as the porous medium equation and the nonlinear Schrödinger equation, arise in various applications and exhibit rich dynamical behavior
- Reaction-diffusion systems, which couple diffusion processes with nonlinear reaction terms, are used to model pattern formation and self-organization in biology, chemistry, and physics
- The study of nonlinear equations and reaction-diffusion systems on manifolds requires the development of specialized analytic techniques, such as maximum principles, energy methods, and geometric flows

## Regularity and long-time behavior of solutions

- The regularity theory for parabolic and hyperbolic equations aims to understand the smoothness properties of solutions and their dependence on the initial and boundary data
- For parabolic equations, such as the heat equation, solutions instantly become smooth for positive times, while for hyperbolic equations, such as the wave equation, the regularity of solutions is determined by the regularity of the initial data and the geometry of the characteristic curves
- The long-time behavior of solutions to parabolic and hyperbolic equations on manifolds is a central question in geometric analysis, with important applications to the study of curvature flows, the stability of stationary solutions, and the formation of singularities

## Spectral theory and eigenvalue problems

- Spectral theory is the study of the eigenvalues and eigenfunctions of linear operators, such as the Laplace-Beltrami operator on Riemannian manifolds

- Eigenvalue problems arise naturally in the study of vibrating systems, quantum mechanics, and the stability of stationary solutions to partial differential equations
- The spectral theory of geometric operators, such as the Laplacian and the Dirac operator, plays a crucial role in the study of the geometry and topology of manifolds

## Compact self-adjoint operators and spectral theorem

- Compact self-adjoint operators, such as the Laplace-Beltrami operator on compact manifolds, have a discrete spectrum consisting of real eigenvalues with finite multiplicities
- The spectral theorem for compact self-adjoint operators provides an orthonormal basis of eigenfunctions for the underlying Hilbert space, allowing for the representation of functions as series expansions in terms of the eigenfunctions
- The spectral theorem is a powerful tool in the study of partial differential equations and the geometry of manifolds, providing a way to analyze the behavior of solutions and the structure of the solution space

## Weyl's law and eigenvalue asymptotics

- Weyl's law describes the asymptotic behavior of the eigenvalues of the Laplace-Beltrami operator on compact manifolds, stating that the number of eigenvalues less than a given value grows like a power of the value, with the exponent determined by the dimension of the manifold
- The coefficients in Weyl's law are related to the volume and the total scalar curvature of the manifold, providing a link between the spectrum of the Laplacian and the geometry of the manifold
- Eigenvalue asymptotics play a crucial role in the study of the heat kernel, the zeta function, and other spectral invariants of Riemannian manifolds

## Spectral geometry and isoperimetric inequalities

- Spectral geometry is the study of the relationships between the spectrum of geometric operators, such as the Laplacian, and the geometry and topology of the underlying manifold
- Isoperimetric inequalities, such as the Faber-Krahn inequality and the Cheeger inequality, relate the eigenvalues of the Laplacian to geometric quantities, such as the volume and the isoperimetric constant of the manifold
- These inequalities provide a way to control the size of the eigenvalues in terms of the geometry of the manifold and have important applications in the study of the heat kernel, the zeta function, and the determinant of the Laplacian

## Nodal sets and unique continuation principles

- Nodal sets are the zero sets of eigenfunctions of the Laplace-Beltrami operator and provide a way to visualize the structure of the eigenfunctions and the geometry of the manifold
- The study of nodal sets leads to important questions in geometric analysis, such as the nodal domain theorem, which relates the number of nodal domains (connected components of the complement of