

Statics and Strength of Materials

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Contents

- Unit 1 – Intro to Statics and Mechanics - 3 guides
- Unit 2 – Force Systems and Resultants - 4 guides
- Unit 3 – Equilibrium of Particles & Rigid Bodies - 3 guides
- Unit 4 – Structural Analysis & Machines - 3 guides
- Unit 5 – Distributed Forces: Centroids & Centers - 3 guides
- Unit 6 – Friction - 3 guides
- Unit 7 – Stress and Strain - 4 guides
- Unit 8 – Material Properties in Engineering - 3 guides
- Unit 9 – Axial Loading & Deformation - 3 guides
- Unit 10 – Torsion - 3 guides
- Unit 11 – Shear Force & Bending Moment Diagrams - 3 guides
- Unit 12 – Stresses in Beams - 3 guides
- Unit 13 – Beam Deflection - 3 guides
- Unit 14 – Combined Loading & Stress Transformation - 3 guides
- Unit 15 – Columns and Structural Stability - 3 guides

Unit 1 – Intro to Statics and Mechanics

1.1 Fundamental concepts and principles of statics

Statics is the backbone of engineering mechanics, focusing on forces acting on objects at rest. It's crucial for designing structures that stay put, like bridges and buildings. Without it, things would topple over!

In this part, we'll dive into the key ideas of statics. We'll look at force and moment equilibrium, types of forces, and how to use free body diagrams. These concepts are the building blocks for solving real-world engineering problems.

Statics in Engineering Mechanics

Definition and Importance

- Statics is the study of forces and their effects on bodies at rest or in equilibrium
- Statics is a fundamental branch of engineering mechanics that forms the basis for many other branches (dynamics, strength of materials, fluid mechanics)
- The principles of statics are used to analyze and design structures, machines, and other engineering systems to ensure stability and prevent failure

Role in Engineering

- Statics plays a crucial role in various engineering disciplines (civil, mechanical, aerospace)
- Understanding statics is essential for designing and analyzing load-bearing structures (bridges, buildings, cranes)
- Statics principles are applied in the design of machines and mechanisms to ensure proper functioning and avoid failure under load

Principles of Statics

Force Equilibrium

- The principle of force equilibrium states that a body is in equilibrium when the net force acting on it is zero
- For a body to be in equilibrium, the sum of all forces acting on it must be zero in each direction (x, y, z)
- Force equilibrium is represented by the equation: $\sum F = 0$

Moment Equilibrium

- The principle of moment equilibrium states that a body is in equilibrium when the net moment (or torque) acting on it about any point is zero
- For a body to be in equilibrium, the sum of all moments acting on it must be zero about any point
- Moment equilibrium is represented by the equation: $\sum M = 0$

Assumptions in Statics

- Statics assumes that bodies are rigid, meaning that they do not deform under the action of forces
- Rigid body assumption simplifies the analysis by neglecting the effects of deformation
- In reality, all bodies deform to some extent under load, but the deformation is often negligible for statics problems
- Statics assumes that the effects of friction are negligible unless otherwise specified
- Frictionless surfaces are assumed to simplify the analysis
- Friction forces can be included in the analysis when necessary (rough surfaces, wedges, screws)

Types of Forces and Effects

Classification of Forces

- Forces can be classified as external forces (applied by objects outside the body) or internal forces (acting within the body itself)
- External forces include applied loads, reactions, and body forces (gravity)
- Internal forces arise from the interaction between particles within the body (tension, compression, shear)
- Forces can be further categorized as:
 - Concentrated forces (acting at a single point)
 - Distributed forces (spread over an area or volume)
 - Body forces (acting throughout the body, such as gravity)

Effects of Forces

- The effects of forces on a body include translation (linear motion), rotation, or deformation, depending on the nature and magnitude of the forces and the properties of the body
- Translation occurs when a force causes a body to move in a straight line without rotation
- Rotation occurs when a force causes a body to turn about an axis
- Deformation occurs when a force causes a body to change its shape or size
- The effects of forces can be analyzed using the principles of statics to determine the resulting motion, stress, or strain in the body

Equilibrium in Statics

Concept of Equilibrium

- Equilibrium is the state in which a body remains at rest or moves with constant velocity under the action of balanced forces
- Static equilibrium occurs when a body is at rest and the net force and net moment acting on it are zero

- In static equilibrium, the body has no acceleration (linear or rotational)
- The forces acting on the body balance each other, resulting in no net force or moment

Equilibrium Equations

- Equilibrium equations are used to analyze the forces and moments acting on a body and determine the conditions necessary for equilibrium
- Force equilibrium equations: $\sum F_x = 0, \sum F_y = 0, \sum F_z = 0$
- Moment equilibrium equations: $\sum M_x = 0, \sum M_y = 0, \sum M_z = 0$
- Solving equilibrium equations allows for the determination of unknown forces or reactions acting on the body

Free Body Diagrams

- Free body diagrams are used to represent the forces and moments acting on a body and to apply the equilibrium equations
- A free body diagram isolates the body of interest from its surroundings and shows all the external forces and moments acting on it
- The body is treated as a free body, and the effects of the surrounding bodies are represented by forces and moments
- Free body diagrams are an essential tool for visualizing and analyzing the forces acting on a body in equilibrium

Applications of Equilibrium

- The concept of equilibrium is used in the design and analysis of structures, machines, and other engineering systems to ensure stability and prevent failure under the action of forces
- Bridges and buildings must be designed to withstand the loads (dead loads, live loads, wind loads) acting on them without collapsing
- Machines and mechanisms must be designed to operate in equilibrium under the forces and moments generated during their operation
- Understanding and applying the principles of equilibrium is crucial for ensuring the safety, reliability, and performance of engineered systems

1.2 Units, dimensions, and systems of measurement

Units, dimensions, and measurement systems are the building blocks of mechanics. They let you quantify physical quantities and perform calculations that actually mean something. Getting comfortable with these concepts now will save you from frustrating errors throughout the rest of the course.

This section covers fundamental units, derived units, unit conversions, and dimensional analysis, which is your go-to tool for checking whether an equation makes physical sense.

Fundamental Units in Mechanics

Base Units and Their Relationships

Every quantity in mechanics can be traced back to three base dimensions: length, mass, and time. All other units you'll encounter are built from combinations of these three.

The two unit systems you need to know:

- SI (metric) system:
 - Meter (m) for length
 - Kilogram (kg) for mass
 - Second (s) for time
- US customary system:
 - Foot (ft) for length
 - Slug for mass
 - Second (s) for time

A note on the US customary system: the slug is the standard unit of mass in this system when working with Newton's second law. You might also see pound-mass (lbm), but in statics and dynamics courses, slug is what keeps $F = ma$ consistent with pounds-force. One slug is defined so that a 1-slug mass accelerated at 1 ft/s^2 experiences a force of 1 lbf.

Derived Units

Derived units are formed by combining base units through multiplication or division. The most important one for this course is force.

- Newton (N) in SI: $1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$
- Pound-force (lbf) in US customary: $1 \text{ lbf} = 1 \text{ slug} \cdot \text{ft/s}^2$

Other derived units you'll see:

Quantity	SI Unit	US Customary Unit
Force	Newton (N)	Pound-force (lbf)
Pressure	Pascal (Pa) = N/m^2	Pounds per square inch (psi)
Energy	Joule (J) = $\text{N} \cdot \text{m}$	Foot-pound (ft·lbf)
Power	Watt (W) = J/s	Horsepower (hp)

Notice how each derived unit traces back to the three base units. A Pascal, for instance, expands to $\text{kg}/(\text{m} \cdot \text{s}^2)$.

Unit System Conversion

Conversion Factors

Conversion factors let you translate a quantity from one unit system to another without changing its physical meaning. You multiply by a ratio that equals 1 (since the numerator and denominator represent the same quantity).

Common conversion factors to know:

- $1\text{ m} = 3.281\text{ ft}$
- $1\text{ kg} = 0.06852\text{ slug}$
- $1\text{ N} = 0.2248\text{ lbf}$
- $1\text{ Pa} = 1.450 \times 10^{-4}\text{ psi}$

Example: Convert 50 N to pound-force.

$$50\text{ N} \times \frac{0.2248\text{ lbf}}{1\text{ N}} = 11.24\text{ lbf}$$

The "N" cancels, leaving you with lbf. Always write out the units and cancel them explicitly; this is the single best way to catch conversion mistakes.

Consistent Units in Equations

Before plugging values into any equation, check that every term uses the same unit system. Mixing systems is one of the most common errors in mechanics problems.

For example, if you're using $F = ma$ and your mass is in kg but your acceleration is in ft/s^2 , you need to convert the acceleration to m/s^2 first (divide by 3.281), or convert the mass to slugs. Pick one system and stick with it for the entire problem.

Dimensions and Dimensional Homogeneity

Physical Nature of Quantities

Dimensions describe what type of physical quantity something represents, independent of which unit system you use. We denote the basic dimensions with brackets:

- [L] for length
- [M] for mass
- [T] for time

Any derived quantity has dimensions built from these. For example:

- Velocity: $[L][T]^{-1}$
- Acceleration: $[L][T]^{-2}$
- Force: $[M][L][T]^{-2}$
- Pressure: $[M][L]^{-1}[T]^{-2}$

Think of dimensions as the "type" of a quantity. Meters and feet are different units, but they share the same dimension: length [L].

Dimensional Homogeneity

An equation is dimensionally homogeneous if every term has the same dimensions. This is a requirement for any physically valid equation.

Take $v = v_0 + at$. Check each term:

- v : $[L][T]^{-1}$
- v_0 : $[L][T]^{-1}$
- at : $[L][T]^{-2} \times [T] = [L][T]^{-1}$

All three terms have dimensions of $[L][T]^{-1}$, so the equation is dimensionally homogeneous.

Dimensionless quantities like angles (in radians), ratios, and coefficients of friction have no dimensions, represented as [1]. You can't add a dimensionless quantity to one that has dimensions.

Dimensional Analysis for Equation Consistency

Verifying Equation Consistency

Dimensional analysis is a quick way to check whether an equation could be correct. Here's the process:

1. Replace each variable in the equation with its dimensions.
2. Simplify the dimensions on each side using algebra.
3. Compare the left-hand side to the right-hand side. If the dimensions match, the equation is dimensionally consistent. If they don't, something is wrong.

Example: Verify that $s = v_0t + \frac{1}{2}at^2$ is dimensionally consistent.

- Left side: s has dimensions $[L]$
- Right side, first term: $v_0t = [L][T]^{-1} \times [T] = [L]$
- Right side, second term: $\frac{1}{2}at^2 = [L][T]^{-2} \times [T]^2 = [L]$

Both terms on the right reduce to $[L]$, matching the left side. The equation checks out.

Limitations and Applications

Dimensional analysis can catch errors, but it has limits. An equation can be dimensionally correct and still be wrong. For instance, $s = 2v_0t + 3at^2$ passes a dimensional check, but the numerical coefficients are wrong. Dimensional analysis catches structural mistakes (like adding a force to a velocity), not numerical ones.

The Buckingham Pi theorem takes dimensional analysis further: any physically meaningful equation with n variables and k independent dimensions can be rewritten using $p = n - k$ dimensionless groups. This is especially useful in fluid mechanics and heat transfer for scaling experiments, but you likely won't need it much in an introductory statics course. Just know it exists as a powerful extension of the dimensional reasoning you're learning here.

1.3 Vector algebra and its applications in mechanics

Vector algebra gives you the mathematical tools to represent forces and moments as quantities with both size and direction. Without it, you can't analyze how structures carry loads or determine whether a system is in equilibrium. This topic covers vector operations, component resolution, and how vectors apply directly to force and moment problems in statics.

Vectors and their properties

Vector definition and representation

A vector is a quantity that has both magnitude (size) and direction. You can picture it as an arrow: the length of the arrow tells you "how much," and the way it points tells you "which way."

- The magnitude is the length of the arrow, representing the intensity of the quantity (e.g., a 500 N force, a 3 m/s velocity).
- The direction is the orientation of the arrow, indicating the line of action along which the quantity acts.
- Physical quantities like force, velocity, and displacement are all vectors because they require both magnitude and direction to be fully described.

Compare this to a scalar, which has magnitude only (like mass or temperature). Scalars don't need a direction, so they can't be represented as arrows.

Equality of vectors

Two vectors are equal if and only if they have the same magnitude and the same direction. Their starting points don't matter.

- You can slide a vector anywhere in space (translate it) without changing what it represents, as long as you keep its length and orientation the same.
- This property is called free vector behavior, and it means equal vectors produce identical mechanical effects regardless of where they're drawn.

Vector operations

Vector addition and subtraction

Vector addition combines two or more vectors into a single resultant vector. There are two common geometric methods:

1. Parallelogram law: Place both vectors tail-to-tail. Draw a parallelogram using the two vectors as adjacent sides. The diagonal from the shared tail point to the opposite corner is the resultant.
2. Triangle rule (tip-to-tail): Place the tail of the second vector at the head of the first. The resultant runs from the tail of the first vector to the head of the second.

Both methods give the same result. The triangle rule extends naturally to adding three or more vectors by chaining them tip-to-tail.

Vector subtraction works by adding the negative of a vector. To compute $\vec{A} - \vec{B}$, you reverse the direction of $\vec{B}$ (same magnitude, opposite direction) and then add it to $\vec{A}$ using either method above.

Scalar multiplication and vector products

Scalar multiplication scales a vector's magnitude by a number while keeping (or flipping) its direction.

- Multiply by a positive scalar: the direction stays the same, the magnitude scales.
- Multiply by a negative scalar: the direction reverses, the magnitude scales by the absolute value.

Dot product (scalar product) produces a scalar from two vectors:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

where θ is the angle between the two vectors. The dot product tells you how much one vector projects onto the other. It's commutative ($\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$) and distributive over addition. A dot product of zero means the vectors are perpendicular.

Cross product (vector product) produces a new vector perpendicular to both input vectors:

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\sin\theta \hat{n}$$

where $\hat{n}$ is the unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$, found using the right-hand rule: curl your fingers from $\vec{A}$ toward $\vec{B}$, and your thumb points in the direction of the result.

- The cross product is anticommutative: $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$.
- It's distributive over addition.
- A cross product of zero means the vectors are parallel.

The cross product is especially important in statics because it's how you calculate moments.

Resolving vectors into components

Vector resolution

Resolving a vector means breaking it into perpendicular parts along coordinate axes. This converts a single angled vector into two (or three) simpler pieces you can work with algebraically.

For a 2D vector with magnitude F at angle θ from the positive x-axis:

1. x-component: $F_x = F\cos\theta$
2. y-component: $F_y = F\sin\theta$

In 3D, you also get a z-component. Each component is the projection of the vector onto that axis. Once resolved, you can write the vector in component form as $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors along the x, y, and z axes.

Calculating magnitude and direction from components

Going the other direction, you can reconstruct magnitude and direction from known components.

Magnitude uses the Pythagorean theorem:

- 2D: $|\vec{F}| = \sqrt{F_x^2 + F_y^2}$
- 3D: $|\vec{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2}$

Direction in 2D:

$$\theta = \text{atan2}(F_y, F_x)$$

Use `atan2` rather than plain `arctan` because it correctly handles all four quadrants. In 3D, direction is typically described by direction cosines: the cosine of the angle between the vector and each coordinate axis.

$$\cos \alpha = \frac{F_x}{|\vec{F}|}, \quad \cos \beta = \frac{F_y}{|\vec{F}|}, \quad \cos \gamma = \frac{F_z}{|\vec{F}|}$$

A useful check: $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

Vector algebra in physics problems

Forces as vectors

Forces are naturally represented as vectors: the magnitude is the force intensity (in Newtons), and the direction is the line of action.

To find the resultant force on an object, add all force vectors. In practice, you'll almost always resolve each force into components first, then sum the components separately:

1. Resolve every force into x, y (and z) components.
2. Sum all x-components: $R_x = \sum F_x$
3. Sum all y-components: $R_y = \sum F_y$
4. Find the resultant magnitude: $R = \sqrt{R_x^2 + R_y^2}$
5. Find the resultant direction: $\theta_R = \text{atan2}(R_y, R_x)$

Equilibrium occurs when the resultant force is zero. This gives you the equations:

- 2D: $\sum F_x = 0$ and $\sum F_y = 0$
- 3D: $\sum F_x = 0$, $\sum F_y = 0$, and $\sum F_z = 0$

These equilibrium equations are the foundation of every statics problem you'll solve.

Moments as vectors

A moment (or torque) measures a force's tendency to cause rotation about a point. The moment vector is calculated using the cross product:

$$\vec{M} = \vec{r} \times \vec{F}$$

where $\vec{r}$ is the position vector from the point of rotation to the point where the force is applied, and $\vec{F}$ is the force vector.

The magnitude of the moment is:

$$|\vec{M}| = |\vec{r}||\vec{F}|\sin \theta$$

where θ is the angle between $\vec{r}$ and $\vec{F}$. The term $|\vec{r}|\sin \theta$ is the perpendicular distance (moment arm) from the line of action of the force to the rotation point. A larger moment arm means a greater rotational effect for the same force.

The direction of $\vec{M}$ (found via the right-hand rule) tells you the axis of rotation and the sense (clockwise vs. counterclockwise).

Moment equilibrium requires:

- 2D: $\sum M = 0$ (about any point)
- 3D: $\sum M_x = 0$, $\sum M_y = 0$, and $\sum M_z = 0$

Combined with force equilibrium, these conditions give you the complete set of equations needed to solve static equilibrium problems.

Unit 2 – Force Systems and Resultants

2.1 Types of force systems and their characteristics

Force systems are the building blocks of engineering mechanics. They come in various types, each with unique characteristics that affect how objects respond to them. Understanding these systems is crucial for analyzing structures and machines.

From concurrent forces that meet at a point to couples that create pure rotational effects, each force system behaves differently. Knowing how to identify and analyze these systems helps engineers predict object behavior and design safer, more efficient structures.

Force Systems: Types and Properties

Classification of Force Systems

- Force systems are classified based on the spatial arrangement and orientation of the forces acting on an object or structure
- Concurrent force systems consist of forces that all pass through a single point, regardless of their magnitude or direction (cables supporting a suspended load)
- Coplanar force systems are composed of forces that lie within the same plane, but do not necessarily pass through a single point (forces acting on a gear train)
- Parallel force systems have forces that act in the same or opposite directions, with their lines of action being parallel to each other (weight distribution on a beam)
- Non-concurrent force systems have forces that do not pass through a single point and may not be coplanar or parallel (forces on a truss bridge)

Special Force System: Couple

- A couple is a special force system consisting of two equal and opposite forces that do not share the same line of action, causing a pure moment or rotational effect
- Couples are characterized by their moment, which is the product of the force magnitude and the perpendicular distance between the lines of action (turning a doorknob)
- The net force of a couple is always zero, as the forces cancel each other out, but the moment is non-zero
- Couples can be represented by a moment vector, with the direction determined by the right-hand rule

Analysis Methods for Force Systems

- Concurrent force systems can be analyzed using vector addition, as the forces can be represented by vectors that all intersect at a common point
- The resultant force of a concurrent force system can be found using the parallelogram law or by resolving the forces into their rectangular components and summing them (finding the resultant of multiple cables supporting a load)
- Coplanar force systems can be analyzed by resolving the forces into their rectangular components and applying equilibrium conditions separately in each direction (analyzing forces on a pulley system)

- Parallel force systems can be simplified by summing the magnitudes of the forces acting in the same direction and subtracting the magnitudes of the forces acting in the opposite direction (determining the net force on a loaded beam)
- The moment caused by a parallel force system can be determined by calculating the moments of each force about a chosen reference point and summing them (calculating the moment caused by parallel forces on a lever)

Force Systems: Effects on Objects

Translational and Rotational Effects

- The effect of a force system on an object depends on the magnitude, direction, and point of application of the forces, as well as the object's geometry and material properties
- Concurrent force systems cause a translational effect on the object, with the resultant force determining the object's acceleration if it is not in equilibrium (a ball being kicked by multiple players simultaneously)
- Coplanar force systems can cause both translational and rotational effects, depending on the location of the forces relative to the object's center of mass (a door subjected to a push and a pull at different points)
- Parallel force systems can cause translational motion if the forces are unbalanced, or a couple moment if the forces are equal and opposite but not collinear (a book sliding across a table due to unequal friction forces)

Stress Distributions and Deformations

- Non-concurrent force systems can induce complex stress distributions within the object, leading to deformation, bending, or twisting (a cantilever beam supporting a distributed load)
- The stress distribution caused by a force system depends on the object's geometry, material properties, and support conditions
- Deformations can include axial elongation or compression, shear distortion, bending, and torsion, depending on the nature of the force system and the object's characteristics (a spring being compressed or stretched)

Stability and Equilibrium

- The stability of an object subjected to a force system depends on the balance of forces and moments, as well as the object's support conditions and geometry
- For an object to be in equilibrium, the net force and net moment acting on it must be zero (a ladder leaning against a wall)
- Stable equilibrium occurs when an object returns to its original position after a small disturbance, while unstable equilibrium occurs when an object tends to move away from its original position (a ball resting on a curved surface)
- The support conditions, such as the number and location of contact points, play a crucial role in determining the stability of an object under a given force system (a table with four legs vs. a three-legged stool)

Force Systems: Selection for Scenarios

Identifying the Appropriate Force System Type

- Identifying the appropriate force system type is crucial for selecting the correct analysis method and obtaining accurate results
- When all forces pass through a single point, the system can be treated as a concurrent force system, simplifying the analysis (a chandelier suspended by multiple chains)
- If the forces lie within the same plane but do not pass through a single point, the system should be analyzed as a coplanar force system (forces acting on a pulley system in a single plane)
- When the forces are parallel to each other, the system can be classified as a parallel force system, allowing for simplification in the analysis process (forces acting on a loaded shelf)
- If the forces do not meet the criteria for concurrent, coplanar, or parallel systems, the problem should be treated as a non-concurrent force system, requiring a more complex analysis approach (forces acting on a space truss)

Considering Couples and Pure Moments

- The presence of couples or pure moments in the system should be identified, as they can significantly influence the behavior and response of the object or structure
- Couples can be analyzed separately from the other forces in the system, as they only contribute to the net moment and not the net force (a steering wheel being turned)
- Pure moments can be combined with other moments caused by forces to determine the overall rotational effect on the object (a gear subjected to both a couple and a tangential force)

Simplifying Assumptions and Accuracy

- The choice of the appropriate force system type may also depend on the desired level of accuracy and the simplifying assumptions that can be made in the analysis
- In some cases, a more complex force system can be approximated by a simpler one to facilitate the analysis, provided that the assumptions are reasonable and the results are sufficiently accurate (treating a distributed load as a single concentrated force)
- Engineers must exercise judgment in selecting the appropriate force system type, considering factors such as the problem's complexity, the required accuracy, and the available computational resources (using a coplanar force system instead of a non-concurrent one for preliminary design)

2.2 Moment of a force and couple moments

Moment of a force

Definition and significance

A moment is the turning effect of a force about a point or axis. It measures how much a force tends to cause rotation. In statics, moments are central to analyzing whether a structure will stay in rotational equilibrium or start to spin.

The moment of a force is a vector quantity. Its magnitude equals the product of the force and the perpendicular distance from the point (or axis) to the line of action of the force:

$$M = F \times d$$

where F is the force magnitude and d is the perpendicular distance (often called the moment arm).

- The standard units are N·m (Newton-meters) or lb·ft (pound-feet).
- Direction is determined by the right-hand rule: curl your fingers in the direction the force would cause rotation, and your thumb points along the moment vector.
- Moments are what let you solve for support reactions and check that a structure won't rotate under load.

Properties and characteristics

- The moment depends on both the force magnitude and the moment arm. A small force with a long lever arm can produce the same moment as a large force with a short one.
- Sliding a force along its own line of action doesn't change the moment about a given point. This is called the principle of transmissibility.
- Sign convention: counterclockwise (CCW) is typically taken as positive, and clockwise (CW) as negative. Be consistent within each problem.
- A couple moment (covered below) is a free vector, meaning it can be applied anywhere without changing its effect. The moment of a single force about a point, however, depends on which point you choose.

Calculating moment of a force

Methods for calculation

There are three main approaches to computing the moment of a force about a point:

1.

Scalar method: Multiply the force magnitude by the perpendicular distance from the point to the force's line of action. $M = F \times d$

2.

Cross-product method: Take the cross product of the position vector $\vec{r}$ (from the point to any point on the force's line of action) and the force vector $\vec{F}$: $\vec{M} = \vec{r} \times \vec{F}$ This is especially useful in 3D problems where finding the perpendicular distance geometrically is awkward.

3.

Varignon's theorem (principle of moments): Break the force into convenient components, compute each component's moment separately, and add them. The total equals the moment of the original force. This is handy when the perpendicular distance to the full force is hard to find, but the distances to its X - and Y -components are straightforward.

When multiple forces act on a body, the net moment about a point is the algebraic sum of the individual moments, respecting sign conventions.

Examples and applications

Example 1: A 100 N force acts on a lever arm at a perpendicular distance of 0.5 m from the pivot.

$$M = 100 \text{ N} \times 0.5 \text{ m} = 50 \text{ N}\cdot\text{m} \text{ (CCW)}$$

Example 2: A beam has two loads: 200 N upward at 1 m from the left end, and 300 N downward at 2 m from the left end. Taking moments about the left end (CCW positive):

$$M = (200)(1) - (300)(2) = 200 - 600 = -400 \text{ N}\cdot\text{m}$$

The negative sign tells you the net rotation is clockwise.

Where moments show up in practice:

- Structural analysis: Checking that the sum of moments about any support equals zero so a bridge or building frame stays stable.
- Biomechanics: Calculating the moment a muscle force creates about a joint to understand loading during activities like lifting or walking.
- Machine design: Sizing levers, gears, and pulleys so they produce the required torque for a given task.

Couple moments and their effects

Concept and properties

A couple consists of two forces that are equal in magnitude, opposite in direction, and parallel but separated by a perpendicular distance d . Because the forces cancel each other out (net force = zero), a couple produces pure rotation with no translation.

The magnitude of a couple moment is:

$$M = F \times d$$

where F is the magnitude of one of the two forces and d is the perpendicular distance between their lines of action.

The critical property that sets couples apart: the couple moment is the same regardless of which point you take moments about. This is why a couple moment is treated as a free vector. You can move it anywhere in (or parallel to) its plane without changing its effect on the body.

Direction follows the right-hand rule, with the moment vector perpendicular to the plane containing the two forces.

Effects on a system

- Couples can balance or counteract rotational effects from other forces, which is how you achieve rotational equilibrium in many real problems.
- Because a couple is a free vector, you can relocate it on a body for convenience during analysis without altering the solution.
- Multiple couple moments combine through vector addition, just like any other vectors.
- In physical systems, couple moments show up as torque: the twisting action of a wrench on a bolt, the output of an electric motor's shaft, or the torsional load on a drive shaft.

- When a couple acts on a deformable body, it can cause torsional deformation (twisting), which becomes a major topic later in strength of materials.

Equilibrium problems with moments

Principles and problem-solving strategies

For a rigid body in static equilibrium, two conditions must hold simultaneously:

$$\sum \vec{F} = 0 \quad (\text{translational equilibrium})$$

$$\sum \vec{M} = 0 \quad (\text{rotational equilibrium})$$

Here's a reliable approach for solving these problems:

1. Draw a free-body diagram (FBD). Show every external force, reaction, and couple moment acting on the body. This step prevents missed forces, which is the most common source of errors.
2. Choose a convenient moment point. Picking a point where an unknown force acts eliminates that unknown from the moment equation, simplifying your algebra.
3. Write the moment equilibrium equation ($\sum M = 0$) about your chosen point.
4. Write force equilibrium equations ($\sum F_x = 0$, $\sum F_y = 0$) to solve for remaining unknowns.
5. For distributed loads, replace the distributed load with its equivalent resultant force acting at the centroid of the loading distribution before computing moments.

If you have more unknowns than equations from a single moment point, write additional moment equations about different points. Make sure each new equation is independent (gives you genuinely new information).

Examples and applications

Example 1: A uniform beam is 6 m long and weighs 1000 N. It's supported by a pin at the left end (point A) and a roller at the right end (point B).

1. The beam's weight acts at its midpoint (3 m from A).
2. Take moments about A to eliminate the pin reactions: $\sum M_A = 0 : R_B(6) - 1000(3) = 0$
 $R_B = 500 \text{ N}$
3. Use $\sum F_y = 0$ to find the vertical reaction at A: $A_y + 500 - 1000 = 0 \Rightarrow A_y = 500 \text{ N}$

Example 2: For a simple truss loaded at a joint, you'd first find the support reactions using moment equilibrium about one support, then apply force equilibrium. Once reactions are known, the method of joints lets you analyze each pin joint's equilibrium to find internal member forces.

Common applications of moment equilibrium:

- Designing structural supports for bridges, cranes, and building frames so they remain stable under expected loads.
- Analyzing joint and muscle forces in biomechanics to understand injury risk during lifting or running.
- Sizing components in mechanical systems (linkages, cams, gear trains) so they function correctly under operating loads.

2.3 Resultants of force systems

Resultant Force Concept

Understanding Resultant Forces

A resultant force is a single force that produces the same effect on a body as an entire system of forces acting together. Think of it as a shortcut: instead of tracking five different forces on a structure, you replace them with one equivalent force and work with that.

This matters because it simplifies your free body diagrams and makes equilibrium analysis far more manageable. The resultant causes the same linear acceleration as the original force system, so nothing is lost in the simplification.

Significance of Resultant Forces

- If the resultant force is zero, the body is in equilibrium (no acceleration).
- If the resultant is nonzero, it tells you the net magnitude and direction of the unbalanced force, which determines how the body accelerates.

In statics, you're almost always setting the resultant equal to zero. But understanding how to compute it is the first step toward recognizing when equilibrium holds and when it doesn't.

Resultant Force Determination

Concurrent Forces

Concurrent forces have lines of action that all pass through a single point. Because they share that common point, the resultant also acts through it. This means you only need to find the magnitude and direction of the resultant; the location is already known.

A classic example: forces acting on a pin joint in a truss. Every member force passes through the pin, so the system is concurrent by definition.

Coplanar Forces

Coplanar forces all lie in the same plane. Their resultant also lies in that plane, but unlike concurrent forces, the lines of action may not meet at one point. When coplanar forces are not concurrent, you'll need to determine the resultant's line of action using moment calculations (covered below under parallel forces).

Calculating Resultant Force Properties

Step 1: Resolve each force into components.

$$F_x = F\cos(\theta), \quad F_y = F\sin(\theta)$$

Step 2: Sum the components in each direction.

$$R_x = \sum F_x, \quad R_y = \sum F_y$$

Step 3: Find the magnitude of the resultant.

$$R = \sqrt{R_x^2 + R_y^2}$$

For the special case of only two perpendicular forces, this reduces directly to $R = \sqrt{F_1^2 + F_2^2}$.

Step 4: Find the direction of the resultant.

$$\theta_R = \tan^{-1} \left(\frac{R_y}{R_x} \right)$$

Be careful with this step. The arctangent function only gives angles in the first and fourth quadrants, so check the signs of R_x and R_y to confirm which quadrant the resultant actually points into.

Step 5: Determine the line of action.

- For concurrent forces, the resultant passes through the point of concurrency.
- For non-concurrent forces, use moment equilibrium to locate where the resultant acts (see the parallel forces section).

Vector Algebra for Resultants

Representing Forces as Vectors

Every force is a vector: it has both magnitude and direction. To work with forces algebraically, you decompose them into unit vector components.

For example, a 10 N force at 30° from the horizontal becomes:

$$\vec{F} = 10\cos(30^\circ)\hat{i} + 10\sin(30^\circ)\hat{j} = 8.66\hat{i} + 5.00\hat{j} \text{ N}$$

Once every force is in component form, finding the resultant is just adding the $\hat{i}$ components together and the $\hat{j}$ components together.

Graphical Methods for Resultants

Two graphical techniques show up frequently:

- Parallelogram law: Place two force vectors tail-to-tail. Complete the parallelogram. The diagonal from the common tail to the opposite corner is the resultant. This works well for two forces.
- Triangle (polygon) rule: Arrange force vectors head-to-tail in sequence. The resultant is the vector drawn from the tail of the first to the head of the last. This extends naturally to any number of forces (becoming a force polygon).

These methods are useful for visualizing the result and for checking your algebraic answers, but for precise work you'll rely on the component method described above.

Trigonometry in Force Analysis

Trigonometric functions connect a force's magnitude and angle to its rectangular components:

- $F_x = F\cos(\theta)$ gives the horizontal component
- $F_y = F\sin(\theta)$ gives the vertical component

And going the other direction, from components back to magnitude and angle:

$$R = \sqrt{F_x^2 + F_y^2}, \quad \theta = \tan^{-1} \left(\frac{F_y}{F_x} \right)$$

A common mistake: using the wrong reference angle for θ . Always be clear about whether your angle is measured from the positive x-axis, from the horizontal, or from some other reference line. Mixing these up is one of the fastest ways to get a wrong answer.

Simplifying Force Systems

Equivalent Force Systems

A force system can be replaced by a simpler equivalent system as long as two conditions are met:

1. The resultant force is the same ($\sum \vec{F}$ is preserved).
2. The resultant moment about any point is the same ($\sum M$ is preserved).

In practice, this means you can replace a distributed set of forces on a beam with a single resultant force and a resultant couple moment at a chosen point. The choice of point is up to you, and picking it wisely (often at a support) can simplify your calculations.

Principle of Transmissibility

The principle of transmissibility states that you can slide a force anywhere along its line of action without changing its external effect on a rigid body. The force produces the same resultant and the same moment about any point, regardless of where along that line you place it.

This is valid only for rigid bodies. It does not apply if you care about internal forces or deformations, which is why strength of materials problems sometimes require more care about where a load is actually applied.

Parallel Forces

When forces are all parallel, finding the resultant magnitude is straightforward: just add them algebraically (with sign conventions for direction).

$$R = \sum F_i$$

The trickier part is locating the line of action. You use the moment condition: the moment produced by the resultant about any point must equal the sum of moments produced by the individual forces about that same point.

$$R \cdot d = \sum F_i \cdot d_i$$

where d is the perpendicular distance from the chosen point to the resultant's line of action, and d_i is the distance from that point to each individual force.

For example, consider two parallel forces of 40 N and 60 N acting in the same direction, separated by 2 m. The resultant is 100 N. Taking moments about the location of the 40 N force:

$$100 \cdot d = 60 \cdot 2 = 120 \text{ N}\cdot\text{m}$$

$$d = 1.2 \text{ m from the 40 N force}$$

The resultant acts closer to the larger force, which makes intuitive sense. This inverse proportionality between force magnitude and distance from the resultant is a reliable pattern to watch for.

2.4 Distributed forces and equivalent point loads

Distributed Forces and Equivalent Point Loads

Distributed forces act over a length, area, or volume rather than at a single point. Nearly every real structural load starts as a distributed force: the weight of a beam along its span, water pressure against a dam, snow piling on a roof. To analyze these structures, you need to know how to replace a distributed force with a single equivalent point load that produces the same external effect. That conversion is the core skill in this section.

Distributed Forces and Applications

Understanding Distributed Forces

A distributed force is any load spread continuously over a region rather than applied at one discrete point. You describe its intensity as force per unit length (N/m or lb/ft), force per unit area (pressure), or force per unit volume (body forces like self-weight).

Distributed forces fall into two categories:

- Uniform: constant intensity across the entire domain (e.g., a concrete slab of constant thickness resting on a beam)
- Non-uniform (varying): intensity changes with position (e.g., hydrostatic pressure on a dam, which increases linearly with depth)

Applications of Distributed Forces

- Bridges must carry the distributed weight of traffic, pedestrians, and the bridge deck itself.
- Storage tanks resist the distributed hydrostatic pressure of the stored liquid on the walls and base.
- Aircraft wings experience distributed aerodynamic pressure that varies along the span and chord, governing both lift and structural demands.
- Roofs are designed for distributed snow and wind loads; retaining walls resist distributed lateral soil pressure; submarine hulls withstand hydrostatic pressure that increases with depth.

In every case, the first analysis step is the same: represent the distributed load as an equivalent point load so you can apply equilibrium equations.

Resultant Force and Centroid

Calculating the Resultant Force

The resultant force is the single concentrated force that has the same net translational effect as the entire distributed load. You find its magnitude by integrating the load intensity function over the domain.

For a load $w(x)$ distributed along a beam of length L :

$$F_R = \int_0^L w(x) dx$$

Two common cases:

1. Uniform load ($W = \text{constant}$): The integral simplifies to $F_R = W \cdot L$. Graphically, this is the area of the rectangular loading diagram.
2. Linearly varying load (triangular distribution from 0 to W_0): $F_R = \frac{1}{2}W_0L$, which is the area of the triangle.

For any shape, the magnitude of the resultant equals the area under the loading diagram (in 2-D problems) or the volume under the pressure surface (in 3-D problems).

Locating the Centroid

The resultant force must be placed at the centroid of the loading diagram so that it produces the same moment about any point as the original distributed load. The centroid coordinate $\bar{x}$ is found from:

$$\bar{x} = \frac{\int_0^L x w(x) dx}{\int_0^L w(x) dx} = \frac{\int_0^L x w(x) dx}{F_R}$$

Quick references for common shapes:

- Rectangular (uniform) load: centroid at the midpoint, $\bar{x} = L/2$
- Triangular load (zero at one end, max at the other): centroid at $\bar{x} = \frac{2}{3}L$ measured from the zero end (i.e., one-third of the way from the maximum end)

For composite loading diagrams, break the load into simple shapes, find each sub-resultant and its centroid, then combine them using the principle of moments.

Distributed Force to Point Load

Equivalent Point Load Concept

Replacing a distributed load with an equivalent point load means:

1. Calculate the magnitude F_R by integrating (or computing the area of) the loading diagram.
2. Locate the line of action at the centroid $\bar{x}$ of that diagram.
3. Apply F_R at $\bar{x}$ in the same direction as the original load.

After this replacement, every equilibrium equation (forces and moments) gives the same result as it would with the original distributed load.

Important caveat: The equivalent point load reproduces the external reactions correctly, but it does not represent what happens internally. Shear and bending moment diagrams must still be constructed using the actual distributed load, not the point-load shortcut.

Why This Conversion Is Useful

- It lets you solve for support reactions quickly using standard equilibrium methods.
- Equivalent point loads can be combined with other concentrated forces through simple vector addition.
- For beams, trusses, and frames, converting distributed loads to point loads is often the first step before drawing free-body diagrams.

Distributed Force Effects on Structures

Internal Stresses and Deformations

While the equivalent point load is great for finding reactions, the actual distributed load governs the internal response of the structure.

- A beam under a distributed load develops shear forces and bending moments that vary continuously along its length.
- You construct shear (V) and bending moment (M) diagrams from the actual load distribution. For a uniform load on a simply supported beam, the shear diagram is linear and the moment diagram is parabolic.
- Critical values (maximum shear, maximum moment) occur at predictable locations: at supports, at points where the shear crosses zero, or at fixed ends of cantilevers.

Structural Response to Distributed Forces

Different structural forms handle distributed loads in distinct ways:

- Cables under distributed load (their own weight, ice, or wind) take a catenary shape described by a hyperbolic cosine function. When the load is uniform per horizontal length (as with a suspension bridge deck), the cable instead forms a parabola.
- Arches and shells are shaped so that distributed loads travel to the supports primarily through compression, making them highly efficient for loads like self-weight or uniform pressure.

Designing any of these elements requires knowing the distributed load's magnitude, distribution pattern, and location so you can determine the required strength, stiffness, and stability of the structure.

Unit 3 – Equilibrium of Particles & Rigid Bodies

3.1 Conditions for equilibrium of particles

Equilibrium of Particles

A particle is in equilibrium when all the forces acting on it cancel out, meaning the net force is zero. The particle either stays at rest (static equilibrium) or moves at constant velocity (dynamic equilibrium). This section covers how to identify forces, express them mathematically, and solve for unknowns using the equilibrium equations.

Forces Acting on Particles

In statics, a particle is an idealized body with mass but negligible size, represented as a single point. This simplification works whenever the forces acting on a body all pass through (or nearly through) the same point.

Forces are vector quantities with both magnitude and direction. They can be represented graphically as arrows or mathematically using $\hat{i}$, $\hat{j}$, $\hat{k}$ unit vectors. Common forces you'll encounter:

- Gravity: always acts downward with magnitude $W = mg$
- Normal force: perpendicular to and away from a contact surface
- Tension: pulls away from the particle through ropes, cables, or wires
- Friction: opposes motion or impending motion along a surface
- Applied forces: external pushes or pulls in any direction

A free body diagram (FBD) isolates the particle as a dot and shows every force acting on it as an arrow. The net force is the vector sum of all those forces. Drawing a correct FBD is the single most important step in any equilibrium problem. If you miss a force or draw one in the wrong direction, everything downstream will be wrong.

Expressing Forces Mathematically

To work with forces algebraically, you resolve each one into rectangular components along your chosen coordinate axes (x, y, and in 3-D problems, z).

If a force F makes angles θ_x , θ_y , and θ_z with the coordinate axes, its components are:

- $F_x = F \cos \theta_x$
- $F_y = F \cos \theta_y$
- $F_z = F \cos \theta_z$

These cosines are called direction cosines, and they satisfy $\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1$.

For 2-D problems (the most common case at this level), you typically measure a single angle θ from the positive x-axis, giving $F_x = F \cos \theta$ and $F_y = F \sin \theta$.

The magnitude of the resultant force from its components is:

$$R = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

As an example, a force $\vec{F} = (200\hat{i} - 300\hat{j} + 100\hat{k})$ N has components of 200 N in the x-direction, -300 N in the y-direction, and 100 N in the z-direction, with a magnitude of $\sqrt{200^2 + 300^2 + 100^2} \approx 374$ N.

Newton's First Law and Equilibrium Conditions

Newton's first law says an object remains at rest or in uniform motion unless acted on by an unbalanced force. For a particle in equilibrium, the net force is zero, which gives us the scalar equilibrium equations:

$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

$$\Sigma F_z = 0$$

Each equation independently requires that the force components in that direction sum to zero. In 2-D problems you get two independent equations; in 3-D you get three. This means:

- In 2-D, you can solve for at most two unknowns.
- In 3-D, you can solve for at most three unknowns.

If there are more unknowns than independent equations, the problem is statically indeterminate and can't be solved with equilibrium alone.

Solving Equilibrium Problems

Problem-Solving Strategy

Follow these steps for every particle equilibrium problem:

1. Draw a free body diagram. Isolate the particle, then draw and label every force acting on it. Include known magnitudes and angles.
2. Set up a coordinate system. Choose axes that simplify the math. Aligning one axis along a surface or along an unknown force can reduce the number of components you need to resolve.
3. Resolve each force into components. Write out F_x and F_y (and F_z in 3-D) for every force.
4. Apply the equilibrium equations. Set $\Sigma F_x = 0$, $\Sigma F_y = 0$ (and $\Sigma F_z = 0$).
5. Solve the resulting system of equations for the unknowns.
6. Check your answer. Substitute back into the original equations. Make sure signs are consistent and magnitudes are physically reasonable (e.g., tension should be positive, meaning the rope pulls rather than pushes).

Solving Connected Particle Systems

When multiple particles are linked by ropes, cables, or rods, each particle gets its own FBD. The key connection between them is Newton's third law: the tension a rope exerts on particle A is equal in magnitude and opposite in direction to the tension it exerts on particle B.

Steps for connected systems:

1. Draw a separate FBD for each particle (or each joint/connection point).
2. Label internal forces (like cable tensions) consistently across diagrams. If cable T_1 pulls particle A to the right, it pulls particle B to the left with the same magnitude.
3. Write equilibrium equations for each particle.
4. Solve the combined system of equations simultaneously.

These problems have more equations and more unknowns, but the strategy is the same. Start with the particle that has the fewest unknowns, solve what you can, and work outward from there.

3.2 Free-body diagrams and their importance

Free-body diagrams (FBDs) are the single most important skill you'll use in statics. They take a real-world object and strip it down to just the forces acting on it, giving you a clean picture you can translate directly into equilibrium equations. Get the FBD wrong, and every calculation that follows will be wrong too.

This section covers how to draw FBDs for both particles and rigid bodies, how to identify and classify the forces that belong on them, and how to connect them to the equilibrium equations you'll solve.

Free-body diagrams for particles and rigid bodies

Graphical representations of isolated objects

A free-body diagram isolates a single object (either a particle or a rigid body) from its surroundings and shows every external force acting on it. The word "free" means you've mentally cut the object free from all supports, cables, and contact surfaces. What's left is just the object and arrows representing forces.

FBDs serve two purposes:

- They let you set up equilibrium equations by making every force visible and organized.
- They act as a visual check to confirm you haven't missed a force or double-counted one before you start solving.

Replacing supports and connections with reaction forces

The key step in drawing an FBD is replacing every support, connection, or contact with the reaction force(s) it provides. You remove the physical constraint and substitute the force it would exert.

- For particles, all forces are drawn as vectors originating from a single point, since a particle has no physical size.
- For rigid bodies, forces act at specific points on the body. Each vector originates from its actual point of application, because where a force acts affects the moments it creates.
- Moments on rigid bodies are shown as curved arrows (clockwise or counterclockwise) at the point where they act.

All forces are drawn as vectors with arrows showing direction. Label each one with a symbol (known magnitude or an unknown variable) so you can reference it in your equations.

Identifying forces on a body

Classifying forces as applied or reaction forces

Every force on an FBD falls into one of two categories:

- Applied forces are forces imposed on the body by the environment or other objects. Examples: the body's own weight ($W = mg$), wind loads, tension from an attached cable, or a person pushing on the object.
- Reaction forces are forces that supports and constraints exert in response to the applied forces. They prevent the body from moving. Examples: the normal force from a surface, the tension in a rope that holds something in place, or the horizontal and vertical forces at a pin joint.

The distinction matters because applied forces are usually known (or given), while reaction forces are usually the unknowns you're solving for.

Common types of supports and their reaction forces

Different supports constrain motion in different ways, so they produce different reaction forces:

Support Type	Reactions Provided	What It Allows
Roller	One force perpendicular to the surface	Free to slide along the surface and rotate
Pin (hinge)	Force in both horizontal and vertical directions (R_x and R_y)	Free to rotate, but cannot translate
Fixed (cantilever)	Horizontal force, vertical force, and a moment (R_x , R_y , M)	No movement or rotation at all

A few additional points:

- Friction forces should appear on the FBD whenever the object contacts a rough surface and sliding (or the tendency to slide) is relevant. For example, a block on an incline has both a normal force perpendicular to the surface and a friction force parallel to it.
- Internal forces (forces between parts within the isolated system) are never shown on an FBD. By Newton's Third Law, they come in equal-and-opposite pairs that cancel out. You only show them if you cut the system apart and isolate one piece.

Free-body diagrams for equilibrium problems

Equilibrium equations derived from Newton's First Law

Newton's First Law tells you that a body in equilibrium has zero net force and zero net moment. Translated into equations:

For a particle (no size, so moments don't apply):

- $\sum F_x = 0$
- $\sum F_y = 0$
- $\sum F_z = 0$

That gives you up to 3 independent equations (or 2 in a 2D problem).

For a rigid body (size matters, so moments come into play):

- $\sum F_x = 0$
- $\sum F_y = 0$
- $\sum F_z = 0$
- $\sum M_A = 0$ (moments about any chosen point A)

In 3D, you can write up to 6 independent equations (3 force, 3 moment). In 2D, you get 3 ($\sum F_x = 0$, $\sum F_y = 0$, and $\sum M_A = 0$).

Solving for unknown forces using equilibrium equations

Here's the general process for using an FBD to solve an equilibrium problem:

1. Isolate the body and draw it separated from all supports and connections.
2. Draw all external forces and moments, replacing each support with its appropriate reaction forces (use the table above). Include the body's weight acting at its center of gravity.
3. Label everything. Known forces get numerical values; unknown forces get variable names (A_x , B_y , etc.) with assumed directions.
4. Choose a coordinate system and, for rigid bodies, choose a convenient point for summing moments (picking a point where an unknown force acts eliminates that unknown from the moment equation).
5. Write the equilibrium equations and solve the system for the unknowns.
6. Check your results. If an unknown comes out negative, the actual direction is opposite to what you assumed. That's fine; just note the correct direction.

One critical constraint: you can only solve for as many unknowns as you have independent equilibrium equations. If you have more unknowns than equations, the problem is statically indeterminate, meaning statics alone can't solve it. You'd need additional relationships, such as material deformation equations from strength of materials.

3.3 Equilibrium of rigid bodies in two and three dimensions

Equilibrium of rigid bodies is all about balance. In 2D, we need zero forces in x and y directions, plus zero total moment. In 3D, it's the same, but we add a z-direction and moments around all three axes.

Free-body diagrams are key to solving these problems. They show all forces and moments acting on an isolated body. Whether forces meet at one point or not, we use equilibrium equations to find unknown forces and keep things steady.

Equilibrium of Rigid Bodies

Conditions for Equilibrium in Two and Three Dimensions

- Equilibrium of a rigid body occurs when the sum of all forces acting on the body is zero and the sum of all moments about any point is zero
- In two dimensions, the conditions for equilibrium are:
 - $\sum F_x = 0$ (sum of forces in the x-direction equals zero)

- $\Sigma F_y = 0$ (sum of forces in the y-direction equals zero)
- $\Sigma M = 0$ (sum of moments about any point equals zero)
- In three dimensions, the conditions for equilibrium are:
 - $\Sigma F_x = 0$ (sum of forces in the x-direction equals zero)
 - $\Sigma F_y = 0$ (sum of forces in the y-direction equals zero)
 - $\Sigma F_z = 0$ (sum of forces in the z-direction equals zero)
 - $\Sigma M_x = 0$ (sum of moments about the x-axis equals zero)
 - $\Sigma M_y = 0$ (sum of moments about the y-axis equals zero)
 - $\Sigma M_z = 0$ (sum of moments about the z-axis equals zero)

Static Equilibrium and Free-Body Diagrams

- A rigid body is considered to be in static equilibrium when it is at rest or moving with a constant velocity, and the net force and net moment acting on the body are zero
- The concept of free-body diagrams is essential in solving equilibrium problems
- Involves isolating the body of interest
- Representing all forces and moments acting on the isolated body
- Example: A ladder leaning against a wall can be analyzed using a free-body diagram, showing the forces acting on the ladder (normal force from the wall, normal force from the ground, and the ladder's weight) and the moments they create

Concurrent vs Non-Concurrent Forces

Concurrent Force Systems

- Concurrent force systems are those in which all forces act through a single point
- For concurrent force systems, the equilibrium equations can be solved by considering the forces acting at the point of concurrency
- Example: Multiple cables attached to a single point on a suspended object create a concurrent force system at that attachment point

Non-Concurrent Force Systems

- Non-concurrent force systems have forces that do not pass through a single point
- Non-concurrent force systems require the consideration of both forces and moments to solve for equilibrium
- The moment of a force about a point is the product of the force magnitude and the perpendicular distance from the point to the line of action of the force
- Varignon's theorem states that the moment of a force about a point is equal to the sum of the moments of the force's components about the same point
- Couples are a special case of non-concurrent force systems

- Consist of two equal and opposite forces with a non-zero moment arm
- Produce a pure moment
- Example: A beam supported at both ends with a load applied at its midpoint experiences non-concurrent forces from the supports and the applied load

Equilibrium Applications in Structures

Trusses

- Trusses are structural systems composed of connected members, typically forming triangular units
- Designed to support loads primarily through axial forces (tension or compression) in the members
- The method of joints is used to analyze trusses by considering the equilibrium of each joint, assuming pin connections and no moments at the joints
- The method of sections is an alternative approach to analyzing trusses
- Involves the imaginary cut of the truss into two sections
- Considers the equilibrium of one of the sections
- Example: A bridge truss supporting the weight of vehicles and the bridge deck itself can be analyzed using the method of joints or the method of sections

Frames and Machines

- Frames are structures containing multi-force members, which can support loads through axial forces, shear forces, and bending moments
- Machines are devices that transmit or modify forces and can be analyzed using equilibrium principles
- Consider the forces acting on each component and the connections between them
- Example: A crane frame supporting a heavy load can be analyzed by considering the forces acting on each member and the moments they create at the joints

Moment Equilibrium for Rigid Bodies

Moment Equilibrium Principle

- Moment equilibrium is achieved when the sum of all moments acting on a rigid body about any point is zero
- The choice of the moment center (the point about which moments are calculated) is arbitrary, but selecting a point that simplifies the calculations is often advantageous
- When a rigid body is in equilibrium, the net moment about any point must be zero, regardless of the chosen moment center

Couples and Distributed Loads

- The moment of a couple is independent of the choice of the moment center and is always equal to the product of one of the forces and the perpendicular distance between the two forces

- In problems involving distributed loads, the concept of the center of gravity or centroid is used to determine the equivalent point load for calculating moments
- Example: A rectangular plate subjected to a uniformly distributed load can be analyzed by calculating the moment about a chosen point, considering the total load acting at the plate's centroid

Unit 4 – Structural Analysis & Machines

4.1 Analysis of trusses using method of joints and method of sections

Trusses are key structures in engineering, using interconnected members to support loads efficiently. This section explores two powerful analysis techniques: the method of joints and the method of sections. Each approach offers unique advantages for solving truss problems and determining member forces.

Understanding these methods is crucial for structural analysis. The method of joints systematically examines force equilibrium at each connection, while the method of sections cuts through the truss to analyze larger portions. Mastering both techniques equips engineers to tackle various truss configurations effectively.

Method of Joints vs Sections

Differences in Analysis Techniques

- Method of joints is a truss analysis technique that considers the equilibrium of forces at each joint in the truss, while method of sections analyzes the equilibrium of a portion of the truss created by an imaginary cut
- In the method of joints, equilibrium equations are written for each joint, considering forces in the members connected at that joint and any external forces or reactions applied there
- In the method of sections, equilibrium equations are written for forces and moments acting on one section of the truss

Differences in Problem Solving Approaches

- The method of joints solves for unknown forces in members by progressing in a logical order from joint to joint where there are at most two unknown member forces at each
- The method of sections can solve for forces in members that may not be solvable by inspection or the method of joints alone
- The method of joints is systematic and can be applied repeatedly until all member forces are known, while the method of sections is a more direct approach for determining the force in a specific member
- The method of joints requires that all external forces, including reactions, are known in advance, while the method of sections can be applied even if some reaction forces are unknown

Method of Joints Application

Determining External Reactions

- Draw a free body diagram of the entire truss, replacing all supports with the forces and moments that they exert on the truss
- Determine the external reactions at the supports using the equations of equilibrium ($\sum F_x = 0$, $\sum F_y = 0$, and $\sum M = 0$)

Solving for Unknown Member Forces

- Identify a joint with at most two unknown member forces, starting at a joint where most information is known (usually where external forces are applied or at the supports)
- Draw a free body diagram of the joint, representing all known and unknown forces as vectors
- Unknown forces should be assumed to act in a direction away from the joint if the member is in tension and towards the joint if in compression
- Write and solve the equilibrium equations ($\sum F_x = 0$ and $\sum F_y = 0$) for the joint to determine the unknown forces
- If one of the unknown forces cannot be found, move to the next joint and repeat the process until you can solve for that unknown
- Proceed to the next joint where there are at most two unknown forces, repeating the process of drawing the free body diagram, writing equilibrium equations, and solving for the unknown forces
- Continue this process, moving systematically from joint to joint until all member forces have been determined
- It may be necessary to begin at another area of the truss where additional information is known

Verifying Results

- Verify the results by checking that the equilibrium equations are satisfied at each joint
- Ensure that the determined forces do not violate the properties of the truss or its supports (e.g., a member in tension should not have a negative force value)

Method of Sections Application

Selecting the Section Cut

- Identify the member whose force needs to be determined, ensuring that this member intersects the section cut at some point along its length
- Make an imaginary cut through the truss, dividing it into two sections
- The cut should pass through no more than three members whose forces are unknown, including the desired member
- Choose one of the two sections to analyze and indicate the portion of the truss that is being removed
- It is often best to select the section that has more known forces or will result in simpler calculations

Analyzing the Section

- Draw a free body diagram of the section of the truss that remains after the cut, showing all external forces, reactions, and unknown member forces exposed by the cut
- Represent the unknown member forces as vectors, assuming they act in the positive direction (tension) until calculations reveal otherwise
- If the calculated force is negative, the member is in compression
- Write and solve the equilibrium equations ($\sum F_x = 0$, $\sum F_y = 0$, and $\sum M = 0$) for the section of the truss

- If needed, take the moment about a point where the lines of action of the other unknown forces intersect to eliminate them from the equation
- Repeat the process with additional sections and cuts as necessary to determine all required member forces

Verifying Results

- Verify the results by checking that the equilibrium equations are satisfied for all sections
- Ensure that the determined forces are consistent with the properties of the truss and its supports (e.g., a member in compression should not have a positive force value)

Truss Analysis Method Selection

Advantages of the Method of Joints

- The method of joints is most efficient for trusses with a high degree of static indeterminacy, where there are many more members than equilibrium equations available
- It is systematic and can be used to find forces in all members
- The method of joints is preferred when the external reactions can be easily determined and when there are joints with only two unknown member forces
- It is also useful for verifying the results of the method of sections

Advantages of the Method of Sections

- The method of sections is most effective for determining the force in a specific member or a small group of members without solving for all member forces
- It is particularly useful when the force in a member cannot be found by inspection or the method of joints alone
- The method of sections is advantageous when the external reactions are not easily determined, as it can be applied without knowing all reaction forces
- It is also helpful for analyzing compound trusses or trusses with complex geometries (e.g., a Pratt truss or a Howe truss)

Combining Methods

- In some cases, a combination of both methods may be necessary to fully analyze a truss
- The method of joints can be used to determine as many member forces as possible, and the method of sections can be employed to find the remaining unknown forces
- The choice of method may also depend on the specific problem requirements, such as the need to find the force in a particular member (e.g., a critical tension member) or to verify the results using a different approach

4.2 Analysis of frames and machines

Frames and machines are essential components in engineering, each serving unique purposes. Frames provide structural support and stability, while machines transfer energy to perform mechanical tasks. Understanding their differences is crucial for effective design and analysis.

This topic explores the forces and moments acting on frames and machines, both externally and internally. By applying equilibrium equations and analyzing structural integrity, engineers can ensure these systems function safely and efficiently under various loading conditions.

Frames vs Machines

Characteristics and Applications

- Frames are stationary structures composed of interconnected members designed to support loads and maintain equilibrium (bridges, roof trusses, scaffolding)
- Machines are mechanisms consisting of moving parts that transmit and modify forces to perform useful work (engines, gears, pulleys, levers)
- Frames experience only forces and moments at the joints, while machines involve forces, moments, and motions of the components
- The main purpose of frames is to provide structural support and stability, whereas machines are designed to transfer energy and perform mechanical tasks

Differences in Loading and Motion

- Frames are subjected to static or quasi-static loads, meaning the forces and moments acting on them change slowly over time
- The loads on frames are typically due to the weight of the structure itself, as well as any external loads such as snow, wind, or the weight of objects supported by the frame
- Machines experience dynamic loads and motions, with forces and moments that can vary rapidly during operation
- The loads on machines can be due to the inertia of moving parts, friction between components, and the forces required to perform the intended mechanical tasks
- Machines often involve the conversion of energy from one form to another, such as the conversion of electrical energy to mechanical energy in an electric motor

Forces and Moments in Structures

Classification of Forces and Moments

- External forces and moments acting on frames and machines can be classified as applied loads, reactions at supports, or inertial forces due to motion
- Applied loads are forces or moments that are directly applied to the structure or machine, such as the weight of a person standing on a bridge or the torque applied to a gear
- Reactions at supports are forces or moments that act on the structure or machine at the points where it is supported or connected to other structures (foundations, bearings, hinges)

- Inertial forces are forces that arise due to the acceleration or deceleration of the structure or machine, such as the centrifugal force acting on a rotating gear
- Internal forces and moments develop within the members of frames and machines as a result of the external loads and the connectivity of the components
- Internal forces and moments are necessary to maintain the equilibrium of the structure or machine and ensure that it does not collapse or fail under the applied loads

Types of Internal Forces and Moments

- Types of internal forces include axial forces (tension or compression), shear forces, and bending moments
- Axial forces act along the length of a member and can be either tensile (pulling) or compressive (pushing)
- Shear forces act perpendicular to the length of a member and tend to cause adjacent parts of the member to slide past each other
- Bending moments cause a member to bend or flex and are often the result of loads that are applied perpendicular to the length of the member
- The distribution and magnitude of internal forces and moments depend on the geometry, material properties, and loading conditions of the frame or machine
- The geometry of the structure or machine, including the length, cross-sectional shape, and connectivity of the members, affects how the loads are distributed and how the internal forces and moments develop
- The material properties, such as the modulus of elasticity and the yield strength, determine how much the members will deform under the applied loads and how much load they can withstand before failing
- Free-body diagrams are used to represent the external and internal forces and moments acting on individual members or components of frames and machines
- A free-body diagram is a simplified representation of a member or component that shows all the forces and moments acting on it, as well as any reactions or constraints at the supports or connections
- Free-body diagrams are an essential tool for analyzing the forces and moments in structures and machines, as they allow the equilibrium equations to be applied to each member or component individually

Equilibrium Analysis of Structures

Equilibrium Equations

- The equilibrium equations for a two-dimensional system are: $\sum F_x = 0$, $\sum F_y = 0$, and $\sum M = 0$, where $\sum F_x$ and $\sum F_y$ represent the sum of forces in the x and y directions, respectively, and $\sum M$ represents the sum of moments about a chosen point
- These equations state that for a structure or machine to be in equilibrium, the sum of all forces and moments acting on it must be zero
- The x and y directions are usually chosen to be perpendicular to each other and to align with the main axes of the structure or machine

- For three-dimensional systems, an additional equilibrium equation is introduced: $\Sigma F_z = 0$, representing the sum of forces in the z-direction
- This equation is necessary when the structure or machine has forces or moments acting in all three spatial dimensions
- The z-direction is usually chosen to be perpendicular to both the x and y directions

Solving for Unknown Forces and Moments

- To solve for unknown forces and moments, apply the equilibrium equations to each member or component of the frame or machine, considering the external and internal forces and moments acting on it
- This process involves creating a free-body diagram for each member or component, identifying the known and unknown forces and moments, and then using the equilibrium equations to set up a system of linear equations
- The resulting system of equations can then be solved using algebraic methods or matrix operations to determine the unknown forces and moments
- When applying equilibrium equations, it is essential to establish a consistent sign convention for forces and moments (positive for counterclockwise moments and tensile forces)
- A sign convention is necessary to ensure that the forces and moments are being added or subtracted correctly in the equilibrium equations
- The most common sign convention is to consider counterclockwise moments and tensile forces as positive, and clockwise moments and compressive forces as negative
- The number of equilibrium equations available must be equal to or greater than the number of unknown forces and moments to obtain a determinate solution
- A determinate solution is one in which all the unknown forces and moments can be uniquely determined from the given information and the equilibrium equations
- If there are more unknowns than equilibrium equations, the system is said to be statically indeterminate, and additional information or constraints are needed to solve for the unknowns

Statically Indeterminate Systems

- In cases where the system is statically indeterminate (i.e., there are more unknowns than equilibrium equations), additional compatibility equations or force-displacement relationships may be required to solve for the unknowns
- Compatibility equations are relationships between the deformations of the members or components that must be satisfied for the structure or machine to remain connected and continuous
- Force-displacement relationships, such as Hooke's law for linear elastic materials, relate the deformations of the members or components to the forces and moments acting on them
- These additional equations, along with the equilibrium equations, form a complete system that can be solved for the unknown forces and moments in a statically indeterminate system
- Advanced methods, such as the force method or the displacement method, are often used to solve statically indeterminate systems, as they can handle the additional complexity and coupling between the equations

Structural Integrity and Functionality

Assessing Structural Integrity

- Compare the calculated internal forces and moments in each member or component to their respective strength limits or allowable stresses to ensure structural integrity
- The strength limits or allowable stresses are determined by the material properties and the cross-sectional geometry of the member or component
- If the calculated forces and moments exceed the strength limits or allowable stresses, the member or component may fail or deform excessively, compromising the structural integrity of the frame or machine
- Identify critical members or components that experience the highest stresses or are most likely to fail under the given loading conditions
- Critical members or components are those that have the highest ratio of actual stress to allowable stress, or those that are subjected to the most severe loading conditions
- These members or components may require additional reinforcement, redesign, or more frequent inspection and maintenance to ensure the overall safety and reliability of the structure or machine

Evaluating Stability and Functionality

- Evaluate the overall stability of the frame or machine by examining the reactions at the supports and ensuring that they are within acceptable limits
- The reactions at the supports represent the forces and moments that are required to keep the structure or machine in equilibrium and prevent it from collapsing or tipping over
- If the reactions exceed the capacity of the supports or foundations, or if they result in excessive deformations or settlements, the stability of the structure or machine may be compromised
- Assess the functionality of the frame or machine by verifying that the calculated forces, moments, and motions are consistent with the intended purpose and operating conditions
- The functionality of a structure or machine refers to its ability to perform its intended task safely, efficiently, and reliably over its expected service life
- This assessment involves comparing the calculated behavior of the structure or machine to its design specifications, performance requirements, and industry standards
- Factors such as deflections, vibrations, fatigue life, and energy consumption may be considered in evaluating the functionality of a structure or machine

Design Considerations and Modifications

- Consider the effects of dynamic loads, fatigue, and environmental factors on the long-term performance and durability of the frame or machine
- Dynamic loads, such as wind gusts, seismic events, or machine vibrations, can cause additional stresses and deformations that may not be captured by static analysis alone
- Fatigue is the progressive damage and failure of a material subjected to repeated loading and unloading cycles, even if the loads are below the static strength limit

- Environmental factors, such as temperature variations, humidity, corrosion, and ultraviolet radiation, can degrade the materials and components of a structure or machine over time
- Propose design modifications or reinforcements to improve the structural integrity, efficiency, or functionality of the frame or machine based on the analysis results
- Design modifications may involve changing the geometry, materials, or connectivity of the members or components to optimize their performance under the given loading and operating conditions
- Reinforcements, such as additional bracing, stiffeners, or redundant load paths, can be added to improve the strength, stability, or resilience of the structure or machine
- Other modifications, such as lubrication, balancing, or vibration damping, may be proposed to enhance the efficiency, precision, or durability of the machine
- These modifications should be based on a thorough understanding of the analysis results, as well as consideration of the economic, environmental, and social factors involved in the design and operation of the structure or machine

4.3 Principle of virtual work and its applications

The principle of virtual work is a powerful tool in structural analysis, allowing engineers to calculate forces and displacements in complex systems. It's based on the idea that work done by forces during tiny, imaginary movements equals zero when a structure is in equilibrium.

This method simplifies calculations for statically indeterminate structures, where traditional equilibrium equations fall short. It's especially useful for trusses, beams, and frames, providing a unified approach to various structural analysis techniques.

Virtual Work in Structural Analysis

Concept and Significance

- Virtual work is the work done by a force acting through a virtual (infinitesimal) displacement that is compatible with the constraints of the system
- The principle of virtual work states that a system is in equilibrium if and only if the total virtual work done by all forces acting on the system is zero for any virtual displacement
- Virtual work is a powerful tool for analyzing structures because it allows for the calculation of displacements and internal forces without the need to solve the equilibrium equations directly
- The principle of virtual work is based on the conservation of energy and is applicable to both linear and nonlinear systems
- Virtual work can be used to derive important structural analysis methods, such as the flexibility method (force method) and the stiffness method (displacement method)

Applications and Advantages

- Virtual work principles can be applied to a wide range of structural systems, including trusses, beams, frames, and cables
- The virtual work method is particularly useful for analyzing statically indeterminate structures, where the number of equilibrium equations is insufficient to determine all the unknown forces and displacements

- By selecting appropriate virtual displacement fields, the virtual work method can simplify the calculation of displacements and internal forces in complex structures
- The virtual work approach provides a unified framework for deriving various structural analysis techniques, such as the moment-area method, the conjugate beam method, and the force method
- Virtual work principles can be extended to dynamic analysis and nonlinear problems, making it a versatile tool in structural mechanics

Virtual Work for Displacements

Calculating Displacements and Rotations

- The virtual work equation can be used to calculate the displacements and rotations of a structure by equating the external virtual work to the internal virtual work
- The external virtual work is the work done by the applied loads through virtual displacements, while the internal virtual work is the work done by the internal forces through virtual deformations
- To calculate displacements using the principle of virtual work, a virtual force (or moment) is applied at the point and in the direction of the desired displacement, and the virtual work equation is solved for the unknown displacement
- The choice of the virtual force system affects the complexity of the calculations, and it is often advantageous to select a virtual force system that simplifies the computation of virtual work

Multiple Degree-of-Freedom Systems

- The principle of virtual work can be extended to calculate the displacements and rotations of structures with multiple degrees of freedom by applying multiple virtual force systems and solving a system of equations
- In a multiple degree-of-freedom system, each independent displacement or rotation requires a separate virtual force system and a corresponding virtual work equation
- The resulting system of equations can be solved simultaneously to determine all the unknown displacements and rotations
- The virtual work method for multiple degree-of-freedom systems forms the basis for the stiffness method, where the structure's behavior is described by a stiffness matrix relating the applied forces to the unknown displacements

Virtual Work for Forces and Stresses

External Forces

- The virtual work done by external forces is calculated by multiplying the applied force by the virtual displacement in the direction of the force
- For concentrated loads (point loads), the virtual work is the product of the force magnitude and the virtual displacement at the point of application
- For distributed loads (line loads, surface loads), the virtual work is computed by integrating the product of the load intensity and the virtual displacement over the length or area of the member

- In the case of moments, the virtual work is the product of the moment and the virtual rotation at the point of application

Internal Stresses

- The virtual work done by internal stresses is calculated by integrating the product of the stress and the virtual strain over the volume of the member
- For axial forces (normal forces), the virtual work is the product of the axial force and the virtual axial deformation, integrated along the length of the member
- For bending moments, the virtual work is the product of the bending moment and the virtual curvature, integrated along the length of the member
- The virtual work done by shear forces is generally neglected in the analysis of slender members, as it is typically much smaller than the virtual work done by bending moments and axial forces

Virtual Work for Indeterminate Structures

Statically Indeterminate Structures

- Statically indeterminate structures have more unknown reaction forces and/or internal forces than available equilibrium equations, requiring additional compatibility equations to solve
- The principle of virtual work can be used to derive the compatibility equations needed to analyze statically indeterminate structures
- Compatibility equations ensure that the deformations of the structure are consistent with the constraints and continuity conditions of the system
- Examples of statically indeterminate structures include continuous beams, frames with fixed supports, and trusses with redundant members

Flexibility Method and Stiffness Method

- The flexibility method (force method), based on the principle of virtual work, involves releasing the redundant forces, calculating the displacements due to the applied loads and redundant forces, and enforcing compatibility to solve for the redundant forces
- The flexibility method is particularly suitable for structures with few redundant forces, as it requires solving a smaller system of equations compared to the stiffness method
- The stiffness method (displacement method), also based on the principle of virtual work, involves expressing the member forces in terms of unknown displacements, applying the virtual work equation to derive the stiffness matrix, and solving for the unknown displacements
- The stiffness method is more appropriate for structures with many degrees of freedom, as it leads to a larger system of equations but with a symmetric and banded stiffness matrix that can be efficiently solved using computational methods

Unit 5 – Distributed Forces: Centroids & Centers

5.1 Centroids of lines, areas, and volumes

Centroids are the geometric centers of shapes and objects. In statics and strength of materials, knowing where the centroid is located lets you predict how a structure will respond to loads, where bending stresses concentrate, and whether a design will be stable. This section covers how to find centroids of lines, areas, and volumes using integration, how symmetry shortcuts the process, and how to handle composite shapes.

Centroids via Integration

Centroid Definition and Formulas

The centroid is the average position of all points within a shape or object. Think of it as the "balance point" of pure geometry (no weight or density involved yet).

The general idea is the same across lines, areas, and volumes: you take a weighted average of position, where the "weight" is the differential element (dL , dA , or dV).

For a line (1D):

$$\bar{x} = \frac{\int x dL}{L}$$

where L is the total arc length. If the line curves through 2D or 3D space, you'll also need $\bar{y}$ (and possibly $\bar{z}$) using the same pattern.

For an area (2D):

$$\bar{x} = \frac{\int x dA}{A}, \quad \bar{y} = \frac{\int y dA}{A}$$

where A is the total area.

For a volume (3D):

$$\bar{x} = \frac{\int x dV}{V}, \quad \bar{y} = \frac{\int y dV}{V}, \quad \bar{z} = \frac{\int z dV}{V}$$

where V is the total volume.

Integration Techniques and Considerations

Setting up the integral correctly is usually the hardest part. Here's a practical approach:

1. Sketch the shape and choose a coordinate system. Place axes to take advantage of any symmetry.
2. Select your differential element. For areas, decide whether to use a vertical strip ($dA = y dx$), a horizontal strip ($dA = x dy$), or a thin ring/disk for shapes with radial symmetry.
3. Identify the integration limits from the geometry of the shape (where the boundary curves intersect, or where the shape starts and ends along your chosen axis).

4. Write $\tilde{x}$ and $\tilde{y}$ for the element. This is the centroid location of the differential element itself. For a vertical strip of height y , the element's centroid is at $\tilde{y} = y/2$, not at y . This is a common mistake.
5. Evaluate the integrals. Use substitution, integration by parts, or other techniques as needed.

For volumes with complex geometry, you may need double or triple integrals. However, many textbook problems can be reduced to single integrals by choosing a smart differential element (thin disks, shells, etc.).

Symmetry in Centroid Calculations

Applying Symmetry to Simplify Calculations

Symmetry is the single biggest time-saver in centroid problems. The core rule is straightforward: if a shape is symmetric about an axis, the centroid lies on that axis.

- One axis of symmetry: The centroid coordinate perpendicular to that axis equals the axis value. For example, if a shape is symmetric about the y -axis, then $\bar{x} = 0$ (assuming the y -axis is at $x = 0$). You only need to integrate for $\bar{y}$.
- Two axes of symmetry: The centroid sits at the intersection of those axes. No integration needed at all for a shape like a rectangle or ellipse centered at the origin.
- Rotational symmetry: Shapes like circles, spheres, and regular polygons have their centroid at the center of rotation.

Recognizing symmetry before you start integrating can cut your work in half (or eliminate it entirely).

Examples of Symmetry in Centroid Calculations

- A rectangle centered at the origin is symmetric about both axes, so its centroid is at the intersection of its diagonals. No integration required.
- A semicircle of radius r with its diameter along the x -axis is symmetric about the y -axis, so $\bar{x} = 0$. You still need to integrate for $\bar{y}$, which works out to $\bar{y} = \frac{4r}{3\pi}$.
- An equilateral triangle with a vertical line of symmetry has $\bar{x} = 0$. The centroid sits at one-third of the height measured from the base.
- A sphere is symmetric about all three coordinate axes, so its centroid is at the geometric center.

Centroid Significance in Engineering

Centroid as the Balance Point

For a body with uniform density, the centroid coincides with the center of gravity. That means it's the point where you could support the object and it would balance perfectly without tilting. This connection between geometry and physical behavior is why centroids show up constantly in engineering analysis.

Centroid Applications in Different Engineering Fields

- Structural engineering: The centroid of a beam's cross-section is the reference point for calculating the moment of inertia and bending stress distribution. An I-beam's cross-sectional centroid determines its neutral axis under bending.

- Fluid mechanics: The centroid of a submerged surface determines where the resultant hydrostatic pressure force acts. For submerged objects, it's also tied to where buoyant forces are applied.
- Mechanical engineering: Locating the centroid of rotating components (gears, flywheels) is essential for minimizing vibrations and ensuring proper balance.
- Aerospace engineering: The centroid of wing cross-sections and fuselage sections directly affects aircraft stability and control characteristics.

Centroids for Composite Shapes

Calculating Centroids of Composite Shapes and Solids

Most real engineering cross-sections aren't simple rectangles or circles. They're composite shapes built from several simpler pieces. Instead of integrating over the entire complex shape, you use a weighted-average approach.

The method works in clear steps:

1. Decompose the composite shape into simple parts whose individual centroids you already know (rectangles, triangles, semicircles, etc.).
2. Tabulate the area (or volume) and centroid coordinates of each part. A table with columns for A_i , $\bar{x}_i$, $\bar{y}_i$, $A_i\bar{x}_i$, and $A_i\bar{y}_i$ keeps things organized.
3. Apply the weighted average formulas:

For composite areas:

$$\bar{x} = \frac{\sum A_i \bar{x}_i}{\sum A_i}, \quad \bar{y} = \frac{\sum A_i \bar{y}_i}{\sum A_i}$$

For composite volumes:

$$\bar{x} = \frac{\sum V_i \bar{x}_i}{\sum V_i}, \quad \bar{y} = \frac{\sum V_i \bar{y}_i}{\sum V_i}, \quad \bar{z} = \frac{\sum V_i \bar{z}_i}{\sum V_i}$$

Combining and Subtracting Shapes or Solids

Holes and cutouts are handled by treating the removed material as a component with negative area or volume. This is one of the most useful tricks in composite centroid problems.

For example, to find the centroid of a plate with a circular hole:

1. Treat the full plate (no hole) as component 1 with positive area A_1 .
2. Treat the hole as component 2 with negative area $-A_2$.
3. Plug both into the weighted average formula. The negative area automatically shifts the centroid away from the hole.

Common composite shapes you'll encounter include:

- I-beams: three rectangles (two flanges + one web)
- T-sections: two rectangles
- Channel sections: rectangles with subtracted regions
- Complex cross-sections: combinations of standard shapes with cutouts

The key to getting these right is being careful with your coordinate system. Pick a single reference origin and measure every component's centroid from that same point. Mixing up reference points is the most common source of errors in composite centroid problems.

5.2 Center of gravity and mass moment of inertia

Center of gravity and mass moment of inertia are crucial concepts in understanding how objects behave under forces. They help us figure out where an object's weight acts and how it resists rotational motion.

These ideas are key in designing stable structures and efficient machines. By grasping these concepts, we can predict how objects will balance, rotate, and respond to applied forces in real-world situations.

Centroids vs Centers of Gravity

Geometric Center and Mass Distribution

- Centroids are the geometric center of a shape or object, calculated solely based on the geometry of the object
- It is the point where all the area or volume could be concentrated and still have the same first moment of area or volume about any axis
- Centers of gravity are the point where an object's total weight can be considered to act
- It factors in both the geometry and the density distribution of the object

Uniform vs Non-Uniform Density

- For objects with uniform density, the centroid and center of gravity coincide (solid cylinder)
- For objects with non-uniform density, the center of gravity depends on the distribution of mass and may not be at the geometric center (car engine)
- The center of gravity is important for understanding the stability and balance of objects (leaning tower of Pisa)
- It also determines their behavior when subjected to forces and moments (tipping of a crane)

Center of Gravity Calculation

Dividing Objects into Smaller Parts

- To calculate the center of gravity for objects with non-uniform density, the object must be divided into smaller parts or elements with known or approximated densities (sliced bread loaf)
- The center of gravity for each individual part is determined based on its geometry and density (individual slices of bread)
- The overall center of gravity is then calculated using a weighted average of the individual centers of gravity, with the weights being the masses of each part

Integration Techniques for Continuous Objects

- For continuous objects with varying density, integration techniques may be necessary to determine the center of gravity (graduated cylinder filled with liquid)

- The formula for calculating the center of gravity in a particular direction (x, y, or z) is:

$$X_{cg} = (\int x dm) / (\int dm)$$
, where dm is the differential mass element and the integrals are taken over the entire object
- This formula represents the weighted average of the position coordinates, with the mass acting as the weighting factor (density variations in a metal rod)

Mass Moment of Inertia

Resistance to Rotational Acceleration

- Mass moment of inertia is a measure of an object's resistance to rotational acceleration about a specific axis
- It depends on the distribution of mass within the object relative to the axis of rotation (figure skater spinning with arms extended vs tucked in)
- The mass moment of inertia is analogous to mass in translational motion, as it relates the applied torque to the resulting angular acceleration through the equation: $T = I * \alpha$, where T is the torque, I is the mass moment of inertia, and α is the angular acceleration
- Objects with a larger mass moment of inertia require more torque to achieve the same angular acceleration compared to objects with a smaller mass moment of inertia (rotating a long rod vs a short rod)

Importance in Rotational Dynamics

- The mass moment of inertia is important in understanding the rotational dynamics of objects (spinning top)
- It plays a crucial role in the design of flywheels, gears, and other rotating machinery (car engine components)
- The parallel axis theorem allows for the calculation of the mass moment of inertia about any parallel axis, given the mass moment of inertia about an axis passing through the center of gravity and the perpendicular distance between the two axes (rotating a door about its hinges)

Mass Moment of Inertia for Shapes

Formulas for Simple Geometric Shapes

- The mass moment of inertia for various shapes and solids can be determined using predefined formulas or by integration techniques
- For simple geometric shapes, such as rectangular plates, circular disks, and solid cylinders, the mass moment of inertia can be calculated using the following formulas:
 - Rectangular plate (about an axis perpendicular to the plate and passing through its center):

$$I = (1/12) * m * (a^2 + b^2)$$
, where m is the mass, a is the length, and b is the width
 - Circular disk (about an axis perpendicular to the disk and passing through its center):

$$I = (1/2) * m * r^2$$
, where m is the mass and r is the radius
 - Solid cylinder (about its central axis): $I = (1/2) * m * r^2$, where m is the mass and r is the radius

Complex Shapes and Integration Techniques

- For more complex shapes or composite objects, the mass moment of inertia can be determined by dividing the object into simpler parts, calculating the mass moment of inertia for each part, and then using the parallel axis theorem to find the total mass moment of inertia about the desired axis (I-beam)
- In some cases, integration techniques may be necessary to calculate the mass moment of inertia, especially for objects with non-uniform density or irregular shapes (airplane wing)
- These techniques involve setting up and evaluating appropriate integrals based on the object's geometry and density distribution (3D-printed object with varying infill density)

5.3 Composite bodies and Pappus-Guldinus theorems

Centroids and Centers of Gravity

Composite Bodies and Their Centroids

A composite body is a complex shape built from simpler shapes you already know: rectangles, triangles, semicircles, etc. Instead of integrating over the entire complex shape, you find the centroid by treating each simple piece separately and then combining the results with a weighted average.

The centroid formulas for a composite body are:

$$\bar{x} = \frac{\sum A_i \bar{x}_i}{\sum A_i}$$

$$\bar{y} = \frac{\sum A_i \bar{y}_i}{\sum A_i}$$

where A_i is the area of each component shape, and $\bar{x}_i, \bar{y}_i$ are the centroid coordinates of that component (values you can look up in standard tables for common shapes).

A few things to keep in mind:

- Every component must be referenced to the same coordinate system. Pick an origin (usually a corner or axis of symmetry) and stick with it.
- Holes and cutouts count as negative areas. If your shape is a rectangle with a circular hole, include the circle with a negative A_i in your summation.
- The centroid of each simple shape is measured from your chosen origin, not from the shape's own corner.

Centers of Gravity in Composite Bodies

For objects with uniform density, the center of gravity is the same point as the centroid. The center of gravity is where the body would balance if you could support it at a single point.

If density varies between components, you'd replace area A_i with weight W_i in the formulas:

$$\bar{x} = \frac{\sum W_i \bar{x}_i}{\sum W_i}$$

For 3D composite bodies, you use volumes V_i instead of areas (or weights, if density isn't uniform), and you add a $\bar{z}$ equation.

Step-by-step process for composite centroid/center of gravity problems:

1. Sketch the composite body and break it into simple shapes (including any holes as negative shapes).
2. Set up a coordinate system with a clearly defined origin.
3. For each component, record its area A_i and the coordinates of its centroid $(\bar{x}_i, \bar{y}_i)$ relative to your origin.
4. Organize everything in a table. Columns for shape, A_i , $\bar{x}_i$, $\bar{y}_i$, $A_i\bar{x}_i$, and $A_i\bar{y}_i$ make the arithmetic much cleaner.
5. Sum the columns and apply the centroid formulas.
6. Sanity check: the result should fall within the physical boundary of the shape. If it doesn't, look for a sign error or a misplaced centroid coordinate.

Pappus-Guldinus Theorem for Surface Area

Pappus-Guldinus First Theorem

This theorem gives you the surface area of a solid of revolution without integration. The idea: when a plane curve revolves around an external axis, the surface it sweeps out has an area equal to the curve's arc length multiplied by the distance its centroid travels.

$$SA = 2\pi\bar{r} \cdot L$$

- L = arc length of the generating curve
- $\bar{r}$ = perpendicular distance from the centroid of that curve to the axis of revolution
- $2\pi\bar{r}$ = the circumferential distance the centroid travels in one full revolution

The generating curve must not cross the axis of revolution (otherwise the theorem doesn't apply directly).

Applying the First Theorem

1. Identify the generating curve (the line or curve that sweeps out the surface) and the axis of revolution.
2. Calculate the arc length L of the generating curve. For straight segments, this is just the length. For curves, you may need the arc length formula or a table value.
3. Find $\bar{r}$, the distance from the centroid of the generating curve to the axis of revolution. For composite curves, use the weighted-average centroid approach with arc lengths as weights.
4. Plug into $SA = 2\pi\bar{r} \cdot L$.

Example: A straight line segment of length 5 m sits 3 m away from (and parallel to) an axis. Revolving it produces a cylindrical surface with area $SA = 2\pi(3)(5) = 30\pi \approx 94.2 \text{ m}^2$. You can verify this matches the standard cylinder lateral area formula $2\pi rh$.

Pappus-Guldinus Theorem for Volume

Pappus-Guldinus Second Theorem

This theorem gives you the volume of a solid of revolution. When a plane region revolves around an external axis, the volume it generates equals the region's area multiplied by the distance its centroid travels.

$$V = 2\pi\bar{r} \cdot A$$

- A = area of the generating plane region
- $\bar{r}$ = perpendicular distance from the centroid of that region to the axis of revolution

Again, the generating region must not cross the axis.

Applying the Second Theorem

1. Identify the generating plane region (the 2D shape being revolved) and the axis of revolution.
2. Calculate the area A of the generating region.
3. Find $\bar{r}$, the distance from the region's centroid to the axis of revolution.
4. Plug into $V = 2\pi\bar{r} \cdot A$.

Example: A semicircle of radius $R = 5$ m revolves around its diameter. The area of the semicircle is $A = \frac{1}{2}\pi(5)^2 = \frac{25\pi}{2}$ m². The centroid of a semicircle sits at $\bar{r} = \frac{4R}{3\pi} = \frac{4(5)}{3\pi} = \frac{20}{3\pi}$ m from the diameter. So:

$$V = 2\pi\left(\frac{20}{3\pi}\right)\left(\frac{25\pi}{2}\right) = \frac{500\pi}{3} \approx 523.6 \text{ m}^3$$

This matches the sphere volume formula $\frac{4}{3}\pi R^3 = \frac{4}{3}\pi(5)^3 = \frac{500\pi}{3}$, which confirms the theorem works.

Applying Pappus-Guldinus Theorems to Composite Bodies

When the generating curve or region is itself a composite shape, combine the composite centroid technique with the Pappus-Guldinus theorems.

For surface area with a composite generating curve: find the overall centroid of the curve using arc-length weighting:

$$\bar{r} = \frac{\sum L_i \bar{r}_i}{\sum L_i}$$

Then apply $SA = 2\pi\bar{r} \cdot L_{\text{total}}$.

For volume with a composite generating region: find the overall centroid of the region using area weighting:

$$\bar{r} = \frac{\sum A_i \bar{r}_i}{\sum A_i}$$

Then apply $V = 2\pi\bar{r} \cdot A_{\text{total}}$.

Example: A composite region consists of a rectangle (4 m $\times$ 2 m) with a semicircle (radius 2 m) attached to one of its short sides. This region revolves around the opposite short side of the rectangle (4 m away from the semicircle). To find the volume of the resulting solid:

1. Calculate each component's area and centroid distance from the axis of revolution.
2. Find the composite centroid distance $\bar{r}$ using the area-weighted formula.
3. Apply $V = 2\pi\bar{r} \cdot A_{\text{total}}$.

The key advantage of Pappus-Guldinus is that you never need to set up an integral. As long as you know the area (or arc length) and centroid location of the generating shape, you can find the volume (or surface area) directly. This is especially useful on exams where time is limited.

Unit 6 – Friction

6.1 Laws of dry friction and applications

Types of Friction

Friction as an Opposing Force

Friction is a force that opposes the relative motion between two surfaces in contact. It acts parallel to the contact surfaces and points opposite to the direction of motion or attempted motion. Every friction problem in this course starts by identifying this direction correctly on your free body diagram.

Static, Kinetic, and Rolling Friction

Static friction acts between two surfaces that are at rest relative to each other. It prevents motion from starting, and its magnitude adjusts to match whatever force is trying to cause sliding. This is a critical detail: static friction is not a fixed value. It ranges from zero up to a maximum given by:

$$f_{s, \max} = \mu_s \cdot N$$

where μ_s is the coefficient of static friction and N is the normal force. Until the applied force exceeds $f_{s, \max}$, the object stays put, and the friction force simply equals whatever it needs to be to maintain equilibrium.

Kinetic friction acts between two surfaces that are already sliding relative to each other. Unlike static friction, kinetic friction has a single, constant magnitude for a given pair of surfaces:

$$f_k = \mu_k \cdot N$$

where μ_k is the coefficient of kinetic friction. For nearly all material pairs, $\mu_k < \mu_s$, which is why it takes more force to start sliding something than to keep it sliding.

Rolling friction acts when an object rolls along a surface. It's caused by slight deformation at the contact zone and is typically much smaller than sliding friction. You won't encounter it as often in a statics course, but it explains why wheels are so much more efficient than dragging something.

Coefficient of Friction

The coefficient of friction (μ) is a dimensionless number that characterizes how "grippy" a pair of surfaces is. Some reference values to build intuition:

- Rubber on dry concrete: $\mu_s \approx 0.6 - 0.8$
- Steel on steel (dry): $\mu_s \approx 0.6$
- Steel on ice: $\mu_s \approx 0.03 - 0.05$

These values are determined experimentally. The coefficient depends on the material pair, not on the contact area. A wider block on the same surface has the same μ as a narrow one.

Laws of Dry Friction

Direction and Magnitude of the Friction Force

Coulomb's model of dry friction (the one used throughout this course) rests on a few key principles:

- The friction force always acts tangent to the contact surface, opposing the direction of motion or the tendency toward motion.
- The maximum static friction force is proportional to the normal force: $f_{s, \max} = \mu_s \cdot N$.
- The kinetic friction force is proportional to the normal force: $f_k = \mu_k \cdot N$.
- The friction force is independent of the apparent contact area.

The angle of friction ϕ_s is sometimes useful. It's defined as $\tan(\phi_s) = \mu_s$. Geometrically, ϕ_s is the angle between the resultant contact force (friction + normal) and the normal direction at the point of impending motion. If an applied force's line of action falls within the "cone of friction" defined by ϕ_s , the object won't slide no matter how large the force is.

Determining Sliding Conditions

To figure out whether an object will slide, follow these steps:

1. Draw a complete free body diagram, including weight, normal force, applied forces, and the friction force (direction opposing the tendency of motion).
2. Assume the object is in equilibrium and solve for the required friction force using $\Sigma F_x = 0$ and $\Sigma F_y = 0$.
3. Compare the required friction force to $f_{s, \max} = \mu_s \cdot N$.
4. If the required friction force $\leq f_{s, \max}$, the object stays in equilibrium. The actual friction force is whatever you solved for in step 2.
5. If the required friction force $> f_{s, \max}$, the object slides, and the friction force becomes $f_k = \mu_k \cdot N$.

A common mistake: plugging in $f = \mu_s \cdot N$ right away. Only do this when the problem states "impending motion" or you've confirmed that the object is on the verge of sliding. Otherwise, friction is an unknown you solve for.

Friction's Effect on Motion

Friction's Influence on Velocity and Acceleration

When friction is the only horizontal force on a sliding object, it produces a deceleration. Using Newton's second law:

$$a = \frac{f_k}{m} = \frac{\mu_k \cdot N}{m}$$

For an object on a flat surface where $N = mg$, this simplifies to $a = \mu_k \cdot g$. The deceleration depends only on μ_k and gravity, not on the object's mass.

When a driving force is also present (like a car's engine), friction reduces the net force available for acceleration. The net acceleration becomes:

$$a = \frac{F_{\text{applied}} - f_k}{m}$$

Friction and Direction of Motion

If a force is applied at an angle to a surface, friction still acts along the surface opposing the component of motion (or tendency of motion) parallel to that surface. For a block pushed at an angle θ below the horizontal on a flat surface, the normal force increases because the vertical component of the push adds to the weight:

$$N = mg + F \sin \theta$$

This means the friction force also increases, which is why pushing something at a steep downward angle can make it harder to slide.

Work Done by Friction and Stopping Distance

Friction always does negative work on a sliding object because the force and displacement point in opposite directions. The energy removed by friction is converted to heat.

For an object sliding to a stop on a flat surface from initial speed v_0 , you can find the stopping distance d using the work-energy theorem:

$$\frac{1}{2}mv_0^2 = \mu_k \cdot mg \cdot d$$

Solving for d :

$$d = \frac{v_0^2}{2\mu_k g}$$

Notice that mass cancels out. A heavy box and a light box with the same initial speed on the same surface will slide the same distance.

Friction Force in Equilibrium

Equilibrium Conditions

In statics, you're mostly dealing with objects that aren't moving. For a rigid body in equilibrium, three conditions must hold:

- $\Sigma F_x = 0$
- $\Sigma F_y = 0$
- $\Sigma M_O = 0$ (moments about any point)

Friction is one of the unknown forces you solve for using these equations.

Determining the Friction Force in Equilibrium

Here's the process for a typical equilibrium-with-friction problem:

1. Draw the free body diagram. Include the weight at the center of gravity, normal forces at each contact, friction forces opposing the tendency of motion at each contact, and any applied loads.
2. Choose a coordinate system and a convenient moment point (often at a contact point to eliminate unknowns).
3. Write the three equilibrium equations.
4. Solve for the unknowns, including the friction force(s).

Checking the Validity of the Equilibrium Solution

After solving, you must verify that your answer is physically possible:

1. Check that each friction force satisfies $f \leq \mu_s \cdot N$ at its respective contact.
2. Check that each normal force is compressive ($N \geq 0$). A negative normal force means the surface would need to pull on the object, which a simple contact can't do.
3. If either check fails, the assumed equilibrium configuration is impossible, and the object will move.

Classic example: a ladder against a wall. A uniform ladder of length L and weight W leans against a smooth (frictionless) wall at angle θ with the floor. The floor has friction with coefficient μ_s . Taking moments about the base eliminates the floor reactions and lets you solve for the wall's normal force. Then the horizontal equilibrium equation gives you the required friction at the floor. The ladder stays in place only if that required friction doesn't exceed μ_s times the floor's normal force. If you decrease θ (make the ladder more horizontal), the required friction increases, which is why shallow-angle ladders are more likely to slip.

6.2 Wedges, screws, and belt friction

Wedges and screws are simple machines that use friction to our advantage. They make tasks like splitting wood or lifting heavy objects easier by converting small input forces into larger output forces. It's like having a superpower for moving stuff!

Belt friction is another cool trick engineers use in machines. By wrapping a belt around a pulley, we can transfer power and motion between rotating parts. Understanding how tension and friction work together helps us design better machines and solve real-world problems.

Wedges and screws: Principles and applications

Wedges: Triangular-shaped tools for splitting and lifting

- Wedges are triangular-shaped tools with a sharp edge that can be driven into materials to split them apart or lift heavy objects
- Common applications of wedges include axes, knives, chisels, and nails
- Wedges convert a small input force into a larger output force, enabling easier movement of objects or materials
- The mechanical advantage of wedges is the ratio of the output force to the input force, which is always greater than 1

Screws: Helical inclined planes for linear motion

- Screws are helical inclined planes wrapped around a cylinder, converting rotational motion into linear motion
- Screws are used in various applications, such as jacks, presses, clamps, and threaded fasteners (bolts)
- The pitch of a screw is the distance between adjacent threads, which determines the amount of linear motion per rotation
- Lead is the distance a screw advances in one complete rotation

- The mechanical advantage of screws is the ratio of the output force to the input force, which is always greater than 1

Mechanical advantage of wedges and screws

Calculating the mechanical advantage of wedges

- The mechanical advantage (MA) of a wedge is the ratio of the length of the slope to the width of the wedge: $MA = L/w$, where L is the length of the slope and w is the width
- The ideal mechanical advantage assumes frictionless conditions, while the actual mechanical advantage accounts for friction losses, resulting in a lower value
- The effort force required to overcome a given load can be calculated using the mechanical advantage: $EffortForce = LoadForce/MA$

Calculating the mechanical advantage of screws

- The mechanical advantage of a screw is the ratio of the circumference of the circle traced by the effort force to the pitch of the screw: $MA = 2\pi r/p$, where r is the radius of the circle and p is the pitch
- The ideal mechanical advantage assumes frictionless conditions, while the actual mechanical advantage accounts for friction losses, resulting in a lower value
- The effort force required to overcome a given load can be calculated using the mechanical advantage: $EffortForce = LoadForce/MA$

Forces in belt and pulley systems

Friction force and angle of wrap

- Friction-driven systems, such as belt drives and clutches, rely on the friction force between the belt and the pulley to transmit power
- The friction force depends on the coefficient of friction (μ) between the belt and pulley materials, the angle of wrap (θ) of the belt around the pulley, and the tension in the belt
- The angle of wrap is the angle subtended by the belt on the pulley, measured in radians. A larger angle of wrap increases the friction force and prevents slippage

Belt tension and the capstan equation

- The tight side of the belt (T_1) has higher tension than the slack side (T_2). The difference in tension ($T_1 - T_2$) is the net force that drives the pulley
- The belt tension ratio (T_1 / T_2) is related to the coefficient of friction and angle of wrap by the capstan equation: $T_1/T_2 = e^{\mu\theta}$, where e is the mathematical constant (approximately 2.718)

Belt friction and tension problems

Problem-solving steps

- To solve problems involving belt friction, first identify the given parameters, such as the coefficient of friction, angle of wrap, and either the tight side or slack side tension

- Use the capstan equation ($T_1/T_2 = e^{\mu\theta}$) to calculate the unknown tension, ensuring that the angle of wrap is in radians
- If the power transmitted by the belt drive is given, use the equation $P = (T_1 - T_2) \times v$, where P is the power and v is the belt velocity, to solve for the tension difference or velocity
- When multiple pulleys are involved, analyze each pulley individually, using the tight side tension of the previous pulley as the slack side tension of the next pulley

Considering belt material limitations

- Consider the maximum allowable tension in the belt material to avoid excessive stretching or breakage
- Ensure that the calculated tensions are within the safe limits of the belt material
- Different belt materials (rubber, leather, fabric) have varying coefficients of friction and tensile strengths, which should be taken into account when designing belt drive systems

6.3 Rolling resistance

Rolling Resistance

Rolling resistance is the force that opposes the motion of a wheel or cylinder as it rolls across a surface. Understanding it matters in statics and strength of materials because it directly affects force equilibrium in systems with rolling contact, and it shows up in problems involving wheels, bearings, and conveyor systems.

Definition and Causes

When a rigid or semi-rigid object rolls on a surface, neither the object nor the surface is perfectly rigid. The contact zone deforms slightly, creating a small indentation ahead of the rolling object. This deformation shifts the normal force's line of action forward of the center of contact, producing an asymmetric pressure distribution. That asymmetry is what generates a net force opposing the direction of rolling.

Several factors contribute to the magnitude of rolling resistance:

- Deformation of materials at the contact zone (this is the dominant factor)
- Surface roughness of both the rolling object and the surface
- Material properties like elasticity and hardness (softer materials deform more, increasing resistance)
- Contaminants or lubricants on the contact surface

The key distinction from sliding friction: rolling resistance is typically much smaller. This is exactly why wheels are far more efficient than sleds for moving loads across a surface.

The Rolling Resistance Model

In a statics context, rolling resistance is often modeled using the concept of a rolling resistance arm (sometimes called the coefficient of rolling resistance with units of length). Here's the idea:

The normal force N shifts forward by a small distance a from directly beneath the center of the wheel. This offset distance a is the rolling resistance arm. For a wheel of radius r , the moment about the contact point gives the relationship:

$$F_r = \frac{a \cdot N}{r}$$

where:

- F_r is the rolling resistance force
- a is the rolling resistance arm (units of length, typically mm)
- N is the normal force
- r is the radius of the wheel

This is the physically accurate model you'll encounter in statics courses. The offset a depends on the materials and surface conditions, and it's typically very small (fractions of a millimeter for hard surfaces).

Simplified Force Equation

A common simplified approach treats rolling resistance similarly to sliding friction:

$$F_r = C_{rr} \times N$$

where C_{rr} is a dimensionless coefficient of rolling resistance. Note that C_{rr} and the rolling resistance arm a are related by $C_{rr} = a/r$, so the coefficient depends on wheel radius.

Typical values of C_{rr} :

Surface Combination	C_{rr} Range
Hardened steel on hardened steel (ball bearings)	0.001 – 0.005
Railroad steel wheel on steel rail	0.001 – 0.002
Pneumatic car tire on concrete	0.01 – 0.02
Pneumatic car tire on loose sand	0.06 – 0.10

Normal Force on Horizontal and Inclined Surfaces

For a horizontal surface, the normal force equals the weight of the object:

$$N = m \times g$$

For a surface inclined at angle θ from horizontal, the component of weight perpendicular to the surface is reduced:

$$N = m \times g \times \cos(\theta)$$

A 1000 kg cart on a flat surface has $N = 9810$ N. On a 15° incline, $N = 9810 \times \cos(15^\circ) = 9475$ N. The rolling resistance force decreases on the incline, but now you also have a gravity component pulling the cart downhill, which usually dominates.

Solving Rolling Resistance Problems

When you encounter a rolling resistance problem in statics, follow these steps:

1. Draw a free body diagram of the rolling object, showing the weight, normal force, applied forces, and the rolling resistance force opposing the direction of motion (or impending motion).
2. Identify the contact geometry. Note the wheel radius r and whether you're given the rolling resistance arm a or the dimensionless coefficient C_{rr} .

3. Calculate the normal force using equilibrium in the direction perpendicular to the surface.
4. Calculate the rolling resistance force using $F_r = a \cdot N/r$ or $F_r = C_{rr} \times N$.
5. Apply equilibrium equations (sum of forces, sum of moments) to solve for unknowns like the force needed to maintain constant velocity or the force needed to start rolling.

Power to Overcome Rolling Resistance

Once an object is moving at velocity V , the power required to overcome rolling resistance is:

$$P_r = F_r \times v$$

For example, a vehicle with $F_r = 200$ N traveling at 25 m/s (about 90 km/h) requires $P_r = 200 \times 25 = 5000$ W, or 5 kW, just to overcome rolling resistance. At highway speeds, air resistance typically dominates, but at low speeds rolling resistance is the larger factor.

Applications in Engineering

Bearings: Ball and roller bearings replace sliding contact with rolling contact, dramatically reducing resistance. Material hardness and surface finish are selected to minimize the rolling resistance arm a .

Vehicle design: Rolling resistance, air resistance, and grade resistance are the three main forces opposing vehicle motion. Tire manufacturers balance tread pattern, rubber compound, and recommended inflation pressure to minimize C_{rr} without sacrificing traction or durability. Under-inflated tires increase the contact patch deformation and raise rolling resistance noticeably.

Conveyor systems and industrial rollers: Selecting harder roller materials and smoother surfaces keeps rolling resistance low, reducing the motor power needed to drive the system. Proper lubrication of bearing surfaces further reduces energy losses at the roller supports.

Unit 7 – Stress and Strain

7.1 Concept of stress and its types

Stress is the internal force per unit area in a material when subjected to external loads. It's crucial for designing strong, stable structures. Engineers use stress analysis to select materials and ensure components can withstand various loads without failing.

Normal stress acts perpendicular to a material's surface, causing compression or tension. Shear stress acts parallel, making adjacent planes slide. Materials often experience both types. Understanding these helps engineers design safer, more reliable structures and components.

Stress in Engineering Materials

Definition and Significance

- Stress is the internal force per unit area that a material experiences when subjected to external loads or forces
- Stress is a crucial concept in engineering that determines the strength, stability, and durability of materials and structures
- Understanding stress helps engineers design components and structures that can withstand applied loads without failure or excessive deformation
- The units of stress are typically expressed in pascals (Pa), megapascals (MPa), or gigapascals (GPa) in the SI system, and pounds per square inch (psi) in the US customary system
- Stress analysis is essential for selecting appropriate materials and designing components with adequate factors of safety to ensure reliable performance under various loading conditions

Normal vs Shear Stress

Normal Stress

- Normal stress acts perpendicular to the surface of a material, causing either compression (pushing together) or tension (pulling apart)
- Normal stress is denoted by the Greek letter sigma (σ)
- The formula for normal stress is $\sigma = F/A$, where F is the normal force and A is the cross-sectional area
- Examples of normal stress include the stress in a rope supporting a weight (tension) or the stress in a column supporting a building (compression)

Shear Stress

- Shear stress acts parallel to the surface of a material, causing adjacent planes to slide relative to each other
- Shear stress is denoted by the Greek letter tau (τ)
- The formula for shear stress is $\tau = F/A$, where F is the shear force and A is the area over which the force is applied

- Examples of shear stress include the stress in a bolt subjected to a shearing force or the stress in the adhesive layer of a bonded joint
- In many loading scenarios, materials experience a combination of normal and shear stresses, which can be analyzed using Mohr's circle or other stress transformation techniques

Types of Stress

Compressive and Tensile Stress

- Compressive stress occurs when a material is subjected to forces that push its particles closer together, causing a decrease in length or volume
- Tensile stress occurs when a material is subjected to forces that pull its particles apart, causing an increase in length or volume
- Examples of compressive stress include the stress in a concrete foundation supporting a building or the stress in a gear tooth during meshing
- Examples of tensile stress include the stress in a stretched rubber band or the stress in a metal wire under tension

Torsional and Bending Stress

- Torsional stress (or shear stress due to torsion) occurs when a material is subjected to a twisting force, causing angular deformation
- Bending stress is a combination of compressive and tensile stresses that occurs when a material is subjected to a bending moment, causing it to curve
- Examples of torsional stress include the stress in a shaft transmitting power or the stress in a torsion spring
- Examples of bending stress include the stress in a beam supporting a load or the stress in a cantilever subjected to a concentrated force at its free end

Thermal and Residual Stress

- Thermal stress is induced in a material due to changes in temperature, which cause expansion or contraction of the material
- Residual stress is the stress that remains in a material after the external loads have been removed, often due to manufacturing processes such as welding or heat treatment
- Examples of thermal stress include the stress in a heated pressure vessel or the stress in a bi-metallic strip used in a thermostat
- Examples of residual stress include the stress in a welded joint after cooling or the stress in a heat-treated gear after quenching

Stress Distribution Analysis

Stress Distribution and Equilibrium

- Stress distribution refers to the variation of stress magnitude and direction throughout a structural element

- To analyze stress distribution, engineers use free-body diagrams to identify the external forces and moments acting on the element
- The stress distribution can be determined by applying equilibrium equations, constitutive relationships (such as Hooke's law), and compatibility conditions
- For simple loading cases, such as uniaxial tension or compression, the stress distribution is uniform throughout the cross-section of the element

Stress Concentration and Finite Element Analysis

- For more complex loading scenarios, such as bending or torsion, the stress distribution varies across the cross-section, with maximum stresses occurring at specific locations (the outer fibers in bending)
- Stress concentration factors are used to account for the increased stress levels at geometric discontinuities, such as holes, notches, or sharp corners, which can lead to localized failure
- Examples of stress concentration include the stress around a hole in a plate subjected to tension or the stress at the root of a gear tooth
- Finite element analysis (FEA) is a powerful numerical method for analyzing stress distribution in complex geometries and loading conditions, providing detailed insights into the stress state of a component or structure

7.2 Normal and shear strain

Normal vs Shear Strain

Definition and Characteristics

Normal strain measures how much a material stretches or compresses along the direction of an applied force. You calculate it by dividing the change in length by the original length. If you pull on a bar and it gets longer, that elongation relative to its starting length is the normal strain.

Shear strain measures angular distortion. When a force acts parallel to a surface rather than perpendicular to it, the material doesn't stretch or compress; it skews. Shear strain is defined as the tangent of the deformation angle, representing how much the shape distorts without changing volume.

Key Differences

The core distinction is the direction of deformation relative to the applied force:

- Normal strain occurs parallel to the force. It changes the length of the material but not its shape. Think of a rubber band stretching when you pull it: it gets longer, but it's still a band-shaped object.
- Shear strain occurs perpendicular to the force. It changes the shape of the material but not its volume. Picture a deck of cards sitting on a table: push the top card sideways, and the whole deck deforms into a parallelogram. No card got longer or shorter, but the overall shape changed.

Strain Calculation

Normal Strain Formula

Normal strain (ϵ) is the ratio of change in length to original length:

$$\epsilon = \frac{L - L_0}{L_0} = \frac{\delta}{L_0}$$

where L is the final length, L_0 is the original length, and $\delta = L - L_0$ is the change in length (often called the deformation or elongation).

Example: A steel bar has an initial length of 1 m. Under tension, it elongates to 1.001 m.

1. Find the change in length: $\delta = 1.001 - 1.000 = 0.001$ m
2. Divide by the original length: $\epsilon = \frac{0.001}{1.000} = 0.001$
3. Express as a percentage if needed: $0.001 \times 100 = 0.1\%$

Normal strain is dimensionless (length divided by length), but you'll often see it reported in units like mm/mm, in/in, or as a percentage. A positive value means tension (elongation); a negative value means compression (shortening).

Shear Strain Formula

Shear strain (γ) is defined as:

$$\gamma = \tan(\theta)$$

where θ is the angle of deformation (the angle the deformed edge makes with its original orientation). For small angles, $\tan(\theta) \approx \theta$ (in radians), so you'll often see shear strain written simply as the angle in radians.

An equivalent form that's useful in practice:

$$\gamma = \frac{\Delta x}{y}$$

where Δx is the lateral (transverse) displacement and y is the height (perpendicular distance from the fixed face to the displaced face).

Example: A rectangular block deforms by an angle of 2° under shear stress.

1. Convert to radians if using the small-angle approximation: $\theta = 2^\circ \times \frac{\pi}{180^\circ} = 0.0349$ rad
2. Calculate: $\gamma = \tan(2^\circ) = 0.0349$
3. As a percentage: $0.0349 \times 100 \approx 3.5\%$

Notice that for this small angle, $\tan(\theta)$ and θ in radians are nearly identical. This is why the small-angle approximation works well for most engineering applications where deformations are tiny.

Consistency in Units

Strain is dimensionless, but you still need to be careful: the numerator and denominator must use the same units. If your original length is in meters, your change in length must also be in meters. Mixing mm and m is one of the most common mistakes in strain calculations.

Strain and Deformation

Normal Strain Deformation

Picture a rectangular bar fixed at one end with a tensile force pulling on the other end. The bar elongates along its length, and every cross-section along the bar moves slightly away from the fixed end. Under compression, the opposite happens: the bar shortens. In both cases, the cross-sectional shape stays the same; only the length changes.

A real-world example: a concrete column shortening under the weight of a building above it. The column gets slightly shorter (compressive normal strain), even though the deformation is too small to see with the naked eye.

Shear Strain Deformation

Now picture a rectangular block glued to a table with a horizontal force pushing across its top face. The block skews into a parallelogram. The bottom face stays fixed, the top face slides sideways, and the angle between the vertical edge and its new tilted position is the shear strain angle θ .

A simple analogy: a book on a shelf tilting when you push its top edge sideways. The pages slide relative to each other, and the book's profile goes from a rectangle to a parallelogram.

Combined Deformations

In textbook diagrams, deformations are exaggerated so you can actually see what's happening. Real engineering strains are tiny: a normal strain of 0.001 (0.1%) is common for metals under working loads.

In practice, materials often experience normal and shear strains simultaneously. A beam under bending, for instance, has normal strain (tension on one side, compression on the other) and shear strain (from internal shear forces). Analyzing these combined states is where strain concepts become especially powerful.

Stress and Strain in Materials

Definitions

- Stress (σ for normal, τ for shear) is the internal force per unit area acting within a material.
- Strain (ϵ for normal, γ for shear) is the resulting deformation.

Stress is the cause; strain is the effect. You apply a load (creating stress), and the material deforms (producing strain).

Hooke's Law

Within the elastic region, stress and strain are directly proportional. This linear relationship is Hooke's Law:

For normal stress and strain:

$$\sigma = E \cdot \epsilon$$

where E is the modulus of elasticity (also called Young's modulus). A higher E means the material is stiffer and deforms less for a given stress. Steel, for example, has $E \approx 200$ GPa, while aluminum has $E \approx 70$ GPa.

For shear stress and strain:

$$\tau = G \cdot \gamma$$

where G is the shear modulus (also called the modulus of rigidity). For steel, $G \approx 77$ GPa.

Both E and G are material properties, meaning they depend on what the material is, not on the geometry of the part.

Elastic Limit and Plastic Deformation

- Elastic behavior: Below the elastic limit, the material returns to its original shape when the load is removed. The stress-strain relationship is linear (Hooke's Law applies).
- Plastic behavior: Beyond the elastic limit, permanent deformation occurs. The stress-strain curve becomes nonlinear, and removing the load leaves the material in a deformed state.

A familiar example: bend a metal paperclip slightly and it springs back (elastic). Bend it far enough and it stays bent (plastic deformation). The transition point between these behaviors is the yield point.

Stress-Strain Curve

The stress-strain curve plots stress (vertical axis) against strain (horizontal axis) for a material tested to failure. For a ductile material like mild steel, the key regions are:

- Elastic region — linear, governed by Hooke's Law
- Yield point — where plastic deformation begins
- Strain hardening (plastic region) — stress continues to increase but the relationship is nonlinear
- Ultimate tensile strength — the maximum stress the material can sustain
- Fracture point — where the material breaks

This curve is one of the most important tools for understanding how a material will behave under load. Different materials produce very different curves: brittle materials like cast iron fracture with little plastic deformation, while ductile materials like steel show significant yielding before failure.

7.3 Stress-strain diagrams and Hooke's law

Stress-Strain Diagrams for Materials

Stress-strain diagrams are the primary tool for understanding how a material responds to loading. They plot the relationship between applied stress and the resulting strain, letting you identify exactly when a material will deform elastically, yield plastically, or fracture. If you can read one of these diagrams, you can predict material behavior under load and make informed design decisions.

Hooke's law governs the linear elastic portion of these diagrams. It gives you a simple proportional relationship between stress and strain, which is the basis for most structural analysis in this course.

Graphical Representation and Key Components

A stress-strain diagram is built from tensile test data. Strain (unitless, or length/length) goes on the x-axis, and stress (force/area, in Pa or psi) goes on the y-axis.

Two features of the curve carry physical meaning:

- Slope of the linear portion = the modulus of elasticity (Young's modulus, E). A steeper slope means a stiffer material.
- Area under the curve = energy absorbed during deformation, which relates to the material's toughness. A material with a large area under its curve can absorb more energy before fracture.

Variation in Stress-Strain Diagrams for Different Materials

The shape of a stress-strain diagram depends heavily on the type of material tested.

Ductile materials (most metals) show a clear yield point followed by a large plastic deformation region before fracture. Steel is the classic example: it has a distinct yield point, a long plastic region, and undergoes significant necking before breaking.

Brittle materials (ceramics, glass) behave very differently. Their curve is nearly linear all the way up, then the material fractures suddenly with little to no plastic deformation. Glass is a good example: it stretches elastically, then snaps without warning.

This distinction matters for design. Ductile materials give visible warning before failure (they bend and deform). Brittle materials don't.

Key Points on Stress-Strain Diagrams

Elastic and Plastic Deformation Regions

Several critical points appear on every stress-strain diagram, and you need to know what each one means:

- Proportional limit: The highest stress at which stress is directly proportional to strain. This marks the end of the straight-line portion of the curve. Beyond this point, Hooke's law no longer applies.
- Elastic limit: The maximum stress a material can sustain and still return to its original shape when unloaded. This may or may not coincide with the proportional limit. The distinction is subtle but worth noting: a material can behave nonlinearly yet still recover elastically.
- Yield point (yield strength): The stress at which permanent (plastic) deformation begins. Once you pass this point, the material won't fully return to its original dimensions. In some steels, the yield point shows up as a sharp drop in stress on the diagram, called the upper and lower yield points.

Ultimate Strength and Fracture

- Ultimate tensile strength (UTS): The highest stress value on the entire diagram. This is the maximum stress the material can withstand. On the curve, it's simply the peak.
- Fracture point: Where the material breaks completely. For ductile materials, this occurs after significant necking (localized thinning). For brittle materials, the fracture point is close to the UTS because there's almost no plastic deformation.

Some typical behaviors to keep in mind:

- Aluminum alloys have moderate ultimate strength with good ductility, so they deform noticeably before fracture.
- Concrete has relatively low tensile strength and fractures in a brittle manner (which is why it's reinforced with steel in structural applications).

Hooke's Law and Elasticity

Hooke's Law in the Elastic Region

Hooke's law states that within the proportional limit, stress is directly proportional to strain:

$$\sigma = E\varepsilon$$

where σ is the applied stress, E is the modulus of elasticity (Young's modulus), and ε is the resulting strain.

This relationship only holds in the linear elastic region of the stress-strain diagram. Within this region, deformation is completely reversible: remove the load, and the material returns to its original shape.

Hooke's law also assumes the material is homogeneous (uniform properties throughout) and isotropic (same properties in all directions), and that deformations remain small.

Deviations from Hooke's Law

Once stress exceeds the proportional limit, the stress-strain relationship becomes nonlinear and Hooke's law no longer applies. This happens in the plastic deformation region for metals.

Some materials deviate from Hooke's law even within what might look like an "elastic" range:

- Rubber stretches elastically to very large strains, but the relationship between stress and strain is nonlinear from the start.
- Polymers can exhibit viscoelastic behavior, meaning their stress-strain response depends on how fast the load is applied and how long it's held. Time becomes a factor.

Modulus of Elasticity Calculation

Calculating the Modulus of Elasticity from Stress-Strain Data

You can determine E directly from a stress-strain diagram using the slope of the linear region. Here's the process:

1. Identify the linear (straight-line) portion of the stress-strain curve.
2. Pick two points on that line: $(\varepsilon_1, \sigma_1)$ and $(\varepsilon_2, \sigma_2)$. Choose points that are well-separated for better accuracy, and make sure both fall within the proportional limit.
3. Calculate the slope:

$$E = \frac{\sigma_2 - \sigma_1}{\varepsilon_2 - \varepsilon_1}$$

Since strain is unitless, E carries the same units as stress (Pa or psi).

Interpreting and Using the Modulus of Elasticity

The modulus of elasticity is a material property, not a structural property. It tells you how stiff a material is: how much it resists deformation per unit of applied stress.

- High E = stiff material. Diamond has one of the highest moduli of elasticity (~1,220 GPa), meaning it barely deforms under load.
- Low E = flexible material. Rubber has a very low modulus (~0.01–0.1 GPa), so it deforms easily.

- Steel falls in between at roughly 200 GPa, which is why it's the workhorse of structural engineering: stiff enough to limit deflection, ductile enough to warn before failure.

Once you know E for a material, you can use Hooke's law to predict stress or strain for any loading condition within the elastic region. This is the foundation for most of the structural analysis you'll do in this course.

7.4 Poisson's ratio and thermal effects

Poisson's ratio and thermal effects play crucial roles in understanding material behavior under stress. Poisson's ratio measures how materials deform perpendicular to applied forces, while thermal effects cause materials to expand or contract with temperature changes.

These concepts are essential for predicting how structures respond to loads and temperature variations. By considering both mechanical and thermal stresses, engineers can design safer, more efficient structures that withstand real-world conditions.

Poisson's ratio and material behavior

Definition and significance

- Poisson's ratio is the negative ratio of transverse strain to axial strain in a material under uniaxial loading
- Measures the Poisson effect, which describes how a material tends to expand in directions perpendicular to the direction of compression
- Fundamental material property that relates the deformation in the transverse direction to the deformation in the axial direction
- Ranges from -1 to 0.5 for isotropic materials, with most materials having a positive value between 0 and 0.5 (rubber-like materials: ~ 0.5 , cork: ~ 0)
- Affects the overall deformation and stress distribution in a structure under loading

Variations in Poisson's ratio

- The value of Poisson's ratio varies depending on the material properties
- Materials with a higher Poisson's ratio experience a greater change in the transverse dimensions relative to the axial dimension
- Rubber-like materials have a Poisson's ratio close to 0.5, indicating that they maintain a nearly constant volume under deformation
- Cork has a Poisson's ratio close to 0, indicating minimal transverse deformation under axial loading
- Understanding the Poisson's ratio of a material is crucial for predicting its behavior and selecting appropriate materials for specific applications

Calculating dimensional change

Transverse dimension change

- The change in the transverse dimension (Δd) is calculated using the formula: $\Delta d = -\nu * (\Delta L / L) * d$

- ν is Poisson's ratio
- ΔL is the change in length
- L is the original length
- d is the original transverse dimension
- Under uniaxial tension, the transverse dimensions decrease (negative change)
- Under uniaxial compression, the transverse dimensions increase (positive change)

Axial dimension change

- The change in the axial dimension (ΔL) is calculated using the formula: $\Delta L = (P * L) / (A * E)$
- P is the applied load
- L is the original length
- A is the cross-sectional area
- E is the modulus of elasticity
- Under uniaxial tension, the axial dimension increases (positive change)
- Under uniaxial compression, the axial dimension decreases (negative change)
- The magnitude of the change in dimensions depends on the value of Poisson's ratio and the applied load

Thermal effects on stress and strain

Thermal strain

- Temperature changes cause materials to expand or contract, resulting in thermal strain (ϵ_T)
- Thermal strain is calculated using the formula: $\epsilon_T = \alpha * \Delta T$
- α is the coefficient of thermal expansion
- ΔT is the change in temperature
- The coefficient of thermal expansion is a material property that describes the extent of expansion or contraction with temperature changes
- Materials with a higher coefficient of thermal expansion experience greater thermal strain for a given temperature change

Thermal stress

- Thermal stress (σ_T) develops when a material is restrained from expanding or contracting freely due to temperature changes
- The magnitude of thermal stress is calculated using the formula: $\sigma_T = -E * \alpha * \Delta T$
- E is the modulus of elasticity
- α is the coefficient of thermal expansion

- ΔT is the change in temperature
- Compressive thermal stress (negative) occurs when a material is heated and restrained from expanding
- Tensile thermal stress (positive) occurs when a material is cooled and restrained from contracting
- Thermal stresses can lead to material failure if they exceed the yield strength or ultimate strength of the material

Combined mechanical and thermal stresses

Superposition of stresses and strains

- When a structural element is subjected to both mechanical and thermal loads, the total stress is the sum of the mechanical stress (σ_M) and the thermal stress (σ_T)
- The total strain is the sum of the mechanical strain (ϵ_M) and the thermal strain (ϵ_T)
- The mechanical stress is calculated using the formula: $\sigma_M = (P / A) \pm (M * y) / I$
- P is the applied load
- A is the cross-sectional area
- M is the bending moment
- y is the distance from the neutral axis
- I is the moment of inertia
- The total stress (σ_{total}) is the sum of the mechanical stress and the thermal stress: $\sigma_{total} = \sigma_M + \sigma_T$
- The total strain (ϵ_{total}) is the sum of the mechanical strain and the thermal strain: $\epsilon_{total} = \epsilon_M + \epsilon_T$

Complex stress distributions

- The combined effects of mechanical and thermal stresses lead to complex stress distributions in a structural element
- Advanced analysis techniques, such as finite element analysis (FEA), may be required to accurately predict the stress state and potential failure modes
- FEA involves discretizing the structural element into smaller elements and solving for the stress and strain in each element
- The results of FEA can help identify critical stress concentrations and guide the design of structural elements to withstand combined mechanical and thermal loads

Unit 8 – Material Properties in Engineering

8.1 Elastic and plastic behavior of materials

Materials can behave elastically or plastically when stressed. Elastic deformation is temporary: the material returns to its original shape when the load is removed. Plastic deformation is permanent, occurring when stress exceeds the material's yield point.

Stress-strain curves capture both behaviors in a single graph, showing the transition from elastic to plastic deformation. The yield strength and ultimate tensile strength are the key parameters you'll pull from these curves, and they directly guide material selection and component design.

Elastic vs Plastic Deformation

Types of Deformation

Elastic deformation is temporary. The material returns to its original shape and size once the load is removed. In this region, the stress-strain relationship is linear, meaning stress and strain increase proportionally. Think of stretching a rubber band or compressing a spring: release the force, and the object snaps back.

Plastic deformation is permanent. The material does not return to its original shape after the load is removed. Internally, atoms have shifted to new positions, and the stress-strain relationship is no longer linear. Bending a metal paperclip past a certain point or reshaping clay are everyday examples: the new shape stays.

Yield Point and Material Behavior

The yield point is the stress at which a material transitions from elastic to plastic behavior. Below it, deformation is recoverable. Above it, permanent deformation begins. This makes the yield point one of the most critical values in engineering design.

You can find the yield point on a stress-strain curve in two ways:

1. Proportional limit method - Identify where the curve first deviates from a straight line.
2. 0.2% offset method - Draw a line parallel to the linear elastic portion of the curve, but starting at a strain of 0.002 (0.2%) on the x-axis. The stress where this line intersects the curve is the offset yield strength. This method is more common because many materials don't have a sharp, obvious yield point.

Ductile materials (steel, aluminum, copper) exhibit both elastic and plastic deformation. They can undergo significant plastic deformation before fracture, which means they can be shaped, formed, and often give visible warning before failure.

Brittle materials (glass, concrete, cast iron) show little to no plastic deformation. They fracture suddenly once their strength is exceeded, with minimal warning.

Stress-Strain Curves Analysis

Stress-Strain Curve Characteristics

A stress-strain curve plots the relationship between applied stress (y-axis) and resulting strain (x-axis) during a tensile or compressive test. The shape of this curve tells you nearly everything about how a material behaves under load.

The slope of the linear (elastic) portion of the curve is Young's modulus (E), also called the elastic modulus. It quantifies stiffness.

- A steeper slope means a higher E and a stiffer material (diamond $\approx 1,200$ GPa, steel ≈ 200 GPa).
- A shallower slope means a lower E and a more flexible material (rubber ≈ 0.01 – 0.1 GPa, low-density polyethylene ≈ 0.2 GPa).

Key Material Properties from Stress-Strain Curves

Yield strength is the stress where plastic deformation begins. It marks the upper boundary of the elastic region. Materials with higher yield strengths (high-strength steels, titanium alloys) can carry greater loads before permanently deforming.

Ultimate tensile strength (UTS) is the maximum stress the material can withstand, corresponding to the highest point on the stress-strain curve. It represents the peak load-carrying capacity before the material begins to neck and weaken toward fracture.

Toughness is represented by the total area under the stress-strain curve. It measures the energy a material can absorb before fracturing. Tough materials (metals, composites) resist crack propagation well. Brittle materials (ceramics, glass) have small areas under their curves and absorb little energy before breaking.

Elongation at fracture (also called percent elongation) is the maximum strain the material reaches before breaking. It's a direct measure of ductility.

- High elongation: copper ($\sim 50\%$), gold ($\sim 45\%$), aluminum alloys (~ 10 – 40%)
- Low elongation: concrete (~ 0.01 – 0.02%), glass ($\sim 0\%$), cast iron ($\sim 0.5\%$)

Hooke's Law Application

Hooke's Law Formula

Hooke's Law describes the linear relationship between stress and strain in the elastic region:

$$\sigma = E\epsilon$$

where: - σ = stress (force per unit area, in Pa or MPa) - E = Young's modulus (same units as stress, Pa or GPa) - ϵ = strain (change in length divided by original length, dimensionless)

You can rearrange this to solve for any of the three variables: - Strain: $\epsilon = \sigma/E$ - Young's modulus: $E = \sigma/\epsilon$

Calculating Stress and Strain using Hooke's Law

Finding stress from strain:

A steel bar ($E = 200 \text{ GPa}$) experiences a strain of 0.001. What is the stress?

1. Write Hooke's Law: $\sigma = E\epsilon$
2. Substitute values: $\sigma = (200 \times 10^9 \text{ Pa})(0.001)$
3. Solve: $\sigma = 200 \times 10^6 \text{ Pa} = 200 \text{ MPa}$

Finding strain from stress:

An aluminum rod ($E = 70 \text{ GPa}$) is subjected to a stress of 100 MPa. What is the strain?

1. Rearrange: $\epsilon = \sigma/E$
2. Substitute values: $\epsilon = (100 \times 10^6 \text{ Pa})/(70 \times 10^9 \text{ Pa})$
3. Solve: $\epsilon = 0.00143$

Validity limit: Hooke's Law only applies in the elastic region where the stress-strain curve is linear. Once you exceed the yield point and enter the plastic region, the linear relationship breaks down and permanent deformation occurs. Non-linear elastic materials like rubber or biological tissues also don't follow Hooke's Law, even in their elastic range, and require more complex models.

Yield & Tensile Strength Determination

Yield Strength Determination Methods

The yield strength defines the maximum stress a component can experience without permanently deforming. Engineers design parts to operate below this value under normal loading conditions.

Two standard methods for determining yield strength from a stress-strain curve:

1. Proportional limit method - Find the point where the curve first deviates from a straight line. This marks the end of perfectly linear-elastic behavior.
2. 0.2% offset method - Draw a line parallel to the elastic slope, offset by 0.002 strain (0.2%) along the x-axis. The intersection of this line with the stress-strain curve gives the yield strength. Most engineering references report yield strength using this method.

Several factors influence yield strength: - Composition and processing - Alloying elements, heat treatment (quenching, tempering), and cold working can all increase yield strength in metals. - Strain rate - Faster loading generally increases yield strength. - Temperature - Lower temperatures tend to raise yield strength; higher temperatures lower it.

Ultimate Tensile Strength (UTS) Determination

The UTS is the highest stress value on the engineering stress-strain curve. It corresponds to the maximum load the specimen sustains, divided by the original cross-sectional area.

$$\text{UTS} = \frac{F_{\max}}{A_0}$$

Example: A tensile specimen with a cross-sectional area of 100 mm^2 fails at a maximum load of 10 kN.

1. Convert units: $A_0 = 100 \text{ mm}^2 = 100 \times 10^{-6} \text{ m}^2$
2. Apply the formula: $\text{UTS} = \frac{10 \times 10^3 \text{ N}}{100 \times 10^{-6} \text{ m}^2}$
3. Solve: $\text{UTS} = 100 \times 10^6 \text{ Pa} = 100 \text{ MPa}$

For ductile materials (mild steel, aluminum alloys), there's typically a clear yield point and a large gap between yield strength and UTS. The material undergoes significant plastic deformation and necking before fracture.

For brittle materials (glass, concrete, acrylic), the yield strength and UTS are close together, or the material fractures before any measurable yielding occurs. There's little to no warning before failure.

Design takeaway: Engineers apply a factor of safety so that working stresses stay well below both the yield strength (to prevent permanent deformation) and the UTS (to prevent fracture). The choice of which limit to design against depends on whether the application can tolerate any plastic deformation.

8.2 Yield criteria and failure theories

Yield Criteria and Failure Theories

Yield criteria and failure theories provide the mathematical framework for predicting when a material will begin to deform permanently or fracture under load. Without them, you'd have no systematic way to check whether a complex, real-world stress state is safe or dangerous.

These tools connect what you already know about stress analysis (Mohr's circle, principal stresses) to actual design decisions. The core question is always the same: given a multi-axis stress state, how do you compare it to a simple uniaxial yield strength from a tensile test?

Yield Criteria for Material Failure

Concept and Role of Yield Criteria

A yield criterion is a mathematical rule that takes a general stress state and tells you whether the material has started to yield (deform plastically). The problem it solves is straightforward: tensile test data gives you yield strength under uniaxial loading, but real components experience combined stresses in multiple directions. Yield criteria bridge that gap.

- The yield strength (σ_y) is the uniaxial stress at which plastic deformation begins. Every yield criterion ultimately compares some function of the applied stresses back to this single value.
- Yield criteria work with principal stresses, which strip away shear and let you evaluate the most critical normal stress combinations.
- The right criterion to use depends on whether the material is ductile or brittle, and on the type of loading (uniaxial, biaxial, or triaxial).

Common Yield Criteria

Three yield criteria appear most often:

- Tresca (maximum shear stress criterion)
- von Mises (maximum distortion energy criterion)
- Drucker-Prager (a pressure-dependent extension, used for soils and some polymers)

Each has different assumptions about what physical mechanism drives yielding. Tresca focuses on shear stress alone; von Mises accounts for the total distortion energy from all three principal stress differences; Drucker-Prager adds sensitivity to hydrostatic pressure.

Tresca vs. von Mises Yield Criteria

Tresca Yield Criterion

The Tresca criterion says yielding begins when the maximum shear stress in the material equals the shear stress at yielding in a simple tensile test. Since the maximum shear stress in uniaxial tension is $\frac{\sigma_y}{2}$, the criterion becomes:

$$\max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) = \sigma_y$$

where σ_1 , σ_2 , and σ_3 are the principal stresses.

- Geometrically, the Tresca yield surface is a hexagonal prism in principal stress space.
- It's the more conservative of the two criteria: if Tresca says you're safe, von Mises will agree.
- Tresca is best suited for ductile materials where shear-driven slip is the dominant yielding mechanism.

Note on a common error: The original guide listed cast iron and ceramics as suitable for Tresca. That's incorrect. Cast iron is brittle, and ceramics are brittle. Tresca is designed for ductile materials (steel, aluminum, copper). Brittle materials need different failure theories (covered below).

von Mises Yield Criterion

The von Mises criterion says yielding begins when the distortion energy (the portion of strain energy that changes shape, not volume) reaches the value it would have at yielding in uniaxial tension:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$$

An equivalent and often more convenient form defines the von Mises equivalent stress:

$$\sigma_v = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}$$

Yielding occurs when $\sigma_v \geq \sigma_y$.

- Geometrically, the von Mises yield surface is a cylinder in principal stress space (it circumscribes the Tresca hexagon).
- It fits experimental data for most ductile metals better than Tresca because it accounts for the combined effect of all three principal stress differences, not just the largest one.
- It applies to isotropic, ductile materials with roughly equal tensile and compressive yield strengths.

Applying Tresca and von Mises Criteria

Here's the step-by-step process for checking whether yielding occurs:

1. Determine the stress state. Identify all normal and shear stress components (σ_x , σ_y , τ_{xy} , etc.) at the point of interest.
2. Find the principal stresses. Use Mohr's circle or the eigenvalue method to compute σ_1 , σ_2 , and σ_3 . Order them so that $\sigma_1 \geq \sigma_2 \geq \sigma_3$.
3. Substitute into the criterion. - Tresca: Compute $|\sigma_1 - \sigma_3|$ and compare to σ_y . - von Mises: Compute σ_v and compare to σ_y .
4. Interpret the result. If the criterion value meets or exceeds σ_y , the material is predicted to yield at that point.

Quick example: A steel shaft ($\sigma_y = 250$ MPa) has principal stresses $\sigma_1 = 200$ MPa, $\sigma_2 = 50$ MPa, $\sigma_3 = -30$ MPa.

- Tresca: $|\sigma_1 - \sigma_3| = |200 - (-30)| = 230$ MPa. Since $230 < 250$, no yielding predicted.
- von Mises: $\sigma_v = \sqrt{\frac{1}{2}[(200 - 50)^2 + (50 - (-30))^2 + ((-30) - 200)^2]}$
 $= \sqrt{\frac{1}{2}[22500 + 6400 + 52900]} = \sqrt{40900} \approx 202$ MPa. Since $202 < 250$, no yielding predicted.

Notice that von Mises gives a lower equivalent stress than Tresca here, which is typical. Tresca is always equal to or more conservative than von Mises.

Principal Stresses and Failure Theories

Principal Stresses

Principal stresses are the normal stresses on planes where shear stress is zero. They represent the extreme values of normal stress at a point.

- The three principal stresses are ordered $\sigma_1 \geq \sigma_2 \geq \sigma_3$, where σ_1 is the maximum and σ_3 is the minimum.
- You find them using Mohr's circle (graphical) or by solving the characteristic equation of the stress tensor (analytical).
- Every yield and failure criterion is written in terms of principal stresses because they capture the most critical stress combinations regardless of how you originally set up your coordinate system.

Relationship to Failure Theories

Principal stresses feed directly into failure theories in different ways:

- Tresca uses the difference $\sigma_1 - \sigma_3$ (the largest principal stress difference, which equals twice the maximum shear stress).
- von Mises uses all three differences symmetrically.
- Maximum normal stress (Rankine) compares σ_1 directly to the ultimate tensile strength, or $|\sigma_3|$ to the ultimate compressive strength.
- Mohr-Coulomb uses both σ_1 and σ_3 along with separate tensile and compressive strengths.

One important physical insight: the hydrostatic stress, defined as $\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$, changes volume but does not cause shape distortion. For most metals, hydrostatic stress alone does not cause yielding. That's exactly why the von Mises criterion works well for metals: it isolates the distortion (shape-changing) part of the stress state.

Selecting Failure Theories for Materials

Factors Influencing Failure Theory Selection

Choosing the right failure theory comes down to matching the theory's assumptions to your actual situation:

- Material behavior: Is it ductile (significant plastic deformation before fracture) or brittle (little to no plastic deformation)?

- Strength symmetry: Does the material have similar strengths in tension and compression, or are they very different?
- Loading type: Is the load static, or does it cycle repeatedly (fatigue)?
- Isotropy: Does the material behave the same in all directions, or is it anisotropic (like composites or wood)?

Failure Theories for Different Materials and Loading Conditions

Ductile materials (most metals: steel, aluminum, copper): - Use the von Mises criterion as the default. It matches experimental results well for isotropic metals with similar tensile and compressive yield strengths. - Tresca is a simpler, slightly more conservative alternative. Some design codes prefer it for that reason.

Brittle materials (ceramics, cast iron, concrete, some polymers): - Use the Maximum Normal Stress (Rankine) criterion. Failure is predicted when σ_1 reaches the ultimate tensile strength (S_{ut}) or when $|\sigma_3|$ reaches the ultimate compressive strength (S_{uc}). - The Mohr-Coulomb criterion is better when tensile and compressive strengths differ significantly (which is common for brittle materials). It accounts for the interaction between normal stress and shear stress on the failure plane.

Cyclic or fatigue loading: - Static yield criteria don't apply directly. Instead, use fatigue-specific theories like the Goodman criterion or Soderberg criterion, which account for both the mean stress and the alternating stress amplitude over many cycles.

Anisotropic materials (fiber composites, rolled metals, wood): - Standard isotropic criteria won't work. Use the Hill criterion (for metals with directional properties from rolling) or the Tsai-Wu criterion (widely used for fiber-reinforced composites).

Importance of Validation

No failure theory is perfect. Each one is a model built on simplifying assumptions. Whenever possible, validate your chosen theory against experimental data for the specific material and loading you're designing for. If test data shows the theory is non-conservative (predicts safety when failure actually occurs), you need a different theory or an added safety factor.

8.3 Fatigue, creep, and impact loading

Fatigue, Creep, and Impact Loading

Fatigue, creep, and impact loading are three ways materials can fail that don't show up on a simple stress-strain curve. A part can break well below its ultimate tensile strength if it's loaded repeatedly, held at high temperature, or struck suddenly. Understanding these failure modes is essential for designing components that survive real-world service conditions.

Fatigue Failure Mechanisms

Fatigue Crack Initiation and Propagation

Fatigue failure happens when a material is subjected to repeated or fluctuating stresses below its ultimate tensile strength. Over many cycles, microscopic cracks form and grow until the part breaks. The process unfolds in two stages:

1. Crack initiation. Tiny cracks nucleate at stress concentration sites: surface scratches, machining marks, inclusions, or grain boundaries. These are the weakest links where localized plastic deformation

accumulates cycle after cycle.

2. Crack propagation. Once a crack has formed, each loading cycle opens and extends it slightly. The crack grows steadily until the remaining cross-section is too small to carry the applied load. At that point, final fracture occurs suddenly.

The dangerous part of fatigue is that both stages can happen with no visible warning. The component looks fine right up until it doesn't.

Factors Affecting Fatigue Life

Fatigue life is the number of stress cycles a component can withstand before failure. Many variables influence it:

- Stress amplitude and mean stress. Higher stress amplitude shortens fatigue life dramatically. A tensile mean stress also reduces life compared to fully reversed loading.
- Stress ratio $R = \frac{\sigma_{min}}{\sigma_{max}}$. This describes the nature of the loading cycle. An $R = -1$ cycle is fully reversed (equal tension and compression). As R increases toward +1, the loading becomes more tensile-dominated, which generally reduces fatigue life.
- Material properties. Higher strength and ductility both help, though they trade off against each other. Notch sensitivity matters too: a high-strength steel may be more sensitive to small defects than a tougher, lower-strength alloy.
- Surface condition. Rough surfaces, corrosion pits, and sharp geometric features all act as crack initiation sites. Surface treatments like shot peening (which introduces compressive residual stresses) and case hardening can significantly extend fatigue life.
- Environment. Elevated temperature, corrosive media, and fretting (small relative motion between contacting surfaces) all accelerate fatigue damage.
- Loading frequency and load history. Variable-amplitude loading is harder to analyze than constant-amplitude loading and requires cumulative damage methods.

S-N Curve Interpretation

S-N Curve Characteristics

An S-N curve (also called a Wöhler curve) plots stress amplitude S on the vertical axis against the number of cycles to failure N on the horizontal axis. Both axes typically use a logarithmic scale.

- For ferrous metals (steels, cast irons), the curve often flattens out at high cycle counts. The stress level where it flattens is the endurance limit (or fatigue limit): below this stress, the material can theoretically survive an infinite number of cycles. For most steels, the endurance limit is roughly 40–50% of the ultimate tensile strength.
- For non-ferrous metals (aluminum, copper alloys), the S-N curve keeps sloping downward and never truly flattens. These materials don't have a true endurance limit, so engineers define a fatigue strength at a specified number of cycles (commonly 10^7 or 10^8).

S-N curves are generated from laboratory tests on polished, carefully prepared specimens under controlled conditions. Real components differ from lab specimens, so modifying factors are needed.

Fatigue Life Prediction Using S-N Curves

To use an S-N curve for design:

1. Determine the stress amplitude the component will experience in service.
2. Apply modifying factors to account for differences between the lab specimen and the real part. Common factors include surface finish, size, reliability level, and stress concentration (notch factor K_f).
3. Locate the adjusted stress amplitude on the S-N curve and read across to find the predicted number of cycles to failure.

For variable-amplitude loading, where the stress level changes over time, use Miner's rule (Palmgren-Miner linear damage hypothesis). The idea is straightforward: each block of cycles at a given stress level uses up a fraction of the fatigue life. Failure is predicted when the sum of all damage fractions reaches 1:

$$\sum \frac{n_i}{N_i} = 1$$

where n_i is the number of cycles applied at stress level i and N_i is the number of cycles to failure at that stress level from the S-N curve. Miner's rule is simple but approximate; it ignores load sequence effects.

For components under combined static and cyclic loading, the Goodman diagram and Soderberg diagram are used. These plot alternating stress against mean stress and define a safe operating region. The Goodman line connects the endurance limit on the alternating stress axis to the ultimate tensile strength on the mean stress axis. The Soderberg line is more conservative, using the yield strength instead.

Creep Behavior of Materials

Creep Stages and Mechanisms

Creep is time-dependent permanent deformation that occurs under sustained load at elevated temperatures. As a general guideline, creep becomes significant above roughly $0.4 T_m$, where T_m is the material's absolute melting temperature in Kelvin. For steel ($T_m \approx 1800 \text{ K}$), that's about 720 K (roughly 450°C). For lead ($T_m \approx 600 \text{ K}$), creep occurs near room temperature.

A typical creep curve (strain vs. time at constant stress and temperature) has three distinct stages:

1. Primary creep. The strain rate starts high and gradually decreases. The material is strain hardening, which makes further deformation progressively more difficult.
2. Secondary creep (steady-state creep). The strain rate reaches a constant, minimum value. Strain hardening and thermal recovery processes are in balance. This stage usually occupies the longest portion of the creep life, and the steady-state creep rate is the most important parameter for design.
3. Tertiary creep. The strain rate accelerates as internal voids nucleate and grow, the cross-section necks down, and the material weakens. This stage ends in rupture.

The dominant creep deformation mechanisms depend on stress and temperature:

- Dislocation glide and climb at higher stresses
- Grain boundary sliding at intermediate conditions
- Diffusional flow at lower stresses and higher temperatures: Nabarro-Herring creep (diffusion through the grain interior) and Coble creep (diffusion along grain boundaries)

Factors Affecting Creep and Creep-Resistant Materials

Creep rate increases with higher temperature and higher applied stress. Other factors include grain size (larger grains reduce Coble creep by decreasing grain boundary area) and environmental degradation like oxidation.

Creep-resistant materials are engineered to slow dislocation motion at high temperatures. Nickel-based superalloys, widely used in jet engine turbine blades, are a prime example. Their strengthening strategies include:

- Solid solution strengthening. Alloying elements (Mo, W, Re) distort the lattice and impede dislocation movement.
- Precipitation hardening. Stable precipitates (such as the γ' phase in nickel superalloys) act as obstacles to dislocation motion and remain effective at high temperatures.
- Dispersion strengthening. Fine, thermally stable oxide particles (as in oxide dispersion strengthened, or ODS, alloys) pin dislocations and grain boundaries.

Impact Loading and Toughness

Effects of Impact Loading on Materials

Impact loading is the application of a sudden, high-intensity force, producing very high strain rates. Think of a hammer blow, a car crash, or a dropped component. Material behavior under impact differs from behavior under slow (quasi-static) loading in important ways:

- Increased yield strength. At high strain rates, many materials exhibit higher yield and flow stresses because dislocations don't have time to move as easily.
- Reduced ductility. The same rate effect that raises strength tends to reduce the material's ability to deform plastically before fracture.
- Ductile-to-brittle transition. Many BCC metals (including structural steels) undergo a ductile-to-brittle transition as temperature decreases or strain rate increases. The ductile-to-brittle transition temperature (DBTT) is the temperature range where fracture behavior shifts from energy-absorbing ductile tearing to low-energy brittle cleavage. This is critical for structures operating in cold environments (ships, pipelines, bridges).

FCC metals like aluminum and copper generally do not exhibit a sharp DBTT, which is one reason aluminum alloys are used in cryogenic applications.

Toughness and Impact Testing

Toughness is a material's ability to absorb energy and deform plastically without fracturing. It combines both strength and ductility. A material can be strong but brittle (ceramics) or ductile but weak (pure lead); tough materials are both reasonably strong and ductile.

The two standard impact tests are:

- Charpy test. A notched beam specimen is supported at both ends. A pendulum hammer strikes the specimen on the side opposite the notch. The energy absorbed during fracture is measured from the height the pendulum reaches after breaking the specimen.
- Izod test. The specimen is clamped as a vertical cantilever with the notch facing the pendulum strike. It also measures absorbed energy but uses a different specimen geometry and loading configuration.

Both tests produce a single number: absorbed energy in joules (or ft-lbs). By running tests at multiple temperatures, you can plot absorbed energy vs. temperature and identify the DBTT.

For more rigorous analysis, fracture mechanics provides quantitative tools. The stress intensity factor K characterizes the stress field near a crack tip, and the fracture toughness K_{IC} is the critical value at which a crack propagates unstably. The crack tip opening displacement (CTOD) is an alternative measure used especially for materials with significant plasticity at the crack tip. These parameters allow engineers to determine whether an existing crack in a component is safe or will lead to failure under a given load.

Unit 9 – Axial Loading & Deformation

9.1 Stress and strain in axially loaded members

Stress and Strain in Axially Loaded Members

Fundamental Concepts

When you apply a force along the length of a structural member, internal forces develop throughout the material to resist that load. Stress and strain are the two quantities that let you quantify what's happening inside.

Stress (σ) is the internal force per unit area that develops inside a material in response to an applied load. In axially loaded members, this stress acts perpendicular to the cross-section (meaning it pushes or pulls straight through the cut face of the member).

Strain (ϵ) is the deformation per unit length. It tells you how much the member stretches or shortens relative to its original size.

An axially loaded member is any structural element where the load acts along the longitudinal axis. This produces either:

- Tension (pulling force → member elongates)
- Compression (pushing force → member shortens)

The cross-sectional area of the member directly affects how stress distributes internally. Whether the cross-section is circular, rectangular, or irregular, the area is what you divide the force by to get stress.

Types of Stress and Strain

- Tensile stress and strain develop when a pulling force elongates the member. Think of stretching a rubber band: the material gets longer and thinner.
- Compressive stress and strain develop when a pushing force shortens the member, like compressing a spring.
- Shear stress and strain can appear even in "axially loaded" members if the force isn't perfectly aligned with the longitudinal axis. This commonly shows up at bolted or pinned connections.
- Thermal stresses and strains arise from temperature changes that cause expansion or contraction. These aren't from applied loads at all, but they still produce real deformation. Expansion joints in bridges exist specifically to accommodate this.

Calculating Stress and Strain

Stress Calculation

The formula for normal stress in an axially loaded member is straightforward:

$$\sigma = \frac{F}{A}$$

where F is the internal axial force and A is the cross-sectional area.

Common units include pascals (Pa), megapascals (MPa), and pounds per square inch (psi). For reference, $1 \text{ MPa} = 1 \text{ N/mm}^2$.

Two things to watch for:

- The stress distribution is uniform across the cross-section only if the force is applied through the centroid and the cross-section is constant. This is the assumption you'll use for most problems in this course.
- Stress concentrations occur at sudden geometric changes like holes, notches, or sharp corners. At these locations, the local stress can be several times higher than the average F/A value.

Strain Calculation

Normal strain is calculated as:

$$\varepsilon = \frac{\Delta L}{L}$$

where ΔL is the change in length and L is the original (unloaded) length.

- Strain is dimensionless since it's length divided by length. It's often expressed as a decimal (e.g., 0.002), a percentage (0.2%), or in microstrain ($2000 \mu\text{strain}$).
- A positive ΔL means elongation (tension), and a negative ΔL means contraction (compression).
- L is always the original length before any load is applied. This is your reference.

Stress-Strain Relationship for Elastic Materials

Hooke's Law

For most engineering materials loaded below a certain threshold, stress is directly proportional to strain. This is Hooke's Law:

$$\sigma = E \cdot \varepsilon$$

The constant E is the modulus of elasticity (also called Young's modulus). It's a material property that quantifies stiffness. Some typical values:

- Steel: $E \approx 200 \text{ GPa}$ (29,000 ksi)
- Aluminum: $E \approx 70 \text{ GPa}$ (10,000 ksi)
- Concrete (compression): $E \approx 25\text{--}30 \text{ GPa}$

A higher modulus means the material is stiffer and deforms less under the same stress.

The elastic limit is the maximum stress a material can handle and still return to its original shape when unloaded. Beyond this point, permanent (plastic) deformation begins, and the material has yielded.

Stress-Strain Curve

Within the elastic region, the stress-strain curve is a straight line. The slope of that line equals E . This is why a steeper curve means a stiffer material.

Key features you can read from a full stress-strain curve:

- Proportional limit: where the curve stops being linear

- Yield strength: where significant plastic deformation begins
- Ultimate strength: the maximum stress the material can sustain
- Fracture point: where the material breaks

The area under the stress-strain curve up to fracture represents the material's toughness, which is the total energy it can absorb before breaking.

Deformation of Axially Loaded Members

Deformation Calculation

By combining the stress and strain formulas with Hooke's Law, you get the axial deformation formula:

$$\Delta L = \frac{FL}{AE}$$

This is derived in three steps:

1. Start with stress: $\sigma = \frac{F}{A}$
2. Apply Hooke's Law: $\epsilon = \frac{\sigma}{E} = \frac{F}{AE}$
3. Solve for deformation: $\Delta L = \epsilon \cdot L = \frac{FL}{AE}$

This is one of the most-used equations in this unit. Make sure you can identify F , L , A , and E for each segment of a member.

For members with multiple segments (different forces, areas, or materials along the length), calculate ΔL for each segment separately and sum them:

$$\Delta L_{total} = \sum \frac{F_i L_i}{A_i E_i}$$

Keep your sign convention consistent: positive for tension/elongation, negative for compression/shortening.

Thermal Effects on Deformation

Temperature changes cause materials to expand or contract independently of any applied load. The thermal deformation is:

$$\Delta L_{thermal} = \alpha \cdot L \cdot \Delta T$$

where α is the coefficient of thermal expansion (a material property) and ΔT is the temperature change.

Some typical α values:

- Steel: $\alpha \approx 12 \times 10^{-6} / ^\circ\text{C}$
- Aluminum: $\alpha \approx 23 \times 10^{-6} / ^\circ\text{C}$

Aluminum expands nearly twice as much as steel for the same temperature change. This matters in structures exposed to large temperature swings (bridges, pipelines, aircraft components). If the member is free to expand, thermal deformation produces no stress. But if the member is constrained (fixed at both ends, for example), the prevented expansion generates internal thermal stress, which you'd calculate using the deformation compatibility approach.

9.2 Statically indeterminate axial loading problems

Statically indeterminate axial loading problems occur when there are more unknown forces than equilibrium equations. These situations require additional compatibility equations to solve, considering how connected parts deform together. This topic builds on the basics of axial loading, adding complexity to structural analysis.

Understanding statically indeterminate systems is crucial for designing safe and efficient structures. It involves balancing equilibrium, compatibility, and material properties to determine internal forces and deformations. This knowledge helps engineers create structures that can handle complex loading scenarios and thermal effects.

Statically Indeterminate Axial Loading

Characteristics and Identification

- Statically indeterminate axial loading problems occur when the number of unknown forces or reactions exceeds the number of available equilibrium equations
- The degree of static indeterminacy represents the difference between the number of unknown forces and the number of equilibrium equations
- Characteristics of statically indeterminate axial loading problems include:
 - Redundant supports or members that introduce additional constraints
 - Multiple load paths for transferring the applied loads throughout the system
 - Deformation compatibility requirements between connected members to ensure consistent behavior
- Common examples of statically indeterminate axial loading systems include:
 - Trusses with more members than necessary for stability (redundant members)
 - Frames with fixed or continuous connections that provide additional restraint
 - Beams with intermediate supports or constraints that limit deformation

Force-Deformation Relationships

- Force-deformation relationships, such as Hooke's law, relate the unknown forces to the deformations in statically indeterminate systems
- Hooke's law states that the axial stress (σ) is directly proportional to the axial strain (ϵ) within the elastic limit, expressed as $\sigma = E\epsilon$, where E is the modulus of elasticity
- The modulus of elasticity (E) is a material property that describes the stiffness and resistance to deformation under axial loading
- The force-deformation relationships enable the formulation of compatibility equations that relate the deformations of connected members

Solving Statically Indeterminate Axial Loading

Equilibrium Equations

- Solving statically indeterminate axial loading problems requires the use of equilibrium equations derived from the principles of statics
- Force equilibrium equations ensure that the sum of forces in each direction equals zero ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$)
- Moment equilibrium equations ensure that the sum of moments about any point equals zero ($\Sigma M = 0$)
- Equilibrium equations alone are insufficient to solve statically indeterminate problems due to the presence of redundant forces or reactions

Compatibility Equations

- Compatibility equations ensure that the deformations of connected members are consistent and compatible with each other
- Deformation compatibility requires displacement continuity at the connections between members, ensuring that the connected members deform together without separation or overlap
- Strain compatibility requires strain continuity at the interfaces between members, ensuring that the strains in connected members are equal at their shared boundaries
- The number of compatibility equations required is equal to the degree of static indeterminacy, providing additional equations to solve for the unknown forces and deformations

Solution Process

- The solution process for statically indeterminate axial loading problems involves setting up a system of simultaneous equations using both equilibrium and compatibility equations
- The unknown forces and deformations are treated as variables in the system of equations
- The force-deformation relationships (e.g., Hooke's law) are used to express the compatibility equations in terms of the unknown forces and deformations
- The system of equations is solved simultaneously to determine the values of the unknown forces and deformations in the statically indeterminate system
- Techniques such as matrix methods or step-by-step elimination can be employed to solve the system of equations efficiently

Stress and Deformation in Indeterminate Systems

Stress Analysis

- Once the unknown forces in a statically indeterminate axial loading system are determined, the stress in each member can be analyzed
- Axial stress (σ) is calculated using the formula: $\sigma = P/A$, where P is the axial force in the member and A is the cross-sectional area

- The axial force (P) is obtained from the solution of the statically indeterminate problem, considering the equilibrium and compatibility equations
- The cross-sectional area (A) is a geometric property of the member that depends on its shape and dimensions
- Stress analysis helps in assessing the strength and safety of the members under the applied loads, ensuring that the stresses remain within the allowable limits of the material

Deformation Analysis

- Deformation analysis in statically indeterminate axial loading systems determines the elongation or contraction of the members due to the applied loads
- Axial strain (ϵ) is calculated using Hooke's law: $\epsilon = \sigma/E$, where σ is the axial stress and E is the modulus of elasticity of the material
- Deformation (δ) in each member is calculated using the formula: $\delta = (P \times L) / (A \times E)$, where P is the axial force, L is the length of the member, A is the cross-sectional area, and E is the modulus of elasticity
- The principle of superposition can be applied to determine the total deformation in a member subjected to multiple load cases or thermal effects
- Compatibility of deformations at the connections between members must be ensured to maintain the integrity and stability of the structure

Serviceability Considerations

- Stress and deformation analysis helps in assessing the serviceability of the statically indeterminate axial loading system
- Serviceability refers to the ability of the structure to perform its intended function without excessive deformations or displacements
- Excessive deformations can lead to issues such as misalignment, loss of functionality, or aesthetic concerns
- Serviceability limits are often specified in design codes and standards to ensure that the structure remains functional and visually acceptable under the expected loading conditions
- Stiffness requirements may be imposed to limit the deformations within acceptable ranges, considering factors such as the type of structure, occupancy, and user comfort

Thermal Stresses in Indeterminate Systems

Thermal Strain and Stress

- Thermal stresses arise in statically indeterminate axial loading systems when the members are subjected to temperature changes and are restrained from free expansion or contraction
- The thermal strain (ϵ_T) in a member is calculated using the formula: $\epsilon_T = \alpha \times \Delta T$, where α is the coefficient of thermal expansion and ΔT is the temperature change
- The coefficient of thermal expansion (α) is a material property that describes the change in length per unit length for a unit change in temperature

- The thermal stress (σ_T) in a restrained member is calculated using the formula: $\sigma_T = E \times \epsilon_T = E \times \alpha \times \Delta T$, where E is the modulus of elasticity
- Thermal stresses develop in statically indeterminate systems due to the restraint provided by the redundant supports or members, preventing free thermal expansion or contraction

Compatibility Equations with Thermal Effects

- In the presence of thermal effects, the compatibility equations must account for both the mechanical deformations caused by the applied loads and the thermal deformations due to temperature changes
- The total strain in a member is the sum of the mechanical strain (ϵ_M) and the thermal strain (ϵ_T)
- The compatibility equations are modified to include the thermal strain terms, ensuring that the total deformations of connected members are compatible
- The modified compatibility equations are solved simultaneously with the equilibrium equations to determine the unknown forces and deformations in the presence of thermal effects

Strategies for Mitigating Thermal Stresses

- Thermal stresses can be significant in structures exposed to large temperature variations or when materials with different thermal expansion coefficients are connected
- Strategies to mitigate thermal stresses include:
 - Providing expansion joints or sliding supports to allow for thermal movements and reduce restraint
 - Using materials with similar thermal expansion coefficients to minimize differential thermal strains
 - Designing the structure to accommodate thermal deformations without inducing excessive stresses, such as allowing for flexibility or providing adequate clearances
 - Proper consideration of thermal effects during the design and analysis of statically indeterminate axial loading systems helps in preventing excessive thermal stresses and ensuring the structural integrity under varying temperature conditions

9.3 Thermal stresses and deformations

Thermal Stresses and Deformations

When materials heat up or cool down, they naturally want to expand or contract. If they're free to do so, no stress develops. But if something restrains that movement, internal stresses build up. These thermal stresses are a major concern in axial loading problems because temperature swings can push a member into compression or tension without any external force being applied.

This section covers how to calculate thermal deformations and stresses, how to combine them with mechanical loads, and how to handle both determinate and indeterminate systems.

Thermal Stress in Axial Members

Concept and Effects

Thermal stress is the internal stress that develops when a material tries to expand or contract due to a temperature change but is prevented from doing so by constraints.

Think about a steel rod bolted between two rigid walls. If the temperature rises, the rod wants to get longer, but the walls won't let it. That blocked expansion creates compressive stress inside the rod. If the temperature drops, the rod wants to shorten, the walls resist, and tensile stress develops.

The key principle: if an axially loaded member is free to expand or contract, thermal deformation occurs but no thermal stress develops. If the member is restrained, thermal stress arises because the deformation is partially or fully prevented.

Factors Influencing Thermal Stress

- Coefficient of thermal expansion (α): Materials with higher α values expand more per degree of temperature change, leading to greater thermal stresses when restrained.
- Modulus of elasticity (E): Stiffer materials resist deformation more strongly, so the same blocked expansion produces higher stress in a stiffer material.
- Temperature change (ΔT): Larger temperature swings produce proportionally larger thermal effects.
- Degree of restraint:
 - Fully restrained members (both ends fixed) experience the maximum thermal stress
 - Partially restrained members experience reduced thermal stress
 - Unrestrained members (free to move) experience zero thermal stress

Real-world examples include steel bridge girders that expand and contract with daily temperature cycles, and concrete pavements that crack under seasonal temperature variations. Expansion joints in bridges and highways exist specifically to manage these effects.

Calculating Thermal Stresses and Deformations

Thermal Deformation Calculation

When a member is free to change length, the thermal deformation is:

$$\delta_{thermal} = \alpha \times L \times \Delta T$$

- α = coefficient of thermal expansion
- L = original length of the member
- ΔT = temperature change (positive for heating, negative for cooling)

Example: A 2 m long aluminum rod ($\alpha = 23 \times 10^{-6}/^{\circ}C$) is heated by $80^{\circ}C$.

$$\delta_{thermal} = 23 \times 10^{-6} \times 2 \times 80 = 0.00368 \text{ m} = 3.68 \text{ mm (expansion)}$$

Thermal Stress Calculation

When a member is fully restrained (both ends fixed against a rigid support), the thermal deformation is completely blocked. The resulting stress is:

$$\sigma_{thermal} = -E \times \alpha \times \Delta T$$

The negative sign reflects the sign convention: a positive ΔT (heating) causes the member to push against its restraints, producing compressive (negative) stress. A negative ΔT (cooling) produces tensile (positive) stress.

Example: A fully restrained steel rod ($E = 200 \text{ GPa}$, $\alpha = 12 \times 10^{-6}/^\circ\text{C}$) undergoes a temperature increase of 50°C .

$$\sigma_{thermal} = -(200 \times 10^3) \times (12 \times 10^{-6}) \times 50 = -120 \text{ MPa (compressive)}$$

Internal Force from Thermal Restraint

When you need the axial force that develops in a restrained member:

$$\Delta F = -E \times A \times \alpha \times \Delta T$$

where A is the cross-sectional area.

Example: A fully restrained copper bar ($E = 110 \text{ GPa}$, $\alpha = 17 \times 10^{-6}/^\circ\text{C}$, $A = 500 \text{ mm}^2$) cools by 60°C .

$$\Delta F = -(110 \times 10^3)(500)(17 \times 10^{-6})(-60) = 56,100 \text{ N} = 56.1 \text{ kN (tension)}$$

The positive result confirms tension: cooling makes the bar want to shrink, but the restraints pull it back, creating a tensile force.

Combined Mechanical and Thermal Loads

Superposition of Stresses and Deformations

When both mechanical forces and temperature changes act on a member, you find the total response by adding the two effects together (superposition):

$$\sigma_{total} = \sigma_{mechanical} + \sigma_{thermal}$$

$$\delta_{total} = \delta_{mechanical} + \delta_{thermal}$$

This works because we're assuming linear elastic behavior, where effects can be added independently.

Calculation Steps

1.

Calculate mechanical stress and deformation from the applied force: $\sigma_{mechanical} = \frac{F}{A}$ - $\delta_{mechanical} = \frac{F \times L}{E \times A}$

2.

Calculate thermal stress and deformation from the temperature change: $\sigma_{thermal} = -E \times \alpha \times \Delta T$ (if restrained) - $\delta_{thermal} = \alpha \times L \times \Delta T$ (if free to deform)

3.

Add them together, keeping careful track of signs (tension = positive, compression = negative).

Worked Example

A 1.5 m long steel rod ($E = 200 \text{ GPa}$, $A = 800 \text{ mm}^2$, $\alpha = 12 \times 10^{-6}/^\circ\text{C}$) carries an axial compressive force of 120 kN and also undergoes a temperature increase of 40°C . The rod is free to deform axially (not fixed at both ends), so we're looking at total deformation.

Mechanical effects:

$$\sigma_{mechanical} = \frac{120,000}{800} = 150 \text{ MPa (compressive)}$$

$$\delta_{mechanical} = \frac{120,000 \times 1500}{200,000 \times 800} = 1.125 \text{ mm (shortening)}$$

Thermal deformation (free to expand):

$$\delta_{thermal} = 12 \times 10^{-6} \times 1500 \times 40 = 0.72 \text{ mm (expansion)}$$

Total deformation:

$$\delta_{total} = -1.125 + 0.72 = -0.405 \text{ mm}$$

The rod still shortens overall, but the thermal expansion partially offsets the mechanical compression. Note that when the rod is free to expand, the thermal deformation doesn't generate thermal stress; it only contributes to the total change in length.

Thermal Stress in Statically Determinate vs. Indeterminate Systems

Statically Determinate Systems

In a statically determinate system, equilibrium equations alone are enough to find all reactions. A classic example is a rod fixed at one end with the other end free.

The critical point: in a determinate system with a uniform temperature change, the member is free to expand or contract. No thermal stress develops. You only get thermal deformation.

Example: A 2 m aluminum rod ($E = 70 \text{ GPa}$, $\alpha = 23 \times 10^{-6}/^{\circ}\text{C}$, $A = 600 \text{ mm}^2$) is fixed at one end and free at the other. It heats up by 60°C .

Since the free end is unconstrained, the rod simply expands:

$$\delta_{thermal} = 23 \times 10^{-6} \times 2 \times 60 = 2.76 \text{ mm (expansion at the free end)}$$

No reaction force develops at the fixed end beyond what's needed for equilibrium (which is zero, since no external load is applied). The thermal stress is zero.

Statically Indeterminate Systems

A statically indeterminate system has more constraints than equilibrium equations can solve alone. A rod fixed at both ends is the classic case. You need an additional compatibility equation that enforces a geometric condition (e.g., the total change in length must be zero).

Here's the step-by-step approach:

1. Identify the redundant reaction. For a rod fixed at both ends, one of the support reactions is the unknown.
2. Write the compatibility equation. For rigid walls at both ends: $\delta_{total} = \delta_{mechanical} + \delta_{thermal} = 0$
3. Express deformations in terms of the unknown force. The mechanical deformation depends on the unknown reaction force F : $\delta_{mechanical} = \frac{F \times L}{E \times A}$
4. Solve for the unknown force.
5. Calculate stresses from the force you found.

Example: A 3 m copper rod ($E = 110 \text{ GPa}$, $\alpha = 17 \times 10^{-6}/^{\circ}\text{C}$, $A = 400 \text{ mm}^2$) is fixed at both ends and cools by 50°C .

Step 1: The rod wants to shrink, but both walls prevent it. A tensile reaction force F develops.

Step 2: Compatibility requires zero net deformation:

$$\delta_{mechanical} + \delta_{thermal} = 0$$

Step 3: Substitute the deformation expressions:

$$\frac{F \times L}{E \times A} + \alpha \times L \times \Delta T = 0$$

Step 4: Solve for F :

$$F = -E \times A \times \alpha \times \Delta T$$

$$F = -(110 \times 10^3)(400)(17 \times 10^{-6})(-50) = 37,400 \text{ N} = 37.4 \text{ kN (tension)}$$

Step 5: The stress in the rod:

$$\sigma = \frac{F}{A} = \frac{37,400}{400} = 93.5 \text{ MPa (tension)}$$

You can verify this matches the direct thermal stress formula for a fully restrained member:

$$\sigma = E \times \alpha \times |\Delta T| = 110,000 \times 17 \times 10^{-6} \times 50 = 93.5 \text{ MPa}$$

Notice that in the indeterminate case, the mechanical deformation and thermal deformation cancel each other out (total deformation is zero), but the stress is very real. This is why indeterminate systems under temperature changes require careful analysis: the structure looks like nothing moved, but significant internal stresses can be present.

Unit 10 – Torsion

10.1 Torsion of circular shafts

Torsion of circular shafts deals with the stresses and deformations that develop when twisting moments are applied to cylindrical components. This is fundamental to designing anything that transmits rotational power: drive shafts, axles, drill bits, and turbine shafts all depend on torsion analysis. The core challenge is predicting how much a shaft twists and whether the internal shear stresses stay within safe limits.

Circular shafts under torsion

Characteristics and components

Circular shafts exist to transmit power and torque between mechanical components like gears, pulleys, and rotors. When a torsional load (a twisting moment) is applied, the shaft rotates about its longitudinal axis, developing internal shear stresses and angular deformation along its length.

Two categories of properties govern how a shaft responds to torsion:

- Geometric properties: diameter, length, and cross-sectional shape. These determine the polar moment of inertia, which controls how effectively the shaft resists twisting.
- Material properties: the shear modulus G (also called the modulus of rigidity) quantifies stiffness against shear deformation, while the shear yield strength sets the stress limit before permanent deformation begins.

Boundary conditions also matter. How the shaft is supported (fixed at one end, fixed at both ends, etc.) and where torques are applied directly affect the internal torque distribution and resulting stresses.

Torsional behavior and failure

Torsional loading produces shear strains throughout the shaft. Shear strain γ relates to shear stress τ through the shear modulus:

$$\gamma = \tau/G$$

This means a stiffer material (higher G) develops less strain for the same stress level.

Design requirements typically limit both the maximum shear stress and the angle of twist. Excessive twist can cause misalignment between connected components or introduce harmful vibrations. The maximum allowable shear stress is set by the material's shear yield strength, which is typically about 50–60% of the tensile yield strength (this follows from yield criteria like Tresca or von Mises).

Three failure modes to know:

- Yielding: permanent deformation occurs when shear stress exceeds the shear yield strength
- Fatigue: progressive damage under repeated cyclic loading, even at stresses below the yield strength
- Fracture: sudden failure when stress exceeds the ultimate shear strength

Shear stress distribution in torsion

Shear stress variation

One of the key results in torsion theory is that shear stress varies linearly from zero at the center of the shaft to a maximum at the outer surface. This comes from the assumption that plane cross-sections remain plane and radii remain straight during twisting (valid for circular cross-sections within the elastic range).

The torsion formula gives the shear stress at any radial distance ρ from the center:

$$\tau = \frac{T \cdot \rho}{J}$$

where: - T = applied internal torque - ρ = radial distance from the shaft center (use r for the outer radius to find maximum stress) - J = polar moment of inertia of the cross-section

For a solid circular shaft of radius r :

$$J = \frac{\pi r^4}{2}$$

Or equivalently in terms of diameter d :

$$J = \frac{\pi d^4}{32}$$

For a hollow circular shaft with outer radius r_o and inner radius r_i :

$$J = \frac{\pi}{2}(r_o^4 - r_i^4)$$

The maximum shear stress always occurs at the outer surface where $\rho = r$:

$$\tau_{max} = \frac{T \cdot r}{J}$$

Factors affecting shear stress

- Applied torque: a larger twisting moment produces proportionally larger shear stress everywhere in the cross-section.
- Shaft geometry: since J depends on the fourth power of the radius, even a modest increase in diameter dramatically increases torsional resistance. Doubling the diameter of a solid shaft increases J by a factor of 16.
- Material properties: the shear modulus G does not appear directly in the stress formula, so for a given torque and geometry, the shear stress is the same regardless of material. However, G determines how much the shaft deforms at that stress level, which influences whether the elastic torsion formula remains valid.

Angle of twist in shafts

Angular deformation

The angle of twist θ measures the relative rotation between two cross-sections separated by a length L along the shaft. For a shaft with constant torque, constant cross-section, and uniform material:

$$\theta = \frac{T \cdot L}{G \cdot J}$$

where: - T = internal torque (constant along the length considered) - L = length of the shaft segment - G = shear modulus of the material - J = polar moment of inertia

The product GJ is called the torsional rigidity (or torsional stiffness per unit length). It captures how both material and geometry resist twisting.

If the torque or cross-section varies along the shaft, you need to either sum contributions from individual segments or integrate:

$$\theta = \int_0^L \frac{T(x)}{G \cdot J(x)} dx$$

For a shaft with multiple segments of constant T , G , and J , you simply add up the twist from each segment:

$$\theta_{total} = \sum \frac{T_i \cdot L_i}{G_i \cdot J_i}$$

Factors affecting angle of twist

- Applied torque: twist is directly proportional to T
- Shaft length: longer shafts twist more for the same torque and cross-section
- Shear modulus: a stiffer material (higher G) twists less. Steel ($G \approx 80$ GPa) twists roughly three times less than aluminum ($G \approx 26$ GPa) for the same shaft geometry and torque.
- Cross-sectional geometry: increasing the polar moment of inertia reduces twist. Because J scales with r^4 , increasing shaft diameter is the most effective geometric change you can make.

Torque, shear stress, and geometry

Interdependence in torsional loading

The torsion formula $\tau = T\rho/J$ and the twist formula $\theta = TL/GJ$ together capture the full picture of torsional loading. Both depend on the polar moment of inertia J , which links geometry to stress and deformation. A design change that reduces stress (like increasing diameter) will simultaneously reduce the angle of twist.

You can also combine the two formulas to relate shear stress directly to angle of twist:

$$\tau = G \cdot \rho \cdot \frac{\theta}{L}$$

This shows that shear stress at any point depends on how much the shaft is actually twisting per unit length (θ/L), scaled by the material stiffness and the radial position.

Design considerations

Solid vs. hollow shafts: Hollow shafts are more weight-efficient because the material near the center of a solid shaft contributes very little to J (and carries very little stress). Removing that core material barely reduces J but significantly reduces weight. For example, a hollow shaft with an inner radius equal to half the outer radius retains about 94% of the solid shaft's J while using only 75% of the material.

Diameter effects: Since J scales with the fourth power of the radius, small increases in diameter have outsized effects. Increasing the radius by just 20% increases J (and therefore torsional resistance) by about 107%.

Material selection: Common shaft materials include steel (high strength and stiffness), aluminum (lightweight but lower stiffness), titanium (high strength-to-weight ratio), and composites (tailorable properties). The choice depends on the balance between strength, stiffness, weight, cost, and environmental conditions.

Factor of safety: Real designs include a factor of safety (typically 1.5 to 3) to account for uncertainties in loading, material variability, and manufacturing tolerances. The required shaft dimensions are sized so that:

$$\tau_{max} \leq \frac{\tau_{yield}}{n}$$

where n is the factor of safety. Higher consequences of failure (e.g., aircraft vs. hand tools) call for larger safety factors.

10.2 Power transmission and torsional deflection

Power Transmission and Torsional Deflection

Power transmission through shafts is the process of moving mechanical energy from a source (like a motor) to a load (like a machine) via rotational motion. Understanding the relationship between power, torque, and shaft deflection is essential for sizing shafts correctly and preventing mechanical failure.

Power Transmission in Shafts

Fundamentals of Power Transmission

A circular shaft under torsion develops a shear stress distribution that varies linearly from zero at the center to a maximum at the outer surface. This non-uniform distribution is why the outer fibers of a shaft are the most critical for design.

The angle of twist a shaft experiences depends on four things:

- The applied torque (more torque = more twist)
- The shaft's length (longer shaft = more twist)
- The shear modulus of the material, G (stiffer material = less twist)
- The polar moment of inertia of the cross-section, J (larger cross-section = less twist)

These relationships are captured in the angle of twist formula covered below.

Calculating Power Transmission

The core relationship is straightforward: power equals torque times angular velocity.

$$P = T \cdot \omega$$

where P is power in watts, T is torque in N·m, and ω is angular velocity in rad/s.

Since rotational speed is usually given in RPM rather than rad/s, you'll need to convert:

1.

Convert RPM to angular velocity: $\omega = \frac{2\pi N}{60}$ where N is the rotational speed in RPM.

2.

Find torque from power and speed: Rearranging $P = T \cdot \omega$ gives: $T = \frac{P}{\omega} = \frac{60P}{2\pi N}$

3.

Check the shaft's capacity. Once you know the torque, verify that the shaft's material and geometry can handle the resulting torsional stress and deflection without failure.

Always keep your units consistent. If power is in watts, torque comes out in N·m and angular velocity in rad/s. If power is given in horsepower, convert first (1 hp = 745.7 W).

Torsional Deflection of Shafts

Torsional deflection is the angular deformation (twisting) a shaft undergoes when torque is applied. The angle of twist is given by:

$$\phi = \frac{TL}{GJ}$$

where ϕ is the angle of twist in radians, T is the applied torque, L is the shaft length, G is the shear modulus, and J is the polar moment of inertia.

This equation tells you that twist increases with torque and length, and decreases with stiffer materials and larger cross-sections.

Excessive torsional deflection is a real design concern. Even if stresses remain below the yield point, too much twist can cause:

- Misalignment of connected components (gears, couplings)
- Vibration and noise in the system
- Reduced efficiency and premature wear

Shaft Stiffness and Rigidity

Torsional Stiffness

Torsional stiffness measures how much a shaft resists twisting per unit of applied torque. It's defined as:

$$k_t = \frac{T}{\phi} = \frac{GJ}{L}$$

where k_t is in N·m/rad. A higher value means less twist for the same torque.

You'll need the polar moment of inertia for the cross-section:

•

Solid circular shaft: $J = \frac{\pi d^4}{32}$

•

Hollow circular shaft: $J = \frac{\pi(D^4 - d^4)}{32}$ where D is the outer diameter and d is the inner diameter.

Notice that J depends on diameter to the fourth power. Even a small increase in diameter significantly increases stiffness.

Torsional Rigidity

Torsional rigidity is the product GJ , and it represents the shaft's inherent resistance to torsion independent of length. This is the property you're comparing when choosing between shaft designs.

Two practical comparisons:

- A steel shaft ($G = 80$ GPa) vs. an aluminum shaft ($G = 26$ GPa) of the same diameter: the steel shaft has roughly 3 times the torsional rigidity, so it twists about one-third as much under the same torque.

- A solid shaft vs. a hollow shaft with the same outer diameter and material: the solid shaft has higher rigidity because removing the inner core reduces J . However, hollow shafts offer a much better strength-to-weight ratio, which is why they're common in practice.

To increase torsional rigidity, you can either increase the shaft diameter or select a material with a higher shear modulus.

Power, Torque, and Rotational Speed

Relationship between Power, Torque, and Rotational Speed

The equation $P = T \cdot \omega$ connects all three variables. The key takeaway for design:

- For a fixed power output, increasing rotational speed decreases the required torque, and vice versa. This is why high-speed shafts can be smaller in diameter than low-speed shafts transmitting the same power.
- Torque is the rotational equivalent of linear force. It's what the prime mover (motor, engine) supplies to drive the shaft.
- Rotational speed can be modified using gears, belts, or chain drives to match what the driven machinery needs.

Gear Ratios and Speed Changes

Gear ratios and pulley ratios let you trade speed for torque (or vice versa) within a power transmission system.

Gear systems:

The gear ratio equals the number of teeth on the driven gear divided by the number of teeth on the driving gear:

$$\text{Gear Ratio} = \frac{N_{\text{driven}}}{N_{\text{driving}}}$$

- A ratio greater than 1 means speed reduction and torque increase.
- A ratio less than 1 means speed increase and torque reduction.
- Example: A driving gear with 20 teeth meshing with a driven gear of 60 teeth gives a 3:1 ratio. The output shaft spins at one-third the input speed but delivers three times the torque (minus losses).

Belt-driven systems:

The speed ratio equals the diameter of the driven pulley divided by the diameter of the driving pulley:

$$\text{Speed Ratio} = \frac{D_{\text{driven}}}{D_{\text{driving}}}$$

- Example: A 100 mm driving pulley paired with a 200 mm driven pulley gives a 2:1 ratio, cutting the output speed in half while doubling the torque.

In both cases, power is conserved (minus efficiency losses). If speed goes down, torque goes up proportionally.

System Efficiency

Real power transmission systems always lose some energy to friction, slippage, and heat. Efficiency is defined as:

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}}$$

For example, a gearbox rated at 95% efficiency converts 95% of input power to useful output, with the remaining 5% lost as heat from friction between gear teeth, bearing losses, and oil churning.

When sizing a motor or shaft, account for these losses. If you need 10 kW at the output and your gearbox is 95% efficient, the input must be:

$$P_{\text{in}} = \frac{10}{0.95} \approx 10.53 \text{ kW}$$

Selecting low-friction bearings, proper lubrication, and well-matched components all help maximize system efficiency.

10.3 Torsion of non-circular members

Torsional behavior: Circular vs Non-circular

When you twist a non-circular member, the cross-section doesn't stay flat. It warps out of plane, producing a complex, non-uniform shear stress distribution that you can't analyze with the simple $\tau = \frac{T r}{J}$ formula from circular shafts. Understanding how and why this happens is essential for designing any structural member with a rectangular, elliptical, or other non-circular cross-section.

Cross-sectional behavior during torsion

Circular shafts keep their cross-sections perfectly plane during torsion. Every radial line stays straight and in-plane as the section rotates. Non-circular members behave differently: the cross-section warps, meaning points move out of the original plane. A rectangular bar, for instance, develops noticeable out-of-plane bulging at its corners when twisted. This warping is the root cause of the more complicated stress patterns you'll see throughout this topic.

Shear stress distribution

For circular members, shear stress varies linearly from zero at the center to a maximum at the outer surface:

$$\tau = \frac{T r}{J}$$

For non-circular members, the distribution is non-linear and varies across the entire cross-section. The maximum shear stress does not necessarily occur at the point farthest from the center. In an elliptical cross-section, for example, the maximum shear stress occurs at the ends of the minor axis (the points closest to the center), not the major axis. This is counterintuitive and a common exam mistake.

Torsional stiffness

- For circular shafts, torsional stiffness depends on the polar moment of inertia J , which is purely a function of the radius.
- For non-circular members, you use a torsional constant K (sometimes written C or J_t in different texts) instead of J . The value of K depends on the specific shape and its dimensions, and it's always less than J for a circular section of the same area.

- Rectangular cross-sections, for instance, have significantly lower torsional stiffness than circular ones of equal area because energy is "lost" to warping deformation.

Shear stress along the perimeter

On a circular shaft, shear stress is the same magnitude at every point on the outer perimeter. On non-circular members, shear stress varies around the boundary. A key result: shear stress at sharp corners is zero, not maximum. For a triangular cross-section, the maximum shear stress occurs at the midpoints of the sides, while the corners carry no shear stress at all. This is another place where intuition can mislead you.

Membrane analogy for torsion

Membrane analogy concept

The membrane (or soap-film) analogy, introduced by Prandtl, gives you a physical way to visualize torsion of non-circular sections. Imagine stretching a thin elastic membrane over an opening shaped like the cross-section, then applying a small uniform pressure from one side.

The key correspondences are:

- The height of the deflected membrane at any point corresponds to the value of the Prandtl stress function Φ at that point.
- The slope (gradient) of the membrane surface at any point is proportional to the shear stress at that point.
- The volume enclosed between the membrane and the flat base is proportional to the torque carried by the section.

This means steep slopes on the membrane indicate high shear stress, and flat regions (like near corners) indicate low shear stress. The membrane is fixed (zero deflection) along the boundary of the cross-section.

Governing equation and boundary conditions

The Prandtl stress function Φ satisfies the Poisson equation:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2G\theta'$$

where G is the shear modulus and θ' is the rate of twist (angle of twist per unit length, $\theta' = d\theta/dz$). Note that the right-hand side involves the rate of twist, not the torque directly.

Boundary conditions:

- $\Phi = 0$ along the perimeter of the cross-section (the membrane is clamped at the edges).
- The stress function is single-valued throughout the cross-section.

Solving the membrane analogy

Once you solve the Poisson equation for Φ , you can extract everything you need:

1.

Shear stress components from the partial derivatives of Φ : $\tau_{xz} = \frac{\partial \Phi}{\partial y}$ - $\tau_{yz} = -\frac{\partial \Phi}{\partial x}$

2.

Resultant shear stress at any point: - $\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$

3.

Torque from the volume under the stress function: - $T = 2 \iint_A \Phi \, dA$

Analytical solutions exist for simple shapes (ellipses, equilateral triangles). For a rectangle, series solutions or tabulated coefficients are used. For complex shapes, numerical methods (finite elements, finite differences) are typically required.

Shear stress in non-circular members

Shear stress components

The shear stress at any point in a non-circular cross-section under torsion comes from the gradient of the Prandtl stress function:

- $\tau_{xz} = \frac{\partial \Phi}{\partial y}$
- $\tau_{yz} = -\frac{\partial \Phi}{\partial x}$

The resultant shear stress magnitude is:

$$\tau = \sqrt{\left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial x}\right)^2}$$

Wherever the stress function surface is steepest, the shear stress is highest. Wherever it's flat (at corners, for example), the shear stress drops to zero.

Maximum shear stress

Where the maximum shear stress occurs depends on the shape:

Cross-section	Location of τ_{max}
Ellipse	Ends of the minor axis
Rectangle	Midpoints of the longer sides
Equilateral triangle	Midpoints of each side

For a general non-circular section, the maximum shear stress can be written as:

$$\tau_{max} = \frac{T}{\alpha b t^2}$$

where b and t are characteristic dimensions and α is a shape-dependent coefficient found from tables or solutions.

Closed-form solutions for common cross-sections

Elliptical cross-section (semi-axes a and b , with $a > b$):

$$\tau_{max} = \frac{2T}{\pi a b^2}$$

This maximum occurs at the ends of the minor axis (b).

Rectangular cross-section (width a , thickness b , with $a \geq b$):

$$\tau_{max} = \frac{T}{\alpha a b^2}$$

where α is a tabulated coefficient that depends on the ratio a/b . For a square ($a/b = 1$), $\alpha \approx 0.208$. As a/b increases toward a thin rectangle, α approaches $1/3$.

Torsional resistance and stiffness

Torsional constant

The torsional constant K (sometimes called I_t) replaces the polar moment of inertia J for non-circular sections. It's defined through the torque-twist relationship, not as a simple geometric integral of $r^2 dA$.

K is computed from the stress function:

$$T = 2 \iint_A \Phi dA$$

Closed-form results for common shapes:

- Ellipse (semi-axes a, b): $K = \frac{\pi a^3 b^3}{a^2 + b^2}$
- Equilateral triangle (side length S): $K = \frac{S^4 \sqrt{3}}{80}$
- Rectangle (dimensions $a \times b$, $a \geq b$): $K = \beta a b^3$, where β is a tabulated coefficient depending on a/b

Torsional stiffness

The torque-twist relationship for any prismatic member is:

$$\theta = \frac{TL}{GK}$$

where θ is the total angle of twist, T is the applied torque, L is the member length, G is the shear modulus, and K is the torsional constant. This has the same form as the circular shaft equation $\theta = TL/(GJ)$, but with K in place of J .

The torsional stiffness (torque per unit twist) is:

$$\frac{T}{\theta} = \frac{GK}{L}$$

For a rectangular section with $a/b = 1$ (square), $\beta \approx 0.141$, giving $K = 0.141 b^4$. Compare this to a circle of the same area: the square has roughly 84% of the circular section's torsional stiffness.

Comparison with circular members

For the same cross-sectional area, circular sections are always the most torsionally efficient shape.

Non-circular sections pay a penalty because:

- Part of the deformation goes into warping rather than pure rotation.
- The non-uniform stress distribution means material near corners is underutilized (carrying little or no shear stress).

As a practical rule, the more elongated or irregular the cross-section, the greater the stiffness penalty. This is why shafts designed primarily to transmit torque are almost always circular, while non-circular sections appear where other loading or geometric constraints govern the design.

Unit 11 – Shear Force & Bending Moment Diagrams

11.1 Types of beams and loads

Beams and loads are fundamental concepts in structural engineering. Understanding different beam types and support conditions is crucial for analyzing how structures transfer forces and moments. This knowledge forms the basis for creating shear force and bending moment diagrams.

Loads on beams come in various forms, from concentrated forces to distributed pressures. Recognizing these load types and their effects on beam behavior is essential for accurately determining internal forces and designing safe, efficient structures.

Beam types based on supports

Classification and statical determinacy

- Beams are classified based on their support conditions, which determine how they are constrained and how they transfer loads to the supports
- The support conditions also determine whether a beam is statically determinate or indeterminate
- Statically determinate beams have enough equilibrium equations to solve for all unknown reactions (simply supported beams)
- Statically indeterminate beams have more unknown reactions than available equilibrium equations (fixed beams, continuous beams)

Simply supported and cantilever beams

- A simply supported beam is supported at both ends by pinned or roller supports, allowing for rotation but not translation
- Pinned supports resist vertical and horizontal translation but allow rotation
- Roller supports resist translation perpendicular to the support surface but allow rotation and translation along the surface
- A cantilever beam is fixed at one end and free at the other end, with the fixed end providing both translational and rotational resistance
- The free end can deflect and rotate under load, while the fixed end remains constrained

Fixed, overhanging, and continuous beams

- A fixed beam, or built-in beam, is fixed at both ends, preventing both translation and rotation at the supports
- Fixed supports provide both translational and rotational resistance, making the beam statically indeterminate
- Overhanging beams extend beyond one or both supports, with the overhanging portion being free

- The overhanging portion behaves like a cantilever beam, while the supported portion behaves according to its support conditions
- Continuous beams are supported at more than two points, creating multiple spans
- Each intermediate support introduces additional unknown reactions, making the beam statically indeterminate

Truss beams

- Truss beams are composed of interconnected triangular structures, with each member carrying either tension or compression
- The triangular arrangement allows truss beams to efficiently transfer loads through axial forces in the members
- Truss beams are commonly used in bridges, roofs, and other large-scale structures

Loads on beams

Concentrated and moment loads

- Concentrated loads, also known as point loads, are forces applied at a single point on the beam
- Examples include the weight of a person standing on a beam or a column resting on a beam
- Moment loads are concentrated moments applied at a specific point, causing rotation without translation
- Moment loads can be created by offset forces or by external moments applied to the beam

Uniformly and non-uniformly distributed loads

- Distributed loads are forces applied over a length of the beam, either uniformly or non-uniformly distributed
- The intensity of a distributed load is measured in force per unit length ($q = \frac{F}{L}$)
- Uniformly distributed loads have a constant intensity along the beam's length
- Examples include the weight of a beam itself or a constant pressure applied along the beam (snow load on a roof)
- Non-uniformly distributed loads have a varying intensity along the beam's length, often described by a function
- Examples include hydrostatic pressure on a dam or wind pressure on a tall building

Linearly varying loads

- Linearly varying loads are distributed loads with a linear variation in intensity along the beam's length
- The load intensity varies from a minimum value at one end to a maximum value at the other end
- Linearly varying loads can be represented by a trapezoidal or triangular load distribution
- Examples include the soil pressure on a retaining wall or the water pressure on a submerged gate

Reactions at beam supports

Equilibrium equations and statical determinacy

- Support reactions are the forces and moments provided by the supports to maintain equilibrium in the beam
- For statically determinate beams, the support reactions can be calculated using the equations of equilibrium: sum of forces and sum of moments equal to zero
- $\sum F_x = 0, \sum F_y = 0, \sum M = 0$
- The number of equilibrium equations available must be equal to the number of unknown reaction forces and moments for the beam to be statically determinate

Reaction forces and moments at supports

- Pinned supports provide a vertical reaction force, while roller supports provide a reaction force perpendicular to the direction of possible translation
- Pinned supports resist both vertical and horizontal translation, so they provide reaction forces in both directions
- Roller supports only resist translation perpendicular to the support surface, so they provide a reaction force in that direction
- Fixed supports provide both a vertical reaction force and a reaction moment
- The fixed support resists both translation and rotation, introducing a reaction moment in addition to the reaction force

Sign convention for reactions

- The sign convention for reaction forces and moments must be consistently applied, typically with upward forces and counterclockwise moments being positive
- Positive reaction forces act in the opposite direction of gravity (upward)
- Positive reaction moments act in the counterclockwise direction when viewed from the left side of the beam
- When solving for reactions, assume the direction of unknown forces and moments, and solve the equilibrium equations
- If the resulting value is positive, the assumed direction is correct; if negative, the actual direction is opposite to the assumed direction

Distributed loads on beams

Load intensity and resultant force

- Distributed loads are forces applied over a length of the beam, as opposed to concentrated loads applied at a single point
- The intensity of a distributed load is measured in force per unit length ($q = \frac{F}{L}$)
- Common units include N/m (SI) or lb/ft (US customary)

- The resultant force of a distributed load acts at the centroid of the load distribution area
- For a uniformly distributed load, the resultant force is equal to the product of the load intensity and the length ($F_R = q \cdot L$)
- For a linearly varying load, the resultant force is equal to the average load intensity multiplied by the length ($F_R = \frac{q_1 + q_2}{2} \cdot L$)

Effect on shear force and bending moment diagrams

- Distributed loads affect shear force and bending moment diagrams differently than concentrated loads, creating sloped or curved segments instead of instantaneous jumps
- Uniformly distributed loads create linearly sloped segments in the shear force diagram and parabolic segments in the bending moment diagram
- Linearly varying loads create parabolically curved segments in the shear force diagram and cubic segments in the bending moment diagram
- When calculating reactions or drawing diagrams, distributed loads can be replaced by their equivalent resultant force acting at the centroid of the load distribution
- This simplification allows for easier calculation of reactions and construction of diagrams
- However, the actual distributed nature of the load must be considered when analyzing the internal forces and moments along the beam

11.2 Shear force and bending moment relationships

Shear force and bending moment are key concepts in beam analysis. They help engineers understand how internal forces act along a beam, crucial for designing structures that can safely handle loads without failing.

These concepts are closely related through differential equations. By studying their relationships, we can find critical points in beams where failure is most likely, guiding us in creating stronger, safer structures.

Shear Force and Bending Moment

Definition and Context

- Shear force is the internal force acting perpendicular to the longitudinal axis of a beam, causing two segments of the beam to slide relative to each other
- Bending moment is the internal moment acting about the longitudinal axis of a beam, causing the beam to bend or flex
- Shear forces and bending moments are induced in beams subjected to transverse loads, which are forces applied perpendicular to the beam's longitudinal axis (gravity loads, wind loads)
- The magnitude and direction of shear forces and bending moments vary along the length of the beam, depending on the loading conditions (concentrated loads, distributed loads) and support configurations (simply supported, cantilever, fixed)

Importance in Beam Analysis

- Understanding shear force and bending moment is crucial for designing beams that can safely resist the applied loads without failure
- Shear force and bending moment diagrams provide a visual representation of the internal forces acting along the beam, helping engineers identify critical locations where the beam may be prone to failure (maximum shear force, maximum bending moment)
- The maximum values of shear force and bending moment are used to calculate the required size and material properties of the beam to ensure adequate strength and stiffness
- Shear force and bending moment analysis is a fundamental step in the design process for structures such as bridges, buildings, and machinery

Shear Force vs Bending Moment

Relationship Defined by Differential Equations

- The relationship between shear force and bending moment is defined by the differential equations of equilibrium for beams
- The slope of the shear force diagram at any point along the beam is equal to the magnitude of the distributed load acting on the beam at that point
- The slope of the bending moment diagram at any point along the beam is equal to the magnitude of the shear force at that point
- The change in bending moment between two points along the beam is equal to the area under the shear force diagram between those two points

Locating Maximum and Minimum Values

- The maximum or minimum values of the bending moment occur at locations where the shear force is zero, provided there are no concentrated moments applied at those locations
- To find the maximum or minimum bending moment, set the shear force equation equal to zero and solve for the location along the beam
- The maximum or minimum shear force occurs at locations where the slope of the bending moment diagram changes sign (inflection points)
- Concentrated loads and support reactions cause sudden changes (jumps) in the shear force diagram, while the bending moment diagram remains continuous

Sign Conventions for Diagrams

Shear Force Sign Convention

- The sign convention for shear force is positive when the shear force tends to rotate the beam element clockwise, and negative when it tends to rotate the beam element counterclockwise
- Concentrated loads acting upward on a beam are considered positive, while concentrated loads acting downward are considered negative

- Distributed loads acting upward on a beam are considered positive, while distributed loads acting downward are considered negative

Bending Moment Sign Convention

- The sign convention for bending moment is positive when the bending moment causes compression in the top fibers of the beam and tension in the bottom fibers (sagging), and negative when it causes tension in the top fibers and compression in the bottom fibers (hogging)
- Positive bending moments cause the beam to bend concave downward, while negative bending moments cause the beam to bend concave upward
- The sign convention for bending moment is consistent with the sign convention for curvature in beam theory

Equations for Shear Force and Bending Moment

Deriving Equations using Method of Sections

- The equations for shear force and bending moment at any point along a beam can be derived using the method of sections and the equations of equilibrium
- To find the shear force at a point, make an imaginary cut at that point and consider the equilibrium of either the left or right portion of the beam
- The shear force is equal to the sum of the transverse forces acting on the cut section, including support reactions and applied loads
- To find the bending moment at a point, make an imaginary cut at that point and consider the equilibrium of either the left or right portion of the beam
- The bending moment is equal to the sum of the moments acting on the cut section, taken about the cut point, including support reactions, applied loads, and moments

Plotting Shear Force and Bending Moment Diagrams

- The derived equations for shear force and bending moment can be used to plot the shear force and bending moment diagrams
- Shear force and bending moment diagrams provide a visual representation of the internal forces acting along the beam
- To plot the diagrams, evaluate the shear force and bending moment equations at key points along the beam (supports, load locations, midspan) and plot the values on a graph
- Connect the plotted points with straight lines for the shear force diagram and smooth curves for the bending moment diagram, following the sign conventions established earlier
- Identify and label the maximum and minimum values of shear force and bending moment on the diagrams

11.3 Construction and interpretation of shear and moment diagrams

Shear force and bending moment diagrams show internal forces in beams. They help us see where a beam might break or bend too much. These diagrams are key tools for engineers designing safe structures.

Building these diagrams involves calculating reactions, plotting equations, and finding critical points. Understanding how loads affect shear and moment helps us interpret the diagrams and predict beam behavior under different conditions.

Shear Force and Bending Moment Diagrams

Graphical Representations and Internal Forces

- Shear force and bending moment diagrams are graphical representations of the internal forces and moments acting on a beam along its length
- Shear force is the internal force that acts perpendicular to the beam's axis
- Bending moment is the internal moment that causes the beam to bend
- Concentrated loads (point loads), distributed loads, and moments can be applied to a beam
- The magnitudes and locations of these loads and moments affect the shear force and bending moment diagrams

Constructing Diagrams and Sign Conventions

- The sign convention for shear force and bending moment diagrams follows the right-hand rule
- Positive shear force acts upward
- Positive bending moment causes compression on the top of the beam
- Constructing shear force and bending moment diagrams involves:
 - Calculating the reactions at the supports
 - Determining the equations for shear force and bending moment along the beam
 - Plotting these equations
- Discontinuities in the shear force diagram occur at concentrated loads and reactions
- Discontinuities in the bending moment diagram occur at concentrated moments
- The slope of the shear force diagram represents the distributed load acting on the beam
- The slope of the bending moment diagram represents the shear force

Critical Points on Diagrams

Locations and Significance of Critical Points

- Critical points on shear force and bending moment diagrams include:
 - Locations of zero shear force
 - Maximum or minimum shear force
 - Zero bending moment (inflection points)
 - Maximum or minimum bending moment
- Points of zero shear force indicate the locations of maximum or minimum bending moment

- The slope of the bending moment diagram changes sign at these points
- Points of maximum or minimum shear force occur at:
 - Concentrated loads
 - Reactions
 - Discontinuities in the distributed load
- Points of zero bending moment (inflection points) indicate the locations where the beam changes from positive to negative curvature or vice versa

Identifying Critical Points

- Points of maximum or minimum bending moment occur where the shear force diagram crosses the zero axis
- The slope of the bending moment diagram is zero at these points
- To identify critical points:
 - Analyze the shear force diagram for zero crossings, maxima, and minima
 - Examine the bending moment diagram for zero crossings, maxima, minima, and inflection points
 - Consider the locations of concentrated loads, reactions, and discontinuities in the distributed load

Interpretation of Diagrams

Insights from Shape and Magnitude

- The shape and magnitude of shear force and bending moment diagrams provide insights into the internal forces and moments acting on the beam
- They help identify critical regions that may require special attention in design
- The magnitude of the shear force at any point represents the net internal shear force acting at that cross-section of the beam
- The magnitude of the bending moment at any point represents the net internal moment causing bending at that cross-section of the beam
- Regions with high magnitudes of shear force or bending moment indicate areas of the beam that experience significant internal stresses

Deflected Shape and Curvature

- The curvature of the bending moment diagram indicates the beam's deflected shape
- Positive curvature corresponds to sagging (beam bending downward)
- Negative curvature corresponds to hogging (beam bending upward)
- Sudden changes in the slope of the shear force or bending moment diagrams indicate the presence of:
 - Concentrated loads

- Reactions
- Moments acting on the beam
- Analyzing the curvature and slope changes helps understand the beam's behavior under loading

Load, Shear, and Moment Relationships

Interconnections and Derivatives

- The load, shear force, and bending moment diagrams are interconnected
- Understanding their relationships is crucial for analyzing beam behavior
- The distributed load acting on the beam is equal to the slope of the shear force diagram at any point
- The shear force at any point is equal to the slope of the bending moment diagram at that point
- The second derivative of the bending moment diagram with respect to the beam's length is equal to the distributed load acting on the beam

Discontinuities and Integration

- Concentrated loads and reactions appear as discontinuities or jumps in the shear force diagram
- The magnitude of the jump is equal to the load or reaction
- Concentrated moments appear as discontinuities or jumps in the bending moment diagram
- The magnitude of the jump is equal to the applied moment
- By integrating the shear force diagram, the bending moment diagram can be obtained
- By integrating the bending moment diagram, the beam's deflection can be determined
- These relationships allow for the calculation of internal forces, moments, and deflections at any point along the beam

Unit 12 – Stresses in Beams

12.1 Normal stresses in beams

Normal Stress in Beams

Stress Distribution

When a beam bends under load, normal stresses develop across its cross-section. These stresses aren't uniform. They vary linearly from zero at the neutral axis to their maximum values at the outermost fibers.

The pattern works like this: on one side of the neutral axis, fibers are being compressed (shortened), while on the other side, fibers are being stretched (elongated). For a simply supported beam with a downward load, compressive stresses develop on the top and tensile stresses develop on the bottom.

- The stress at any point is directly proportional to its distance from the neutral axis. A point twice as far from the neutral axis experiences twice the stress.
- The distribution flips direction across the neutral axis, creating a linear gradient from maximum compression to maximum tension.
- Both the magnitude and direction of the applied bending moment control this distribution.

Maximum Stress Location

Maximum normal stresses always occur at the extreme fibers, the points farthest from the neutral axis. Where exactly those points are depends on the cross-section shape:

- Rectangular beam: maximum stresses at the top and bottom surfaces
- I-beam: maximum stresses at the top and bottom flanges
- Circular beam: maximum stresses at the very top and bottom of the circle

For a doubly symmetric cross-section, the maximum tensile and compressive stresses have equal magnitudes but opposite signs. Identifying these locations is the first step in checking whether a beam can safely carry its load.

Neutral Axis Location

Symmetric Cross-Sections

The neutral axis is the line across the cross-section where normal stress equals zero. No tension, no compression. Everything pivots around this line.

For symmetric cross-sections (rectangular, circular, I-shaped), the neutral axis passes through the centroid of the cross-section. This makes life easier because you can find it by geometry alone:

- Rectangular beam: the neutral axis sits at mid-height, $h/2$ from either surface
- Circular beam: the neutral axis passes through the center of the circle
- I-beam: the neutral axis coincides with the horizontal centerline of the web

Unsymmetric Cross-Sections

For unsymmetric shapes like T-beams or L-beams, the neutral axis still passes through the centroid, but the centroid is no longer at the obvious geometric center. You need to calculate its location using the first moment of area.

Here's the process:

1. Divide the cross-section into simple shapes (rectangles, etc.).
2. Choose a reference axis (typically the bottom edge of the section).
3. For each sub-shape, calculate its area A_i and the distance $\bar{y}_i$ from the reference axis to its own centroid.
4. Compute the centroid location: $\bar{y} = \frac{\sum A_i \bar{y}_i}{\sum A_i}$

The neutral axis passes through this centroid. For a T-beam, the neutral axis typically sits closer to the flange (the wider part) because the flange contributes more area. For an L-beam, it shifts toward the corner with the larger leg.

Maximum Normal Stress Calculation

Flexure Formula

The flexure formula is the central equation for this topic:

$$\sigma = \frac{My}{I}$$

- σ = normal stress at the point of interest
- M = internal bending moment at the cross-section
- y = distance from the neutral axis to the point of interest
- I = moment of inertia of the cross-section about the neutral axis

To find the maximum normal stress, plug in y_{max} (the distance from the neutral axis to the extreme fiber) along with the bending moment M at that section:

$$\sigma_{max} = \frac{M \cdot y_{max}}{I}$$

You'll sometimes see this written using the section modulus $S = I/y_{max}$, which simplifies the expression to $\sigma_{max} = M/S$. This form is especially handy for design problems where you're selecting a beam size.

Moment of Inertia

The moment of inertia I quantifies how the cross-sectional area is distributed relative to the neutral axis. A larger I means lower stress for the same bending moment, which is why I-beams (with material concentrated far from the neutral axis) are so efficient.

Standard formulas for common shapes:

- Rectangle (base b , height h): $I = \frac{bh^3}{12}$
- Circle (radius r): $I = \frac{\pi r^4}{4}$

- Hollow rectangle / I-shape: subtract the inner rectangle's I from the outer, e.g., $I = \frac{bh^3}{12} - \frac{b_1h_1^3}{12}$

For composite or irregular sections, use the parallel axis theorem:

$$I = \bar{I} + Ad^2$$

where $\bar{I}$ is the moment of inertia of a sub-shape about its own centroid, A is its area, and d is the distance between the sub-shape's centroid and the overall neutral axis. Sum this for each sub-shape to get the total I .

Bending Moment and Normal Stress

Proportional Relationship

The flexure formula $\sigma = My/I$ shows that stress and bending moment are directly proportional (with Y and I held constant at a given point). Double the bending moment, and you double the stress at every point in the cross-section. This linear relationship is what makes superposition work for beam problems: you can analyze load cases separately and add the results.

Stress Distribution along the Beam

The bending moment generally varies along the length of a beam, so the normal stress varies along the length too. The critical design section is wherever the bending moment reaches its maximum magnitude.

Sign convention matters. A positive bending moment (beam sagging) produces tensile stress at the bottom fibers and compressive stress at the top. A negative bending moment (beam hogging) reverses this: compression on the bottom, tension on the top.

Common cases to know:

- Simply supported beam, uniform distributed load: maximum bending moment (and maximum normal stress) occurs at midspan, with $M_{max} = \frac{wL^2}{8}$.
- Cantilever beam, point load at the free end: maximum bending moment occurs at the fixed support, with $M_{max} = PL$.
- Simply supported beam, point load at midspan: maximum bending moment at midspan, with $M_{max} = \frac{PL}{4}$.

At any cross-section, once you know M from your shear and moment diagrams, you can plug it into the flexure formula to find the stress distribution at that location. The sections with the largest $|M|$ are the ones most likely to fail, so those are where you focus your design checks.

12.2 Shear stresses in beams

Shear Stress in Beams

Shear Stress Distribution

When a beam carries a transverse load, internal shear forces develop at every cross-section. Shear stress is the intensity of that internal force per unit area, and it acts parallel to the face of the cross-section rather than perpendicular to it (which would be bending stress).

The distribution of shear stress across a cross-section is not uniform. It varies with position, and the shape of that distribution depends on the geometry of the cross-section. A few key points to remember:

- Shear stress is zero at the top and bottom free surfaces of the beam, because there's no adjacent material to develop a shear flow against.
- Shear stress is typically greatest at or near the neutral axis, where bending stress happens to be zero.
- For a rectangular cross-section, the distribution is parabolic, peaking at the center.

A common misconception from the bending stress topic: students sometimes assume that the most "dangerous" point in a cross-section is always at the extreme fibers (top/bottom). That's true for bending stress, but for shear stress the critical location is usually at the neutral axis. Bending and shear are at their worst at different points in the cross-section.

Factors Influencing Shear Stress

- Cross-section geometry is the dominant factor. The shape determines how Q and t vary across the section, which controls the stress distribution.
- Material properties like the shear modulus (G) relate shear stress to shear strain ($\tau = G\gamma$), but they don't change the stress distribution itself for elastic analysis. They matter more when you're calculating deflections due to shear.
- For symmetrical sections (rectangular, circular), maximum shear stress occurs at the centroid (neutral axis).
- For I-beams and sections with thin webs, the shear stress is carried almost entirely by the web. The maximum shear stress occurs in the web, and it's roughly uniform across the web height. The flanges carry very little shear. This is why the web is the critical element for shear in steel I-beams.

Maximum Shear Stress Location

Where the maximum shear stress falls depends on the cross-section shape:

Cross-Section	Location of τ_{max}	Why
Rectangular	Neutral axis (mid-height)	Q is largest and t is constant
Circular	Neutral axis (center)	Same reasoning; parabolic distribution
I-beam	In the web, near the neutral axis	The web is thin (small t), which drives $\tau = VQ/(It)$ up
T-section or channel	Where the section is thinnest relative to Q	Requires calculation; not always at the neutral axis

Other Factors Affecting the Location

- Loading conditions determine the shear force V at each cross-section along the beam's length. The cross-section with the largest V is where you'll find the highest overall shear stress.
- Non-prismatic beams (beams whose cross-section changes along their length) can shift the critical location because I , Q , and t all change with position.

So finding the true maximum shear stress in a beam is a two-part problem: find where V is largest along the beam's length (from the shear diagram), and find where τ is largest within that cross-section.

Shear Stress Calculation

The Shear Formula

The general shear stress at any point in a beam's cross-section is:

$$\tau = \frac{VQ}{It}$$

where:

- V = internal shear force at the cross-section
- Q = first moment of area of the portion of the cross-section above (or below) the point where you're calculating stress, taken about the neutral axis
- I = second moment of area (moment of inertia) of the entire cross-section about the neutral axis
- t = width of the cross-section at the point where you're calculating stress

Step-by-Step Calculation Process

1. Draw the shear force diagram for the beam and identify the cross-section where V is largest (or wherever you need to evaluate).
2. Locate the neutral axis of the cross-section and compute the moment of inertia I about it.
3. Choose the point in the cross-section where you want the shear stress (often the neutral axis for τ_{max}).
4. Calculate Q : take the area of the cross-section above (or below) your chosen point, and multiply it by the distance from that area's centroid to the neutral axis. Formally: $Q = \bar{y}' \cdot A'$, where A' is the partial area and $\bar{y}'$ is the distance from its centroid to the neutral axis.
5. Measure t : the width of the cross-section at your chosen point.
6. Substitute into $\tau = VQ/(It)$.

Example: Rectangular Cross-Section

For a rectangular beam of width b and height h , find τ_{max} at the neutral axis.

The area above the neutral axis is a rectangle of width b and height $h/2$. Its centroid sits at $h/4$ above the neutral axis.

$$\bullet Q = A' \cdot \bar{y}' = \left(b \cdot \frac{h}{2}\right) \left(\frac{h}{4}\right) = \frac{bh^2}{8}$$

$$\bullet I = \frac{bh^3}{12}$$

$$\bullet t = b$$

Substituting:

$$\tau_{max} = \frac{V \cdot \frac{bh^2}{8}}{\frac{bh^3}{12} \cdot b} = \frac{3V}{2bh} = \frac{3V}{2A}$$

This result is worth memorizing: the maximum shear stress in a rectangular beam is 1.5 times the average shear stress (V/A). For a circular cross-section, the corresponding factor is $4/3$.

Shear Force vs. Shear Stress

These two quantities are related but distinct, and mixing them up is a common mistake.

- Shear force (V) is a resultant internal force acting on an entire cross-section. It has units of force (N or lb). You find it from equilibrium using free-body diagrams and shear diagrams.
- Shear stress (τ) is force per unit area at a specific point within the cross-section. It has units of pressure (Pa or psi). You find it from the shear formula.

The shear formula $\tau = VQ/(It)$ is the bridge between them. A single value of V produces a distribution of τ across the cross-section.

Why Constant V Doesn't Mean Constant τ

Consider a simply supported beam with a single concentrated load P at midspan. The shear diagram shows $V = +P/2$ from the left support to midspan, then $V = -P/2$ from midspan to the right support.

Within any cross-section in the left half, V is the same everywhere: $P/2$. But the shear stress still varies from zero at the top and bottom surfaces to a maximum at the neutral axis. The variation comes from Q and t , which depend on where in the cross-section you look, not on where along the beam you are.

This is the key distinction: the shear diagram tells you how V changes along the beam's length, while the shear formula tells you how τ changes across the beam's cross-section at any given location.

12.3 Combined loading in beams

Combined Bending and Shear Loading

Simultaneous Application of Bending Moments and Shear Forces

In real structures, beams rarely experience pure bending or pure shear alone. Combined loading refers to the simultaneous presence of bending moments and shear forces at a given cross-section. This means both normal stresses (from bending) and shear stresses (from transverse shear) act on the same cross-section at the same time.

Combined loading shows up in all common beam configurations: cantilevers, simply supported beams, and continuous beams. Consider a simply supported beam carrying a distributed load along its length with a concentrated load at midspan. At most cross-sections along that beam, both a nonzero bending moment and a nonzero shear force exist simultaneously.

Stress Distributions in Beams under Combined Loading

The two stress components follow distinctly different distribution patterns across the cross-section:

Normal stress varies linearly with distance from the neutral axis. Bending causes compression on one side and tension on the other. The maximum normal stress occurs at the extreme fibers (top and bottom), and it equals zero at the neutral axis.

Shear stress varies parabolically across the depth of the cross-section (for common shapes like rectangles and wide-flange sections). Shear stress is zero at the top and bottom free surfaces and reaches its maximum at the neutral axis.

This creates an important tradeoff across the cross-section:

- At the extreme fibers: maximum normal stress, zero shear stress

- At the neutral axis: zero normal stress, maximum shear stress
- At points in between: some combination of both normal and shear stress

The resultant stress state at any point is the combination of the normal and shear components acting there. Because the two distributions peak at different locations, the most critical point for failure depends on the failure criterion being used and the relative magnitudes of bending moment and shear force.

Principal Stresses in Beams

Concept of Principal Stresses

At any point in a beam under combined loading, you can find special orientations where the stress state is purely normal, with no shear component. The normal stresses on these planes are called principal stresses, and the planes themselves are called principal planes.

Two key properties of principal planes:

- Shear stress is zero on every principal plane
- The two principal planes at any point are always perpendicular to each other

The principal stresses represent the absolute maximum tensile and maximum compressive normal stresses at that point. This makes them essential for checking failure criteria, since most materials fail based on maximum normal stress or a combination of principal stresses.

Calculation of Principal Stresses

For a 2D (plane stress) state with normal stresses σ_x and σ_y and shear stress τ_{xy} , the principal stresses are:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

where:

- σ_x = normal stress in the x-direction (typically the bending stress $\sigma = My/I$)
- σ_y = normal stress in the y-direction (often zero for a typical beam element)
- τ_{xy} = shear stress on the element (from transverse shear, $\tau = VQ/It$)

The orientation of the principal planes relative to the original axes is found from:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

This equation gives two values of θ_p separated by 90° , corresponding to the two principal planes.

For a typical beam element where $\sigma_y = 0$, the formulas simplify to:

$$\sigma_{1,2} = \frac{\sigma_x}{2} \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

Steps to find principal stresses at a point in a beam:

1. Calculate the bending stress: $\sigma_x = My/I$
2. Calculate the shear stress: $\tau_{xy} = VQ/It$
3. Set $\sigma_y = 0$ (unless another loading produces a vertical normal stress)

4. Substitute into the principal stress formula
5. Find θ_p to determine the orientation of the principal planes

Maximum Stresses in Beams

Maximum Normal Stress

The maximum normal stress from bending alone occurs at the extreme fibers and is calculated with the flexure formula:

$$\sigma = \frac{My}{I}$$

- M = bending moment at the section
- y = distance from the neutral axis to the point of interest (use $y = c$ for the extreme fiber)
- I = second moment of area (moment of inertia) of the cross-section about the neutral axis

At the extreme fibers, shear stress is zero, so the bending stress is the principal stress there. This location often governs design for beams where bending dominates.

Maximum Shear Stress

The maximum transverse shear stress occurs at the neutral axis and is calculated with the shear stress formula:

$$\tau = \frac{VQ}{It}$$

- V = internal shear force at the section
- Q = first moment of area of the portion of the cross-section above (or below) the point of interest, taken about the neutral axis
- I = second moment of area of the entire cross-section
- t = width of the cross-section at the point where shear stress is being calculated

At the neutral axis, normal bending stress is zero, so the stress state is pure shear. The principal stresses there turn out to be equal in magnitude to τ but oriented at 45° to the beam axis (one tensile, one compressive). This is why you sometimes see diagonal cracking near the neutral axis in concrete beams: the concrete fails in tension along those 45° principal planes.

The maximum principal stress in a beam doesn't always occur at the extreme fiber or at the neutral axis. At intermediate points where both σ and τ are nonzero, the combined effect can sometimes produce a larger principal stress. Always check critical points across the section.

Normal vs. Shear Stress Interaction

Simultaneous Action of Normal and Shear Stresses

At most points in a beam cross-section (anywhere between the extreme fibers and the neutral axis), both normal and shear stresses act simultaneously. This combined stress state means the principal stresses are not aligned with the beam's longitudinal and transverse axes. Instead, the principal planes are rotated by the angle θ_p , which depends on the ratio of shear stress to normal stress at that point.

The presence of shear stress always increases the magnitude of the maximum principal stress beyond what the normal stress alone would produce. Even a modest shear stress combined with a normal stress shifts the critical plane to an oblique orientation.

Mohr's Circle Representation

Mohr's circle provides a graphical way to visualize the complete stress state at a point and find principal stresses, maximum shear stress, and their orientations.

How to construct Mohr's circle for a beam element:

1. Plot point A at (σ_x, τ_{xy}) representing the stress state on the x-face. (Convention: plot shear that would cause clockwise rotation as positive upward.)
2. Plot point B at $(\sigma_y, -\tau_{xy})$ representing the stress state on the y-face.
3. Connect A and B. The midpoint of this line is the center of the circle, located at $(\frac{\sigma_x + \sigma_y}{2}, 0)$.
4. Draw the circle through points A and B.

Reading results from the circle:

- Principal stresses σ_1 and σ_2 : where the circle crosses the horizontal axis (shear = 0)
- Maximum in-plane shear stress: the radius of the circle, equal to $\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$
- Principal angle θ_p : half the angle measured on the circle from point A to the σ_1 intersection, with angles on the circle being twice the physical rotation angle

Mohr's circle is especially useful for building intuition. You can see directly how adding shear stress (increasing the radius) increases the difference between σ_1 and σ_2 , and how the principal planes rotate as the ratio of shear to normal stress changes. For beam design, this helps you identify which points in the cross-section are most vulnerable and what type of stress (tension, compression, or shear) governs failure at each location.

Unit 13 – Beam Deflection

13.1 Elastic curve equation and boundary conditions

Elastic Curve Equation and Boundary Conditions

The elastic curve equation describes how a beam deflects under load. It connects the beam's deformed shape to the applied loading, material stiffness, and geometry, making it the starting point for nearly every deflection analysis you'll do in this course. Boundary conditions then let you pin down the specific solution for a given support configuration.

Elastic Curve Equation for Beams

Derivation and Assumptions

The elastic curve is the deformed shape of the beam's neutral axis under load. To derive the governing equation, we rely on Euler-Bernoulli beam theory, which makes two key assumptions:

- Plane sections remain plane and perpendicular to the neutral axis after deformation (no warping of cross-sections).
- The material is linearly elastic, homogeneous, and isotropic (Hooke's law applies everywhere).

These assumptions work well for slender beams with small deflections relative to their length.

Starting from equilibrium and the moment-curvature relationship (covered below), you can write the elastic curve as a fourth-order differential equation:

$$\frac{d^4 y}{dx^4} = \frac{q(x)}{EI}$$

where:

- y = transverse deflection of the beam
- x = position along the beam's length
- $q(x)$ = distributed load intensity (force per unit length)
- E = elastic (Young's) modulus of the material
- I = second moment of area (moment of inertia) of the cross-section

Simplification and Loading Conditions

If the flexural rigidity EI is constant along the beam and you already know the bending moment function $M(x)$, you can work with the simpler second-order form:

$$\frac{d^2 y}{dx^2} = \frac{M(x)}{EI}$$

This is the version you'll use most often. You write $M(x)$ from your shear and moment diagrams, then integrate twice to get $y(x)$.

The equation applies to any loading type. You just need the correct expression for $M(x)$:

- Concentrated (point) loads produce piecewise-linear moment diagrams

- Distributed loads (uniform, linearly varying, etc.) produce curved moment diagrams
- Concentrated moments cause jumps in the moment diagram

Boundary Conditions for Beam Deflection

Types of Boundary Conditions

Boundary conditions capture how the beam is supported and constrained at specific points. Without them, you can't solve for the integration constants that appear when you integrate the elastic curve equation.

The four most common support types and their associated conditions:

Support Type	Deflection	Slope	Moment	Shear
Pin / Roller (simply supported)	$y = 0$	free	$M = 0$	free
Fixed (clamped)	$y = 0$	$\frac{dy}{dx} = 0$	free	free
Free end	free	free	$M = 0$	$V = 0$
Guided end	$y = 0$	free	$M = 0$	free

"Free" in the table means that quantity is not constrained to a known value at that point.

Application of Boundary Conditions

The number of boundary conditions you need matches the number of integration constants. If you start from the second-order form $\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$, two integrations produce two constants (C_1 and C_2), so you need two boundary conditions. If you start from the fourth-order form, you'll have four constants and need four boundary conditions.

Here's how this plays out for two classic cases:

Cantilever beam (fixed at $x = 0$, free at $x = L$): - $y(0) = 0$ (no deflection at the wall) - $\frac{dy}{dx}(0) = 0$ (no rotation at the wall) - If using the fourth-order form, add: $M(L) = 0$ and $V(L) = 0$ at the free end

Simply supported beam (pins at both ends): - $y(0) = 0$ (no deflection at left support) - $y(L) = 0$ (no deflection at right support) - If using the fourth-order form, add: $M(0) = 0$ and $M(L) = 0$

Curvature, Moment, and Rigidity Relationship

Beam Curvature

Curvature (K) measures how sharply the beam bends at a given point. It equals the reciprocal of the radius of curvature ρ :

$$K = \frac{1}{\rho}$$

The exact curvature expression involves both $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$, but under the small-deflection assumption (slopes much less than 1), it simplifies to:

$$K \approx \frac{d^2y}{dx^2}$$

This approximation is what makes the elastic curve equation linear and solvable by direct integration.

Moment-Curvature Relationship

The link between internal bending moment and curvature is:

$$M = EI \kappa$$

or equivalently:

$$M = EI \frac{d^2 y}{dx^2}$$

This is the equation that ties everything together. Flexural rigidity EI quantifies a beam's resistance to bending. It depends on both the material (through E) and the cross-section shape (through I).

Common moment of inertia formulas you'll need:

- Rectangular cross-section (width b , height h): $I = \frac{bh^3}{12}$
- Solid circular cross-section (radius r): $I = \frac{\pi r^4}{4}$

A steel beam with a deep I-section has a much larger EI than a wooden beam of similar weight, which is why steel beams deflect less under the same load.

Integration Constants for Beam Deflection

Solving the Elastic Curve Equation

Each integration of the elastic curve equation introduces one unknown constant. These constants carry physical meaning:

- Integrating $\frac{d^2 y}{dx^2} = \frac{M(x)}{EI}$ once gives the slope equation $\frac{dy}{dx}$, with constant C_1
- Integrating again gives the deflection equation $y(x)$, with constant C_2

C_1 and C_2 encode the beam's slope and deflection at whatever reference point you choose. You find their values by plugging in your boundary conditions.

Determining Integration Constants: Step-by-Step

1. Write $M(x)$ from your free-body diagram and equilibrium equations.
2. Substitute into $EI \frac{d^2 y}{dx^2} = M(x)$ and integrate once to get $EI \frac{dy}{dx} = \int M(x) dx + C_1$.
3. Integrate again to get $EI y = \iint M(x) dx dx + C_1 x + C_2$.
4. Apply boundary conditions by substituting known values of y and $\frac{dy}{dx}$ at specific x -locations.
5. Solve the resulting system of equations for C_1 and C_2 .
6. Substitute back into the general solution to get the specific deflection equation $y(x)$.

Example: Cantilever with a point load P at the free end

The moment function is $M(x) = -P(L - x)$ (taking the fixed end as $x = 0$). After two integrations:

$$EI y(x) = -P\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_1 x + C_2$$

Applying $y(0) = 0$ gives $C_2 = 0$. Applying $\frac{dy}{dx}(0) = 0$ gives $C_1 = 0$. The specific solution is:

$$y(x) = \frac{P}{6EI}(x^3 - 3Lx^2)$$

The maximum deflection occurs at the free end ($x = L$):

$$y_{\max} = -\frac{PL^3}{3EI}$$

The negative sign indicates downward deflection (in the direction of the load). This is one of the most commonly referenced deflection formulas, so it's worth committing to memory.

13.2 Methods for determining beam deflection

Beam Deflection Calculation Methods

Beam deflection describes how much a beam displaces from its original position under load. Predicting this accurately is essential because excessive deflection can cause serviceability failures (cracked finishes, misaligned machinery, uncomfortable vibrations) even when stresses remain well within safe limits. Three classical methods handle this problem: double integration, moment-area, and conjugate beam. Each arrives at the same answer but suits different situations.

Double Integration Method

This is the most general, calculus-driven approach. You start from the fundamental beam equation that relates bending moment to curvature:

$$EI \frac{d^2y}{dx^2} = M(x)$$

Integrating once gives the slope $\frac{dy}{dx}$, and integrating a second time gives the deflection $y(x)$. Each integration introduces a constant, so you need two boundary conditions to pin down the full solution.

This method works for any loading (distributed loads, point loads, applied moments) as long as you can write $M(x)$ as a continuous function. It's the go-to when you want a complete, closed-form equation for the entire deflection curve. The trade-off is that it can get algebra-heavy, especially when the moment expression changes across the span and you have to integrate piecewise.

Moment-Area Method

The moment-area method is a semi-graphical technique built on two theorems that relate the M/EI diagram directly to slopes and deflections. Instead of integrating equations, you compute areas and first moments of areas from the M/EI diagram, which you break into simple shapes (triangles, rectangles, parabolic segments).

This method shines when you only need the slope or deflection at one or two specific points rather than the full curve. It's especially efficient for simply supported and cantilever beams with straightforward loading. For beams with many load changes, the geometry bookkeeping can become tedious.

Conjugate Beam Method

The conjugate beam method converts a deflection problem into a statics problem. You construct an imaginary "conjugate" beam that has the same length as the real beam but carries the real beam's M/EI diagram as its distributed load. The support conditions of the conjugate beam are transformed according to specific rules (more on this below). Once you analyze the conjugate beam with standard equilibrium, its internal shear gives the real beam's slope, and its internal moment gives the real beam's deflection.

This method is particularly powerful for beams with internal hinges, overhangs, or mixed support types that make the other methods awkward.

Double Integration Method for Deflection

Integrating the Bending Moment Equation

1. Write the moment function. Using equilibrium (reactions, free-body diagrams), express $M(x)$ along the beam. If the loading changes at discrete points, you'll need a separate expression for each segment.
2. First integration (slope). Integrate $EI \frac{d^2y}{dx^2} = M(x)$ to get: $EI \frac{dy}{dx} = \int M(x) dx + C_1$ The constant C_1 relates to the slope at whatever reference point you choose.
3. Second integration (deflection). Integrate again: $EI y = \iint M(x) dx dx + C_1 x + C_2$ The constant C_2 relates to the deflection at a reference point.

Applying Boundary Conditions

Boundary conditions are the known physical constraints of the beam. Common ones include:

- Pin or roller support: deflection $y = 0$ at that point (the beam can't move vertically there)
- Fixed (cantilever) support: both $y = 0$ and $\frac{dy}{dx} = 0$ at that point
- Free end: no constraint on y or slope, but shear and moment are zero (these help when writing $M(x)$, not when solving for constants)

Note a common mix-up: at a pinned support the deflection is zero but the slope is generally not zero. The beam is free to rotate there. At a fixed support, both deflection and slope are zero.

Substitute the boundary conditions into your slope and deflection equations to solve for C_1 and C_2 .

Solving for Deflection and Slope

With C_1 and C_2 determined, your equations are complete. Plug in any value of x to find the slope or deflection at that location. To find the point of maximum deflection, set $\frac{dy}{dx} = 0$ and solve for x , then substitute back into the deflection equation.

Plotting $y(x)$ over the full span gives you the elastic curve, which is the deformed shape of the beam. Always do a quick sanity check: deflection should be zero at supports and the curve should bow in the direction you'd physically expect.

Moment-Area Method for Deflection

First Moment-Area Theorem

The change in slope between any two points A and B on the elastic curve equals the area under the M/EI diagram between those points:

$$\theta_B - \theta_A = \int_A^B \frac{M}{EI} dx$$

If EI is constant (prismatic beam), this simplifies to $\frac{1}{EI}$ times the area under the M diagram between A and B.

To get the absolute slope at a point, you need a reference where the slope is known. For a cantilever, the slope at the fixed end is zero. For a simply supported beam, you typically compute the slope at one support first using the second theorem, then work from there.

Second Moment-Area Theorem

The tangential deviation of point B from the tangent drawn at point A equals the first moment of the M/EI diagram area (between A and B) taken about point B:

$$t_{B/A} = \int_A^B \frac{M}{EI} (x_B - x) dx$$

Here $t_{B/A}$ is the vertical distance from point B on the elastic curve to the tangent line drawn at A. Be careful: this is not the deflection itself. It's the deviation from a tangent. You often need to use geometry (similar triangles) to convert tangential deviations into actual deflections, especially for simply supported beams.

Applying the Moment-Area Method

1. Draw the M/EI diagram for the beam. For a prismatic beam, this is just the bending moment diagram scaled by $1/EI$.
2. Break the diagram into simple shapes. Triangles, rectangles, and parabolic segments each have known area and centroid formulas: - Triangle: $\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$, centroid at $\frac{1}{3}$ of the base from the larger end - Rectangle: $\text{Area} = \text{base} \times \text{height}$, centroid at the midpoint - Parabolic segment (from a distributed load): $\text{Area} = \frac{1}{3} \times \text{base} \times \text{height}$, centroid at $\frac{1}{4}$ of the base from the vertex
3. Sum areas for slope changes (first theorem).
4. Sum first moments of areas about the appropriate point for tangential deviations (second theorem).
5. Use geometry to convert tangential deviations into the actual deflections you need. For a simply supported beam, sketch the tangent at one support and use similar triangles to relate $t_{B/A}$ values to true vertical deflections.

Conjugate Beam Method for Deflection

Transforming the Real Beam into a Conjugate Beam

The conjugate beam has the same length as the real beam, but its supports are transformed according to these rules:

Real Beam Support	Conjugate Beam Support
Pin or roller	Pin or roller (same)
Fixed end	Free end
Free end	Fixed end
Internal hinge	Internal support (pin)
Internal support (pin)	Internal hinge

The loading on the conjugate beam is the M/EI diagram of the real beam, applied as a distributed load. Positive bending moment (sagging) is applied as downward load on the conjugate beam.

Analyzing the Conjugate Beam

Once you've set up the conjugate beam with its transformed supports and M/EI loading:

1. Calculate the support reactions of the conjugate beam using equilibrium ($\sum F = 0$, $\sum M = 0$).

2. Determine the shear force $V_c(x)$ and bending moment $M_c(x)$ at the points of interest using standard section cuts.

The key relationships are:

- Slope of the real beam at point x : $\theta(x) = V_c(x)$
- Deflection of the real beam at point x : $y(x) = M_c(x)$

Note that the EI is already baked into the conjugate beam's loading (you divided by EI when you created the load diagram), so you do not divide again when reading off results. The shear in the conjugate beam directly equals the slope, and the moment directly equals the deflection.

Calculating Deflection and Slope

1. Identify the point on the real beam where you need the deflection or slope.
2. Cut the conjugate beam at that same location.
3. Use equilibrium on the cut section to find V_c (gives slope) and M_c (gives deflection).
4. Sign conventions: positive shear in the conjugate beam corresponds to counterclockwise rotation of the real beam; positive moment in the conjugate beam corresponds to downward (positive) deflection.

The conjugate beam method is especially clean for cantilevers. The fixed end of a real cantilever becomes a free end on the conjugate beam, and the free end of the real cantilever becomes a fixed end on the conjugate beam. The "reaction moment" at the conjugate beam's fixed end directly gives you the tip deflection of the real cantilever.

Method Comparisons for Beam Deflection

Advantages and Limitations

- Double Integration Method:
 - Gives a complete equation for deflection at every point along the beam
 - Handles any loading type
 - Algebra-intensive; piecewise moment functions require careful bookkeeping of constants across segments
- Moment-Area Method:
 - Fast for finding slope or deflection at specific points
 - Visual and intuitive once you're comfortable with the M/EI diagram
 - Converting tangential deviations to actual deflections requires geometric reasoning that can trip you up
 - Less practical when the M/EI diagram has many irregular shapes
- Conjugate Beam Method:
 - Turns the problem into pure statics, which you already know well
 - Handles complex supports (internal hinges, overhangs) more naturally than the other methods

- Requires memorizing the support transformation rules and being careful with sign conventions

Choosing the Appropriate Method

The right method depends on what you need and what the beam looks like:

- If you need the full deflection curve or a general formula, use double integration.
- If you need deflection or slope at just one or two points on a simply supported or cantilever beam, the moment-area method is usually fastest.
- If the beam has unusual supports (internal hinges, propped cantilevers, overhangs), the conjugate beam method often simplifies the work.
- On exams, it's worth checking your answer with a second method if time allows. All three methods must give the same numerical result for the same beam and loading.

13.3 Statically indeterminate beams

Statically Indeterminate Beams

Statically indeterminate beams have more unknown reactions than equilibrium equations can handle, so standard statics alone won't get you to a solution. These beams show up constantly in real structures (think continuous bridge spans or fixed-end beams in building frames), and solving them requires combining equilibrium with additional conditions based on how the beam actually deforms.

Identifying Statically Indeterminate Beams

A beam is statically determinate when you can solve for every unknown reaction using just the three equilibrium equations: $\sum F_x = 0$, $\sum F_y = 0$, and $\sum M = 0$. When the number of unknown reactions exceeds three, you've got a statically indeterminate beam.

Common configurations that produce indeterminacy:

- Propped cantilever: A fixed support on one end and a roller or pin on the other gives you four unknowns (three from the fixed support, one from the roller) but only three equilibrium equations.
- Fixed-fixed beam: Both ends are fixed, producing six reaction unknowns total.
- Continuous beams: Multi-span beams over three or more supports. Each interior support adds an unknown vertical reaction.

The degree of static indeterminacy is simply the number of unknowns minus the number of independent equilibrium equations. A propped cantilever is indeterminate to the 1st degree. A fixed-fixed beam is indeterminate to the 3rd degree. The higher the degree, the more additional equations you'll need to solve the problem.

Solving for Reactions in Indeterminate Beams

Compatibility Equations

Since equilibrium alone isn't enough, you need compatibility equations to close the gap. These equations enforce the physical requirement that the beam's deflections and rotations remain continuous and consistent with the support conditions.

For example, if a propped cantilever has a roller at one end, you know the deflection at that roller is zero. That's your compatibility condition. You can then express the deflection at that point in terms of the unknown redundant reaction, set it equal to zero, and solve.

The general approach:

1. Identify the degree of indeterminacy (how many extra unknowns you have).
2. Choose that many reactions as redundants and temporarily remove them, creating a "released" or "primary" structure that is statically determinate.
3. Write compatibility equations that enforce the known displacement or rotation at each removed support.
4. Solve the compatibility equations (using superposition of deflections from the applied loads and the redundant reactions) to find the redundants.
5. Back-substitute into equilibrium equations to find all remaining reactions.

Slope-Deflection Method

The slope-deflection method writes the end moments of each beam segment in terms of the unknown joint rotations and displacements. The general slope-deflection equation for a member spanning from node A to node B is:

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi) + FEM_{AB}$$

where θ_A and θ_B are the rotations at ends A and B, ψ is the chord rotation (relative settlement divided by span length), L is the span, EI is the flexural rigidity, and FEM_{AB} is the fixed-end moment due to the applied loads.

Steps to apply the method:

1. Calculate fixed-end moments (FEMs) for each loaded span, treating every joint as fixed.
2. Write slope-deflection equations for both ends of every span.
3. Apply equilibrium at each joint: the sum of moments from all members meeting at a joint equals zero.
4. Apply boundary conditions (e.g., $\theta = 0$ at a fixed support, $M = 0$ at a pin or roller).
5. Solve the resulting system of equations for the unknown rotations and displacements.
6. Substitute back into the slope-deflection equations to get the final end moments.

This method works well for continuous beams and frames, especially when you're comfortable solving simultaneous equations.

Moment-Distribution Method

The moment-distribution method is an iterative technique developed by Hardy Cross that avoids solving simultaneous equations directly. It's particularly efficient for hand calculations.

Key concepts:

- Stiffness factor: For a prismatic member of length L and flexural rigidity EI , the stiffness is $\frac{4EI}{L}$ if the far end is fixed, or $\frac{3EI}{L}$ if the far end is pinned.

- Distribution factor (DF): At any joint, each member's DF equals its stiffness divided by the total stiffness of all members at that joint. The DFs at a joint always sum to 1.
- Carry-over factor: When a moment is distributed to one end of a member, half of that moment carries over to the far end (for a fixed far end).

Steps:

1. Calculate fixed-end moments for all loaded spans.
2. Compute distribution factors at each joint.
3. Identify the unbalanced moment at each joint (the algebraic sum of FEMs).
4. Distribute the unbalanced moment to each member at the joint according to the DFs (with opposite sign to balance).
5. Carry over half of each distributed moment to the far end of each member.
6. Repeat steps 3–5 until the carry-over moments become negligibly small.
7. Sum all moments at each member end to get the final end moments.

Convergence is usually fast, often requiring only 3–4 cycles for reasonable accuracy.

Deflection and Slope of Indeterminate Beams

Once you've found the reactions and internal moments, you still need to determine deflections and slopes. Several methods work for indeterminate beams.

Moment-Area Method

This method uses two theorems that relate the M/EI diagram to slopes and deflections:

- First theorem: The change in slope between two points A and B equals the area under the M/EI diagram between those points.

$$\theta_B - \theta_A = \int_A^B \frac{M}{EI} dx$$

- Second theorem: The vertical distance of point B from the tangent drawn at point A equals the first moment of the M/EI diagram area (between A and B) taken about point B.

$$t_{B/A} = \int_A^B \frac{M}{EI} \cdot \bar{x}_B dx$$

where $\bar{x}_B$ is the horizontal distance from each differential area element to point B.

To use this method, you sketch the M/EI diagram (often breaking it into simple shapes like triangles and parabolas), compute areas and centroids, and apply the theorems. It works best when the M/EI diagram has a clean geometric shape.

Conjugate-Beam Method

The conjugate-beam method converts the deflection problem into a familiar shear-and-moment problem on an imaginary beam.

The rules:

- Load the conjugate beam with the M/EI diagram of the original beam as a distributed load.

- The shear in the conjugate beam at any point equals the slope of the original beam at that point.
- The moment in the conjugate beam at any point equals the deflection of the original beam at that point.

The support conditions of the conjugate beam are transformed from the original:

Original Beam Support	Conjugate Beam Support
Pin or roller	Pin or roller
Fixed end	Free end
Free end	Fixed end
Internal pin/hinge	Internal support

This transformation ensures the math stays consistent. Once you set up the conjugate beam, you just analyze it like any other loaded beam using equilibrium.

Direct Integration Method

This is the most fundamental approach. You start from the moment-curvature relationship:

$$EI \frac{d^2y}{dx^2} = M(x)$$

1. Express $M(x)$ as a function of position along the beam (you'll need the reactions first).
2. Integrate once to get the slope: $EI \frac{dy}{dx} = \int M(x) dx + C_1$
3. Integrate again to get the deflection: $EI \cdot y = \iint M(x) dx dx + C_1x + C_2$
4. Apply boundary conditions (known deflections and slopes at supports) to solve for the constants C_1 and C_2 .
5. For indeterminate beams with multiple segments, enforce continuity of slope and deflection at the junction between segments.

Direct integration is straightforward when the moment equation is simple, but it gets tedious for beams with multiple load regions.

Principle of Virtual Work

Virtual work (also called the unit-load method) lets you find the deflection at any specific point on the beam.

1. Remove all real loads and apply a virtual unit load ($P = 1$) at the point and in the direction of the desired deflection.
2. Solve for the internal virtual bending moment $m(x)$ throughout the beam due to this unit load.
3. The real deflection Δ at that point is:

$$\Delta = \int_0^L \frac{M(x) \cdot m(x)}{EI} dx$$

where $M(x)$ is the real bending moment from the actual loads.

For slope at a point, apply a virtual unit moment instead of a unit force. This method handles complex loading and variable cross-sections well, making it one of the most versatile tools for indeterminate beam analysis.

Support Settlements in Indeterminate Beams

Effects of Support Settlements

In a determinate beam, if a support settles (moves downward), the beam simply follows it without developing any new internal forces. Indeterminate beams behave very differently. Because the beam is constrained at multiple points, a settlement at one support forces the beam to deform in a way that generates additional bending moments, shear forces, and reaction changes throughout the structure.

The severity of these induced forces depends on:

- The magnitude of the settlement
- The location of the settling support relative to other supports
- The beam's flexural rigidity EI (stiffer beams develop larger forces from the same settlement)
- The span lengths and overall configuration

Analysis Techniques for Support Settlements

The same methods used for loading analysis can be adapted for settlements. The key difference is that the compatibility equations now include known nonzero displacements instead of zeros.

Using the slope-deflection method: The chord rotation term $\psi = \Delta/L$ (where Δ is the relative settlement between two adjacent supports) gets plugged directly into the slope-deflection equations. The rest of the procedure stays the same.

Using the moment-distribution method: Settlements modify the fixed-end moments. You calculate the FEMs caused by the settlement (using $FEM = \pm \frac{6EI\Delta}{L^2}$ for a prismatic member with relative settlement Δ), then distribute and carry over as usual.

Using superposition: You can analyze the beam under applied loads (ignoring settlements) and under settlements alone (ignoring applied loads), then add the results together. This separation often simplifies the work considerably.

Using the conjugate-beam method: The settlements appear as additional boundary conditions on the conjugate beam, modifying the reactions and therefore the slope and deflection results.

Accounting for support settlements is critical in practice because differential settlement between supports is one of the most common causes of distress in continuous beam and frame structures.

Unit 14 – Combined Loading & Stress Transformation

14.1 Principal stresses and maximum shear stress

Principal stresses are the key to understanding a material's stress state. They represent the maximum and minimum normal stresses at a point, helping us identify critical stress orientations. Knowing these values is crucial for designing components that can withstand various loads.

Calculating principal stresses involves solving equations based on the given stress state. For 2D problems, we use simple formulas, while 3D cases require solving a characteristic equation. These calculations help us determine the orientation of principal planes and maximum shear stress.

Principal Stresses and Significance

Definition and Importance

- Principal stresses are the normal stresses acting on planes where shear stresses are zero
- Represent the maximum and minimum normal stresses at a point in a stressed body
- Help identify the most critical stress states and orientations
- Essential for designing and analyzing components for strength and durability

Notation and Uniqueness

- The three principal stresses are denoted as σ_1 , σ_2 , and σ_3 , where $\sigma_1 \geq \sigma_2 \geq \sigma_3$
- The principal stress state is unique for a given stress state
- Independent of the coordinate system used to define the original stresses

Relation to Other Stress Quantities

- Principal stresses are used to calculate other important stress quantities
- Maximum shear stress and von Mises stress are derived from principal stresses
- These quantities are used in failure theories (Tresca and von Mises criteria)

Calculating Principal and Shear Stress

Two-Dimensional State of Stress

- Principal stresses can be calculated using the equations: $\sigma_{1,2} = (\sigma_x + \sigma_y) / 2 \pm \sqrt{((\sigma_x - \sigma_y) / 2)^2 + \tau_{xy}^2}$
- σ_x and σ_y are normal stresses and τ_{xy} is the shear stress
- Maximum shear stress (τ_{max}) is equal to one-half the difference between the maximum and minimum principal stresses: $\tau_{max} = (\sigma_1 - \sigma_3) / 2$

Three-Dimensional State of Stress

- Principal stresses can be found by solving the characteristic equation: $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$
- I_1 , I_2 , and I_3 are stress invariants calculated using the following equations:
- $I_1 = \sigma_x + \sigma_y + \sigma_z$
- $I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$
- $I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$
- Solving the characteristic equation yields the principal stresses (σ_1 , σ_2 , σ_3)

Orientation of Principal Planes

Two-Dimensional State of Stress

- Principal planes are the planes on which principal stresses act and where shear stresses are zero
- The orientation of principal planes (θ_p) can be found using the equation: $\tan(2\theta_p) = 2\tau_{xy} / (\sigma_x - \sigma_y)$
- θ_p is measured counterclockwise from the positive x-axis
- Maximum shear stress planes are oriented at 45° to the principal planes
- The orientation of maximum shear stress planes (θ_s) can be found using the equation: $\theta_s = \theta_p \pm 45^\circ$

Three-Dimensional State of Stress

- The orientation of principal planes is given by the eigenvectors corresponding to the principal stresses
- Eigenvectors can be found by solving the eigenvalue problem
- The eigenvalue problem involves setting up a system of linear equations based on the stress tensor and solving for non-trivial solutions

Interpretation of Principal Stresses and Shear Stress

Physical Meaning of Principal Stresses

- The maximum principal stress (σ_1) represents the maximum normal stress acting on any plane at a given point
- The minimum principal stress (σ_3) represents the minimum normal stress
- The intermediate principal stress (σ_2) acts in a direction perpendicular to both σ_1 and σ_3
- The signs of principal stresses indicate the nature of the stress: positive for tension and negative for compression

Significance of Maximum Shear Stress

- Maximum shear stress (τ_{max}) represents the maximum shear stress acting on any plane at a given point
- Important for predicting yielding in ductile materials according to the Tresca yield criterion

- The planes of maximum shear stress are oriented at 45° to the principal planes
- Indicates the planes on which the material is most likely to experience shear failure (sliding or shearing)

14.2 Mohr's circle for plane stress

Mohr's circle is a powerful tool for visualizing and analyzing plane stress states. It helps engineers determine principal stresses, maximum shear stress, and stress transformations, which are crucial for understanding how materials behave under complex loading conditions.

By using Mohr's circle, you can easily find stresses on any plane and identify critical stress orientations. This knowledge is essential for designing safe and efficient structures, as it allows you to predict potential failure modes and optimize component geometry.

Mohr's Circle Construction

Graphical Representation of Plane Stress

- Mohr's circle represents the state of stress at a point in a material subjected to plane stress
- The x-axis represents normal stress (σ) and the y-axis represents shear stress (τ)
- The center of Mohr's circle is located at $(\sigma_{avg}, 0)$, where $\sigma_{avg} = (\sigma_x + \sigma_y) / 2$
- The radius of Mohr's circle is calculated using the formula: $R = \sqrt{((\sigma_x - \sigma_y)^2 / 4 + \tau_{xy}^2)}$

Stress Orientation on Mohr's Circle

- The angle between the x-axis and the line connecting the center of the circle to a point on the circle represents twice the angle (2θ) between the x-axis and the plane on which the stresses act
- This relationship allows for the determination of stresses acting on planes oriented at different angles to the original coordinate system
- Mohr's circle provides a visual representation of how stress components change as the orientation of the plane changes
- The sign convention for shear stress on Mohr's circle: positive shear stress acts counterclockwise on the positive face of the element, and clockwise on the negative face

Stress Analysis with Mohr's Circle

Principal Stresses and Maximum Shear Stress

- Principal stresses (σ_1 and σ_2) are the maximum and minimum normal stresses acting on a point, occurring on planes where shear stress is zero
- On Mohr's circle, principal stresses are represented by the points where the circle intersects the σ -axis (x-axis)
- The maximum shear stress (τ_{max}) equals the radius of Mohr's circle and occurs on planes oriented at 45° to the principal stress planes
- Principal stresses and maximum shear stress provide crucial information about the critical stress states in a material

Stress Orientation Determination

- The orientation of the principal stress planes (θ_p) is determined by measuring half the angle between the σ -axis and the line connecting the center of the circle to the point representing the principal stress
- The orientation of the maximum shear stress planes (θ_s) is 45° from the principal stress planes
- Knowing the orientation of principal stress and maximum shear stress planes helps engineers design components to withstand critical stress conditions
- Stress orientation information is essential for understanding the behavior of materials under complex loading scenarios (combined axial, bending, and torsional loads)

Stress Transformations with Mohr's Circle

Stress Transformation Concept

- Stress transformation involves determining the normal and shear stresses acting on a plane oriented at a specific angle to the original coordinate system
- Mohr's circle allows for the graphical transformation of stress components from one coordinate system to another
- Stress transformation is crucial for analyzing stresses in rotated or inclined planes (cross-sections of beams, pressure vessels, or structural elements)
- Understanding stress transformations helps engineers assess the strength and stability of components under various loading conditions

Stress Transformation Procedure

- To find the stresses acting on a plane at an angle θ , locate the point on Mohr's circle corresponding to an angle 2θ from the σ -axis
- The normal stress (σ_θ) acting on the plane is the x-coordinate of the point on Mohr's circle
- The shear stress (τ_θ) acting on the plane is the y-coordinate of the point on Mohr's circle
- The transformed stress components (σ_θ and τ_θ) can be used to assess the stress state on any plane of interest
- Stress transformations can also be performed using analytical equations derived from Mohr's circle relationships $\sigma_\theta = (\sigma_x + \sigma_y) / 2 + ((\sigma_x - \sigma_y) / 2) * \cos(2\theta) + \tau_{xy} * \sin(2\theta)$ and $\tau_\theta = -((\sigma_x - \sigma_y) / 2) * \sin(2\theta) + \tau_{xy} * \cos(2\theta)$

Plane Stress Problem Solving with Mohr's Circle

Problem-Solving Steps

- Identify the given stress components (σ_x , σ_y , and τ_{xy}) and the angle of the plane of interest (θ)
- Construct Mohr's circle using the given stress components, following the construction steps (locate center, calculate radius, and plot circle)
- Determine the principal stresses (σ_1 and σ_2) and maximum shear stress (τ_{max}) from Mohr's circle by identifying the intersection points with the σ -axis and measuring the radius

- If required, find the normal and shear stresses (σ_θ and τ_θ) acting on a plane at a specific angle θ using the stress transformation technique (locate the point on the circle corresponding to 2θ)

Interpreting Results and Application

- Interpret the results in the context of the problem, considering the magnitude and orientation of the stresses acting on the plane of interest
- Use the obtained stress values to assess the strength, stability, and failure criteria of the component or structure (compare with yield strength, ultimate strength, or fatigue limits)
- Apply the results to optimize the design of components, ensuring they can withstand the anticipated stress conditions (material selection, geometry modifications, or reinforcements)
- Combine the results from Mohr's circle analysis with other design considerations (functionality, manufacturability, and cost) to develop a comprehensive solution to the engineering problem

14.3 Combined loading scenarios and analysis

Combined Loading Scenarios

Combined loading scenarios involve multiple forces acting on a structure at the same time. In real engineering, structural members almost never experience just one type of load in isolation. A shaft in a machine might be twisted and bent simultaneously, or a column might carry an axial load while also resisting wind-induced bending. Analyzing these combined effects is essential for predicting whether a component will survive its service conditions.

The approach relies on the superposition principle: break the complex loading into individual components, calculate each stress separately, then add them together. From there, you use stress transformation to find the critical stress values (principal stresses and maximum shear stress) and compare them against failure criteria.

Combined Loading Scenarios

Types of Combined Loading

Most structural members experience more than one load type at once. Here are the primary types you need to know:

- Axial loading acts along the longitudinal axis, producing either tension (pulling the member apart) or compression (pushing it together). The stress is uniform across the cross-section.
- Bending loads act perpendicular to the longitudinal axis, causing the member to curve. One side goes into tension while the opposite side goes into compression, with a neutral axis in between where bending stress is zero.
- Torsional loads are twisting moments applied about the longitudinal axis. These produce shear stresses that vary from zero at the center to a maximum at the outer surface of a circular cross-section.
- Transverse shear forces cause internal shear stresses distributed across the cross-section (recall the shear formula $\tau = VQ/(Ib)$).
- Thermal loads arise from temperature changes that cause expansion or contraction, which can generate significant stresses if the member is constrained.

A classic combined loading example: a cantilever beam with an offset load creates bending and torsion simultaneously. A vertical column supporting a load that's slightly off-center experiences axial compression and bending.

Superposition Principle

The superposition principle states that the total stress (or strain) at any point in a member under combined loading equals the algebraic sum of the stresses (or strains) caused by each individual load acting alone.

For example, if a beam carries both an axial force and a bending moment, the normal stress at a point on the cross-section is:

$$\sigma_{total} = \frac{P}{A} + \frac{My}{I}$$

You simply add the axial contribution and the bending contribution. On one side of the neutral axis these may add together; on the other side they may partially cancel.

Superposition is valid only when: - The material behaves linearly and elastically (stress is proportional to strain) - Deformations are small enough that the geometry of the structure doesn't change significantly under load

If either condition is violated (plastic deformation, large deflections, or buckling scenarios), superposition does not apply.

Stress and Strain Analysis

Calculating Stresses

When you encounter a combined loading problem, follow these steps:

1. Identify all external loads acting on the member (axial forces, bending moments, torques, shear forces).
2. Use internal equilibrium (free-body diagrams, section cuts) to find the internal forces and moments at the cross-section of interest.
3. Calculate each stress component separately using the appropriate formula: - Axial stress: $\sigma_{axial} = \frac{P}{A}$ - Bending stress: $\sigma_{bending} = \frac{My}{I}$ - Torsional shear stress: $\tau_{torsion} = \frac{Tr}{J}$ - Transverse shear stress: $\tau_{shear} = \frac{VQ}{Ib}$
4. Combine the stress components at the point of interest. Normal stresses (axial + bending) add algebraically. Shear stresses (torsion + transverse shear) also add algebraically if they act in the same direction at that point.
5. Write the stress state at the point as σ_x , σ_y , and τ_{xy} for use in stress transformation.

The critical point on a cross-section depends on the loading. For combined axial and bending, check the top and bottom fibers where bending stress is maximum. For combined bending and torsion, the point where both contributions are large is typically the critical location.

Analyzing Strains

Once you know the stresses, you can find the corresponding strains using material relationships:

- Hooke's law for normal stress: $\sigma = E\epsilon$, so $\epsilon = \frac{\sigma}{E}$, where E is Young's modulus.
- Axial strain: $\epsilon = \frac{P}{AE}$

- Bending strain varies linearly with distance from the neutral axis: $\epsilon = KY$, where K is the beam curvature ($K = M/(EI)$).
- Shear strain from torsion: $\gamma = \frac{\tau}{G} = \frac{Tr}{JG}$, where G is the shear modulus.

Don't forget the Poisson effect: when a member is stretched axially, it contracts laterally (and vice versa). The lateral strain is $\epsilon_{lateral} = -\nu\epsilon_{axial}$, where ν is Poisson's ratio. This matters when you need the full 2D or 3D strain state.

Stress Transformation for Combined Loads

Principal Stresses

After combining the individual stress components, you'll have a stress state defined by σ_x , σ_y , and τ_{xy} at a particular point. Stress transformation lets you find the stresses on planes oriented at any angle θ relative to the original axes.

Principal stresses are the maximum and minimum normal stresses at a point. They occur on planes where the shear stress is zero. These are the values you'll compare against material strength.

To find principal stresses:

1.

Find the orientation of the principal planes using: $\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$. This gives two angles 90° apart (the two principal directions).

2.

Calculate the principal stress magnitudes using: $\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$. Here, σ_1 is the larger (more tensile) value and σ_2 is the smaller one.

The term $(\sigma_x + \sigma_y)/2$ is the average normal stress, and the square root term is the radius of Mohr's circle.

Maximum Shear Stress

The maximum in-plane shear stress occurs on planes oriented 45° from the principal planes. Its magnitude is:

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Notice that τ_{max} equals the radius of Mohr's circle. On the planes of maximum shear stress, the normal stress is not zero; it equals the average normal stress $(\sigma_x + \sigma_y)/2$.

Mohr's Circle provides a visual way to see all of this at once:

1. Plot point $A = (\sigma_x, \tau_{xy})$ and point $B = (\sigma_y, -\tau_{xy})$ on a σ - τ graph. (Sign convention: positive shear rotates the element clockwise on the face where you're measuring.)
2. The center of the circle is at $C = \left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$.
3. The radius is $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.
4. The circle's intersections with the horizontal axis give σ_1 and σ_2 . The top and bottom of the circle give $\pm\tau_{max}$.

Angles on Mohr's circle are doubled: a physical rotation of θ corresponds to 2θ on the circle. This is a common source of mistakes on exams.

Safety and Performance Evaluation

Failure Criteria

Once you've found the principal stresses, you need to determine whether the material will fail. The appropriate criterion depends on the material type.

For ductile materials (steel, aluminum, most metals):

The von Mises yield criterion is the most widely used. It predicts yielding when the equivalent (von Mises) stress reaches the material's yield strength σ_Y :

$$\sigma_e = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$$

For plane stress ($\sigma_3 = 0$), this simplifies to:

$$\sigma_e = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2}$$

Yielding occurs when $\sigma_e \geq \sigma_Y$.

For brittle materials (cast iron, concrete, ceramics):

- The maximum normal stress criterion (Rankine's criterion) predicts failure when the largest principal stress reaches the material's ultimate tensile or compressive strength.
- The Mohr-Coulomb criterion accounts for the difference between tensile and compressive strengths and defines a failure envelope on the Mohr's circle diagram. This is particularly useful for materials like concrete that are much stronger in compression than in tension.

Safety Factors and Performance Requirements

The factor of safety (FS) quantifies how much stronger the component is compared to what's required:

$$FS = \frac{\text{Material strength}}{\text{Maximum applied stress}}$$

A factor of safety greater than 1 means the component can handle more than the applied load. Typical values range from 1.5 to 4 depending on the application, consequences of failure, and how well the loads are known.

Fatigue is a concern when loads are cyclic (repeated loading and unloading). A component can fail at stresses well below the static yield strength if subjected to enough cycles.

- The stress-life (S-N) approach uses an S-N curve that plots stress amplitude versus number of cycles to failure. For steels, there's often an endurance limit below which the material can theoretically survive infinite cycles.
- The strain-life (ϵ -N) approach accounts for local plastic strains at stress concentrations and uses the Coffin-Manson equation to predict fatigue life. This is more accurate for low-cycle fatigue situations.

Deflection and stiffness requirements can also govern a design. Even if stresses are safe, excessive deflection may make a component unusable. Methods for calculating deflections under combined loading include:

- Direct integration: Solve the differential equation $EI \frac{d^2 y}{dx^2} = M(x)$ with appropriate boundary conditions.
- Moment-area method: Use the first and second moment-area theorems to find slopes and deflections from the M/EI diagram.

In practice, you check all three: stress (against failure criteria), fatigue life (if loading is cyclic), and deflection (against serviceability limits).

Unit 15 – Columns and Structural Stability

15.1 Buckling of columns and critical loads

Column Buckling Modes

Columns under compression don't always fail by being crushed. Instead, they often fail through buckling, a sudden sideways deflection that can be catastrophic. Understanding the different buckling modes and how to calculate critical loads is central to safe structural design.

Types of Buckling Modes

The way a column buckles depends on its end conditions, slenderness ratio, and cross-sectional shape. There are three main modes:

- Euler (elastic) buckling occurs in long, slender columns. The column deflects laterally all at once when the critical load is reached. The material itself hasn't yielded; the failure is purely a stability problem.
- Inelastic buckling occurs in shorter, stockier columns. Here, portions of the material yield before the theoretical elastic critical load is reached, so the column loses stiffness and buckles at a lower load than Euler's formula would predict.
- Torsional buckling occurs in columns with open, thin-walled cross-sections (I-beams, C-channels). Instead of deflecting sideways, the column twists about its longitudinal axis.

Factors Affecting Buckling Mode

End conditions determine the column's effective length, which directly controls buckling behavior:

- Fixed ends resist both rotation and translation, providing the most buckling resistance.
- Pinned ends allow rotation but prevent translation.
- Free ends allow both rotation and translation, offering the least restraint.

A fixed-fixed column has a much shorter effective length than a fixed-free (cantilever) column of the same physical length, so it can carry a significantly higher critical load.

The slenderness ratio L/r (effective length divided by radius of gyration) is the single most important parameter for predicting which buckling mode governs:

- High $L/r \rightarrow$ elastic (Euler) buckling dominates
- Low $L/r \rightarrow$ inelastic buckling or direct yielding dominates

The radius of gyration r depends on the cross-section's moment of inertia and area: $r = \sqrt{I/A}$. A column always tends to buckle about the axis with the smallest radius of gyration.

Critical Load Calculation

Elastic Buckling

The critical load P_{cr} is the maximum axial compressive load a column can support before buckling. For long, slender columns that remain elastic, Euler's formula gives:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

where: - E = modulus of elasticity - I = moment of inertia about the weaker axis - L = effective length (not necessarily the physical length)

To express this as a stress, divide by the cross-sectional area A , which yields the Euler buckling stress:

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

This formula is only valid when σ_{cr} falls below the material's proportional limit. If it doesn't, the column will buckle inelastically and you need a different approach.

Inelastic Buckling

When the slenderness ratio is low enough that the Euler stress exceeds the proportional limit, the material's stiffness is no longer constant. Two classical theories address this:

- Tangent modulus theory: Replace E with the tangent modulus E_t (the slope of the stress-strain curve at the current stress level):

$$P_{cr} = \frac{\pi^2 E_t I}{L^2}$$

This gives a lower-bound estimate of the critical load because it assumes the entire cross-section unloads along the reduced-stiffness curve.

- Reduced (secant) modulus theory: Replace E with a reduced modulus E_r that accounts for the fact that fibers on the convex side of the column actually unload elastically while fibers on the concave side continue loading inelastically:

$$P_{cr} = \frac{\pi^2 E_r I}{L^2}$$

This gives an upper-bound estimate. Experimental results tend to fall between the two, but closer to the tangent modulus prediction.

For practical design of intermediate-length columns, the Johnson formula provides a parabolic transition between yielding and Euler buckling:

$$P_{cr} = A \left[\sigma_y - \frac{\sigma_y^2}{4\pi^2 E} (L/r)^2 \right]$$

where A is the cross-sectional area and σ_y is the yield stress. You use Johnson's formula when L/r is below the transition slenderness ratio, and Euler's formula when L/r is above it.

Factors Influencing Buckling

Material and Geometric Properties

- Modulus of elasticity E : Appears directly in Euler's formula. A stiffer material (higher E) resists buckling more effectively.
- Moment of inertia I : Larger I means greater resistance to bending, and therefore greater buckling strength. Cross-sections like wide-flange shapes and hollow tubes are efficient because they place material far from the centroid, maximizing I .
- Effective length L : Shorter effective lengths yield higher critical loads. This is why end conditions matter so much; a column with both ends fixed ($L_{eff} = 0.5L$) can carry roughly four times the critical load of the same column with both ends pinned ($L_{eff} = L$).

- Residual stresses from manufacturing (hot-rolling, welding) can cause portions of the cross-section to yield prematurely, reducing the actual buckling load below the theoretical value. This effect is most significant in the inelastic range.

Initial Imperfections and Loading Conditions

Real columns are never perfectly straight or perfectly loaded. These imperfections reduce buckling capacity:

- Out-of-straightness: Even a small initial bow creates bending moments as soon as axial load is applied, amplifying deflection and triggering buckling at loads below P_{cr} .
- Eccentric loading: When the load doesn't act through the centroid, it produces a bending moment $M = Pe$ (where e is the eccentricity) from the start. The column then behaves as a beam-column, and its capacity is reduced.
- Lateral loads or end moments: Any transverse force or applied moment adds bending stress to the compressive stress, reducing the axial load the column can carry before instability occurs. End moments can also change the effective buckled shape and, depending on their direction, either increase or decrease the effective length.

Elastic vs. Inelastic Buckling

Elastic Buckling Characteristics

Elastic buckling governs when the column is slender enough that the critical stress stays below the proportional limit. Key features:

- The failure is a stability problem, not a material strength problem. The material itself is never overstressed.
- The process is reversible: remove the load, and the column springs back to its original straight shape.
- Buckling happens suddenly with no visible warning. The column appears fine right up until it deflects sideways.
- The critical stress depends only on E and L/r , not on the yield strength:

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

Inelastic Buckling Characteristics

Inelastic buckling governs for stockier columns where the Euler stress would exceed the proportional limit. Key features:

- Parts of the cross-section yield before the column buckles, reducing its effective stiffness.
- The deformation is permanent and irreversible.
- The critical stress is always lower than what Euler's formula would predict, because the material's stiffness drops once you pass the proportional limit.
- The transition point between elastic and inelastic buckling occurs at a specific slenderness ratio. You can find it by setting the Euler stress equal to the proportional limit (or yield stress, depending on the design code) and solving for L/r .

For inelastic buckling, use the tangent modulus formula ($P_{cr} = \frac{\pi^2 E_t I}{L^2}$) or the Johnson formula for design. The choice between elastic and inelastic formulas comes down to comparing your column's actual L/r to the transition slenderness ratio.

15.2 Euler's formula and its limitations

Euler's Formula for Slender Columns

Calculating Critical Load

Euler's formula gives you the critical load (P_{cr}), the maximum axial load a slender column can carry before it buckles elastically. The formula is:

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

Three quantities drive the result:

- Modulus of elasticity (E): the material's stiffness
- Moment of inertia (I): the cross-section's resistance to bending
- Effective length (L_e): the column's length adjusted for its end conditions

Example: A steel column has $E = 200 \text{ GPa}$, $I = 1.2 \times 10^{-4} \text{ m}^4$, and $L_e = 3 \text{ m}$. Plugging in:

$$P_{cr} = \frac{\pi^2 (200 \times 10^9) (1.2 \times 10^{-4})}{3^2} = \frac{\pi^2 (2.4 \times 10^4)}{9} \approx 26.3 \text{ MN}$$

Notice that L_e is squared in the denominator. That means doubling the effective length cuts the critical load to one-quarter. Length has a huge influence on buckling capacity.

Moment of Inertia and Effective Length

The moment of inertia (I) depends on the cross-sectional shape and dimensions. A column always buckles about the axis with the smallest I , so you need to check both principal axes.

Common formulas:

- Solid circular section (radius r): $I = \frac{\pi r^4}{4}$
- Solid rectangular section (width b , height h): $I = \frac{bh^3}{12}$

The effective length (L_e) is the equivalent pinned-pinned length that would buckle at the same critical load as the actual column with its real end conditions. It accounts for how much the supports restrain rotation and translation. (More on how to calculate L_e in the End Conditions section below.)

Elastic Buckling and Critical Stress

Elastic buckling is a sudden lateral deflection that occurs when the axial load reaches P_{cr} . The column doesn't get progressively more loaded; instead, it snaps sideways at a well-defined load. All of this happens while the material is still in its elastic range (stresses below the proportional limit).

You can convert the critical load to a critical stress by dividing by the cross-sectional area:

$$\sigma_{cr} = \frac{P_{cr}}{A}$$

Example: For a column with $P_{cr} = 200 \text{ kN}$ and $A = 0.005 \text{ m}^2$:

$$\sigma_{cr} = \frac{200 \times 10^3}{0.005} = 40 \text{ MPa}$$

This critical stress must stay below the material's proportional limit for Euler's formula to be valid. If σ_{cr} exceeds the proportional limit, the column will yield before it can buckle elastically, and Euler's formula overpredicts the actual failure load.

Assumptions and Limitations of Euler's Formula

Ideal Column Assumptions

Euler's formula is derived for an ideal column. That means it assumes:

- The column is perfectly straight with no initial curvature or crookedness.
- The material is homogeneous and linearly elastic (obeys Hooke's law), with stresses remaining below the proportional limit.
- The axial load is applied concentrically, exactly through the centroid of the cross-section, with no eccentricity or applied moments.
- The column is free from residual stresses and manufacturing imperfections.

Real columns never satisfy all of these perfectly, which is why actual buckling loads are typically lower than Euler's prediction.

Slenderness Ratio Limitations

The slenderness ratio is defined as L/r , where L is the unsupported length and r is the radius of gyration. Euler's formula is only valid for columns with high slenderness ratios, typically $L/r > 100$ or so (the exact threshold depends on the material).

Why the cutoff? For stocky columns with low slenderness ratios, the calculated σ_{cr} from Euler's formula exceeds the material's yield strength. The material yields before elastic buckling can occur. This is called inelastic buckling, and Euler's formula does not account for it.

Other Limitations

- Shear deformation: Euler's formula neglects shear effects, which can matter in deep, short members.
- Local buckling: Thin-walled or open cross-sections (like channels or angles) can buckle locally at a flange or web before the whole column buckles globally. Euler's formula doesn't capture this.
- Complex cross-sections: Non-uniform material properties or unusual shapes may require finite element analysis or other advanced methods.
- Initial imperfections and residual stresses: These reduce the actual buckling load below the Euler prediction. Design codes handle this by applying safety factors or using empirical column curves.

Applicability of Euler's Formula

Slenderness Ratio and Column Classification

The slenderness ratio (L/r) is the primary tool for deciding whether Euler's formula applies. The radius of gyration is:

$$r = \sqrt{\frac{I}{A}}$$

It represents how far the cross-section's area is spread from the bending axis. A larger r means more resistance to buckling.

Columns are classified into three categories based on slenderness:

Category	Slenderness Ratio	Likely Failure Mode	Euler Applicable?
Slender	$L/r > 100$	Elastic buckling	Yes
Intermediate	$50 < L/r < 100$	Mixed elastic/inelastic	No; use Johnson-Euler or secant formula
Short	$L/r < 50$	Yielding or crushing	No; use yield-based criteria

These boundaries are approximate and shift depending on the material. For example, a high-strength steel column transitions to inelastic behavior at a different slenderness ratio than an aluminum column.

Material Properties and Cross-Sectional Shape

The ratio E/σ_y (modulus of elasticity to yield strength) affects where the transition from elastic to inelastic buckling occurs. Materials with a high E/σ_y ratio (like mild steel or aluminum) have a wider range of slenderness ratios where Euler's formula works well. Materials with a low ratio transition to inelastic buckling sooner.

Cross-sectional shape matters too:

- Solid or compact sections (solid rectangles, solid circles) tend to buckle globally, which is what Euler's formula predicts.
- Thin-walled or open sections (wide-flange I-beams, channels) are more susceptible to local buckling of individual flanges or webs, which can trigger failure before the Euler load is reached. For these shapes, you need to check local buckling limits in addition to the global Euler load.

End Conditions and Critical Load

Effective Length Factor

End conditions have a dramatic effect on a column's buckling capacity. The effective length factor (K) converts the actual unsupported length L into the effective length used in Euler's formula:

$$L_e = K \cdot L$$

Since L_e is squared in the denominator of Euler's formula, even small changes in K significantly change P_{cr} .

Ideal End Conditions

The four standard cases and their K values:

End Condition	Description	K	L_e	Relative Strength
Fixed-Fixed	No rotation or translation at either end	0.5	$0.5L$	Strongest (4× pinned-pinned)
Fixed-Pinned	Fixed at one end, pinned at the other	0.7	$0.7L$	~2× pinned-pinned

End Condition	Description	K	L_e	Relative Strength
Pinned-Pinned	Free rotation, no translation at both ends	1.0	L	Baseline case
Fixed-Free (Cantilever)	Fixed at base, completely free at top	2.0	$2L$	Weakest (¼ of pinned-pinned)

A fixed-fixed column is four times stronger than a pinned-pinned column of the same length and cross-section, because its effective length is half as long and P_{cr} depends on L_e^2 .

A cantilever column (fixed-free) is the weakest configuration. Its effective length is twice the actual length, so its critical load is only one-quarter that of a pinned-pinned column.

Partial Fixity and End Restraints

In practice, connections are rarely perfectly pinned or perfectly fixed. Most real supports provide partial rotational restraint, giving an effective length factor somewhere between the ideal cases.

- A column with partial fixity at both ends might have K between 0.5 and 1.0.
- Example: A column bolted at both ends with some rotational stiffness might use $K = 0.8$, giving $L_e = 0.8L$.

When in doubt, design codes often recommend using conservative (higher) K values. Overestimating K gives a lower P_{cr} , which is the safe direction for design.

15.3 Design considerations for columns

Column Cross-Section Selection

Cross-Section Shapes and Factors Influencing Selection

Columns come in several common cross-section shapes: rectangular, circular, hollow structural sections (HSS), I-shaped (wide-flange), and built-up sections. Each shape has trade-offs, and the right choice depends on several practical factors:

- Load-carrying capacity: Select a cross-section that efficiently resists the applied loads. For example, hollow sections resist buckling well in all directions, while wide-flange sections are more efficient when buckling is restrained about one axis.
- Aesthetics: The column's visual appearance matters in exposed structures like building lobbies or bridges.
- Ease of fabrication: Some shapes are simpler to manufacture, cut, and assemble on site.
- Connection requirements: The cross-section must be compatible with the intended connection methods (bolting, welding). Wide-flange columns, for instance, offer flat surfaces that simplify bolted connections.

Strength and Stability Considerations

Strength considerations involve ensuring the column can withstand applied axial loads and bending moments without exceeding the material's yield stress or buckling stress. Stability requirements relate to the column's ability to resist buckling under compression, which depends heavily on two geometric properties:

- Moment of inertia (I): Measures the cross-section's resistance to bending. Higher values of I directly improve buckling resistance because they increase the critical buckling load.
- Radius of gyration (r): Defined as $r = \sqrt{I/A}$, where A is the cross-sectional area. A higher r means a lower slenderness ratio, which reduces the risk of buckling.

Compact sections have a higher local buckling resistance compared to slender sections, making them better suited for columns under high compressive stress. The width-to-thickness ratios of cross-section elements (flanges and webs) control local buckling behavior and must stay within limits specified by design codes.

Slenderness Ratio Influence

Definition and Effects on Column Behavior

The slenderness ratio is defined as the effective length of the column divided by its least radius of gyration: L_e/r . This single ratio largely determines how a column will fail.

- Slender columns (high L/r) are governed by global (Euler) buckling, where the entire column deflects laterally as a single unit.
- Stocky columns (low L/r) fail by material yielding or local buckling of individual elements like flanges or webs.
- Intermediate columns fall between these extremes, and their behavior involves an interaction of yielding and instability. This is the range where most real columns land, and where design curves from codes become especially important.

As the slenderness ratio increases, the column's load-carrying capacity decreases because the risk of elastic buckling grows.

Buckling Load and Design Code Provisions

The critical buckling load is inversely proportional to the square of the slenderness ratio, as given by Euler's formula:

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

where E is the modulus of elasticity, I is the moment of inertia about the buckling axis, K is the effective length factor, and L is the actual column length.

Design codes set upper limits on slenderness ratios:

- AISC (USA): Limits the slenderness ratio to 200 for main compression members.
- Eurocode 3 (Europe): Provides different limiting slenderness ratios depending on column type and material grade.

The effective length factor K accounts for end restraint conditions and their influence on the buckled shape:

- Pinned-pinned: $K = 1.0$
- Fixed-fixed: $K = 0.5$
- Fixed-pinned: $K = 0.7$
- Fixed-free (cantilever): $K = 2.0$

These are theoretical values. In practice, true fixity is hard to achieve, so real K values often fall between the idealized cases. Design codes provide alignment charts (nomographs) to estimate K for columns in frames.

Design Codes and Standards

Codes and Their Provisions

Several design codes govern column design, including AISC 360 (USA), Eurocode 3 (Europe), and CSA S16 (Canada). While the details differ, they all specify:

- Material properties and characteristic strengths
- Load combinations and resistance/safety factors
- Design equations for axial compression, tension, and combined loading

For columns under combined axial load and bending, interaction equations check whether the combined demand exceeds the column's capacity. The AISC specification uses two equations, and you apply whichever controls based on the axial load ratio:

When $P_r/P_c \geq 0.2$, use AISC H1-1a:

$$\frac{P_r}{P_c} + \frac{8}{9} \left(\frac{M_{rx}}{M_{cx}} + \frac{M_{ry}}{M_{cy}} \right) \leq 1.0$$

When $P_r/P_c < 0.2$, use AISC H1-1b:

$$\frac{P_r}{2P_c} + \left(\frac{M_{rx}}{M_{cx}} + \frac{M_{ry}}{M_{cy}} \right) \leq 1.0$$

Here, P_r is the required axial strength, P_c is the available axial strength, and the M terms represent required and available flexural strengths about each axis.

Local Buckling and Effective Length Considerations

Before a column can develop its full global buckling or yielding capacity, its individual plate elements (flanges, webs) must not buckle locally. Codes handle this through width-to-thickness ratio limits:

- AISC: Provides tables of limiting width-to-thickness ratios (λ_r and λ_p) based on cross-section shape and applied stress. Sections are classified as compact, noncompact, or slender.
- Eurocode 3: Classifies cross-sections into Class 1, 2, 3, or 4 based on width-to-thickness ratios. Class 1 and 2 sections can develop full plastic capacity, while Class 4 sections require effective cross-section properties due to local buckling.

If a section's plate elements exceed these limits, the column's capacity must be reduced to account for local buckling, typically by using an effective cross-sectional area or reduced strength.

Imperfections and Eccentricities

Initial Imperfections and Their Effects

No real column is perfectly straight or stress-free. Two types of initial imperfections reduce load-carrying capacity:

Out-of-straightness is the deviation of the column's centerline from a perfectly straight line. Even a small bow introduces bending moments under axial load, which grow as the load increases (a second-order

effect). Codes account for this with assumed imperfection magnitudes:

- AISC: Assumes an initial out-of-straightness of $L/1000$ for design purposes.
- Eurocode 3: Uses equivalent geometric imperfections tied to the column's assigned buckling curve (curves a, b, c, or d), which vary by cross-section type and fabrication method.

Residual stresses are internal stresses locked into the cross-section from manufacturing processes like hot rolling or welding. They cause portions of the cross-section to yield earlier than expected under load, reducing the effective stiffness and therefore the buckling capacity. Hot-rolled wide-flange sections, for example, typically have residual stresses on the order of 30-50% of the yield stress at the flange tips.

Eccentricities and Design Considerations

Eccentricities occur when the applied load doesn't line up with the column's centroidal axis, producing bending moments in addition to axial compression.

- Accidental eccentricities arise from construction tolerances, load misalignment, or connection geometry that shifts the load path slightly off-center.
- Intentional eccentricities are introduced by design, such as in moment-resisting frames where beams frame into one side of a column.

Both imperfections and eccentricities reduce the column's effective buckling strength. Design codes provide several methods to account for them: applying additional equivalent moments, using reduced material properties, or incorporating their effects directly into the interaction equations discussed above.

Axial Loads vs. Bending Moments

Combined Loading Effects

Columns in real structures rarely carry pure axial compression. Eccentricities, lateral loads (wind, seismic), and frame action all introduce bending moments alongside the axial force.

Bending moments cause additional compressive stress on one side of the column and tensile stress on the other. The combined stress on the compression side can exceed the yield stress or trigger premature buckling well before the pure axial capacity is reached. This is why the interaction equations (AISC H1-1a and H1-1b) are so central to column design: they check that the combined demand from axial load and bending stays within the column's capacity.

Second-Order Effects and Design Methods

When an axial load acts on a column that has already deflected (from bending moments or imperfections), the axial force creates additional bending moment equal to $P \times \delta$, where δ is the lateral deflection. This is the P-delta effect, and it amplifies the first-order moments.

The moment magnification factor B_1 approximates this amplification:

$$B_1 = \frac{C_m}{1 - \frac{P_r}{P_{e1}}} \geq 1$$

where C_m is the equivalent uniform moment factor (accounts for the shape of the moment diagram), P_r is the required axial strength, and P_{e1} is the Euler buckling load for the member. Notice that as P_r approaches P_{e1} , the magnification factor grows rapidly, which is why columns loaded near their buckling capacity are so sensitive to any bending.

Two primary design methods handle these effects:

1. Effective length method: Uses the K factor to approximate the column's behavior within a frame. First-order moments are amplified by B_1 (and B_2 for sway effects). This method is simpler but less accurate for complex frames.

2. Direct analysis method: Models second-order effects and geometric nonlinearities directly in the structural analysis software. It uses $K = 1.0$ for all members and applies notional loads to represent imperfections. This approach is more rigorous and is the preferred method in current AISC practice.

Proper consideration of end restraints, bracing conditions, and lateral loads is essential regardless of which method you use. A column braced at mid-height, for example, has a very different effective length (and therefore capacity) than the same column unbraced over its full height.